Strong Comultiplication Modules

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ABSTRACT

In this paper, we introduced the concepts of strong comultiplication modules and copure submodules and some related results more obtained.

Key Words: Comultiplication modules, Strong comultiplication modules, Copure submodules

INTRODUCTION

Throughout this paper, R will denote a commutative ring with identity and □ will denote the ring of integers. The dual notion of multiplication modules was introduced by Ansari-Toroghy and Farshadifar (2007) and the first properties of this class of modules had been considered. We recall that M is a comultiplication module (Ansari-Toroghy and Farshadifar, 2007) if for every submodule N of M there exists an ideal I of R such that $N = (0:_M I)$. Also, it is shown that (Ansari-Toroghy and Farshadifar, 2007, 3.7) M is a *comultiplication module* if and only if for each submodule N of M, $N = (0_M Ann_R(N))$. Let M be an R-module. In Section 3 of this paper, we will introduce the concepts of strong comultiplication modules and copure submodules. M is said to be a *strong comultiplication module* if M is a comultiplication R-module which satisfies the double annihilator conditions (see 1.1 (d)). Furthermore, a submodule N of M is said to be *copure* if $(N:_M I) = N + (0:_M I)$ for each ideal I of R. Now let M be an R-module and let N be a submodule of M. Among the other results, it is shown (see 2.5) that whenever M is a strong comultiplication module, M / N is a comultiplication R-module if and only if $Ann_R (N)\, Ann_R (K / N) = Ann_R (K)$ for each submodule K of M with $N \subseteq K$ Moreover, it is shown (see 2.12) that pure and copure submodules of M are the same over a principal ideal domain. Also it is proved (see 2.13) that whenever M is a strong comultiplication module, N is a copure submodule of M if and only if $Ann_R (N)$ is a pure ideal of R. Moreover, it is shown (see 2.13) that if N is a copure submodule of a strong comultiplication module M, then $(N:_R M) = Ann_R\, Ann_R (N)$ and $Ann_R (N)$ is the intersection of all ideals I of R such that $N = (N:_M I)$. Finally, it is proved (see 2.15) that if M is a comultiplication (resp. multiplication) module such that $Soc(M)$ (resp. $Rad(M)$) is a pure (resp. copure) submodule of M, then $M = Soc(M)$ (resp. $Rad(M)=0$).

1 Auxiliary results

Definition 1.1.

(a) An R -module M is said to be a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$.

(b) An R -module M is said to be a *comultiplication module* (Ansari-Toroghy and Farshadifar, 2007) if for every submodule N of M there exists an ideal I of R such that $N = (0:_M I)$.

(c) A submodule N of an R -module M is said to be *pure* (Anderson and Fuller, 1974) if $IN = IM \cap N$ for every ideal I of R.

(d) An R -module M satisfies *the double annihilator conditions* (DAC for short) (Faith (1995)) if for each ideal I of R, we have $I = Ann_R\,(0:_M I)$.

(e) An R -module L is said to be *cocyclic* (Yassemi, 1998) if L is isomorphic to a submodule of $E(R\,/\,P)$ for some maximal ideal P of R.

(f) An endomorphism f of an R -module M is said to be *trivial* (Choi and Smith, 1994) if there exists $a \in R$ such that $f(m) = am$ for all $m \in M$.

(g) A submodule N of an R -module M is said to be *large* (resp. *small*) if for every submodule L of M, $N \cap K \neq 0$ (resp. $N + L = M$ implies that $L = M$ (Anderson and Fuller, 1974).

(h) A non-zero submodule N of an R -module M is said to be a *second submodule* of M (Yassemi, 2001) if for each $a \in R$, the homothety $N \xrightarrow[a]{} N$ is either surjective or zero. Also M is said to be a *second module* if M is a second submodule of itself.

(i) A proper submodule N of an R -module M is said to be *prime* if for each $a \in R$, the homothety $M\,/\,N \xrightarrow[a]{} M\,/\,N$ is either injective or zero. Also M is said to be a *prime module* if the zero submodule of M is prime.

Remark 1.2 (Ansari-Toroghy and Farshadifar, 2008a). Let M be a comultiplication R -module. Then every non-zero submodule of M contains a minimal submodule of M. Moreover, every minimal submodule of M is of the form $(0:_M P)$ where P is a maximal ideal of R.

2 Main results

Definition 2.1. We say that an R -module M is a *strong comultiplication module* if M is a comultiplication R -module and satisfies the DAC conditions.

Example 2.2. By (Sharpe and Vamos, 1972), $E(R\,/\,m)$ satisfies the DAC conditions, where (R,m) is a local ring. Hence by (Ansari-Toroghy and Farshadifar, 2007), $E(R/m)$ where (R,m) is a complete Noetherian local ring is a strong comultiplication R -module.

Example 2.3. Let p be a prime number and n be a positive integer. Then $\mathbb{Z}_{p^\infty}$ and $\mathbb{Z}_n$ are comultiplication $\mathbb{Z}$ -modules but they are not strong comultiplication $\mathbb{Z}$ -modules.

Proposition 2.4. Let M be an R -module. Then we have the following.

(a) Let M be a faithful cogenerator for R and let $S = End_R(M)$. If every $f \in S$ is trivial, then M is a strong comultiplication R -module.

(b) If R is a Noetherian ring and M is a strong comultiplication R -module, then M is an injective R -module.

Proof. (a) Let N be a submodule of M. By (Faith, 1995, Theorem 7), $N = (0:_M Ann_S(N))$, and $I = Ann_R(0:_M I)$. Now since M is faithful and every $f \in S$ is trivial, $(0:_M Ann_R(N)) = (0:_M Ann_S(N))$. Therefore, M is a strong comultiplication R -module.

(b) By (Ansari-Toroghy and Farshadifar, 2008b, 3.3), for a collection $\{M_\lambda\}_{\lambda\in\Lambda}$ of submodules of M, we have

$$(0:_M \bigcap_{\lambda\in\Lambda} Ann_R(M_\lambda)) = \sum_{\lambda\in\Lambda}(0:_M Ann_R(M_\lambda)).$$

Now the result follows by (Faith, 1976, 23.22) and the fact that for each ideal I, we have $I = Ann_R(0:_M I)$.

Theorem 2.5. Let M be a strong comultiplication R -module and let I be an ideal of R. Let N be a submodule of M.

(a) M / N is a comultiplication R -module if and only if $Ann_R(N)Ann_R(K / N) = Ann_R(K)$ for each submodule K of M with $N \subseteq K$.

(b) If M / N is a comultiplication R -module, then $Ann_R(N)$ is a multiplication ideal of R.

(c) If $M /(0:_M I)$ is a comultiplication R -module, then I is a multiplication ideal of R.

Proof. (a) Suppose that M / N is a comultiplication R -module and K is a submodule of M with $N \subseteq K$. Then

$$K / N = (0:_{M/N} Ann_R(K/N)) = (0:_M Ann_R(N)Ann_R(K/N))/N$$

Hence $K = (0:_M Ann_R(N)Ann_R(K / N))$. It follows that

$$Ann_R(K) = Ann_R(N)Ann_R(K/N)$$

because M is a strong comultiplication R -module. Conversely, suppose that K / N is a submodule of M / N. Then $N \subseteq K$ and by hypothesis, $Ann_R(N)Ann_R(K / N)) = Ann_R(K)$. Thus

$$\begin{aligned}(0:_{M/N} Ann_R(K/N)) &= (0:_M Ann_R(N)Ann_R(K/N))/N \\ &= (0:_M Ann_R(K))/N = K/N\end{aligned}$$

as desired.

(b) Let I be an ideal of R contained in $Ann_R(N)$. Then $IN = 0$. Hence $N \subseteq (0:_M I)$. Thus by part (a),

$$Ann_R(N)Ann_R((0:_M I)/N) = Ann_R(0:_M I).$$

Since M satisfies the DAC conditions, $Ann_R(N)Ann_R((0:_M I) / N) = I$ as desired.

(c) Since M satisfies the DAC conditions, the result follows by part (b).

Example 2.6. Let $A = K[x, y]$ be the polynomial ring over a field K in two indeterminates x, y. Then $\overline{A} = A / (x^2, y^2)$ is a strong comultiplication $\overline{A}$ -module. But $\overline{A} / \overline{A}\overline{xy}$ is not a comultiplication $\overline{A}$ -module by (Faith, 1976, 24.4). Furthermore, this example shows that every homomorphic image of a strong comultiplication module is not a comultiplication module in general.

Definition 2.7. We say that a submodule N of an R -module M is *copure* if $(N:_M I) = N + (0:_M I)$ for each ideal I of R.

Example 2.8. Every submodule of □ $_n$ (as a □ -module) is copure, where n is square-free.

Theorem 2.9. Let M be an R -module and let N and K be submodules of M such that $N \subseteq K \subseteq M$.

(a) If K is a copure submodule of M and N is a copure submodule of K, then N is a copure submodule of M.

(b) If N is a copure submodule of M, then N is a copure submodule of K.

(c) If K is a copure submodule of M, then K / N is a copure submodule of M / N.

(d) If N is a copure submodule of M and K / N is a copure submodule of M / N, then K is a copure submodule of M.

(e) If N is a copure submodule of M, then there is a bijection between the copure submodules of M containing N and the copure submodules of M / N.

Proof. (a) Let I be an ideal of R. Then since K is a copure submodule of M,

$$(N :_M I) = (N \cap K :_M I) = (N :_M I) \cap (K :_M I) =$$
$$(N :_M I) \cap (K + (0 :_M I)) = (N :_K I) + (0 :_M I).$$

Now since N is a copure submodule of K, we have

$$(N :_M I) = N + (0 :_K I) + (0 :_M I) = N + (0 :_M I).$$

(b) Let I be an ideal of R. Then

$$(N :_K I) = K \cap (N :_M I) = K \cap (N + (0 :_M I)) =$$
$$K \cap N + K \cap (0 :_M I) = N + (0 :_M I).$$

(c) Let I be an ideal of R. Then

$$(K / N :_{M/N} I) = (K :_M I) / N = ((K :_M I) + K \cap (N :_M I)) / N$$
$$(K + (0 :_M I) + K \cap (N :_M I)) / N = K / N + ((N :_M I) \cap (K + (0 :_M I)) / N$$
$$K / N + ((N :_M I) \cap (K :_M I)) / N = K / N + (N :_M I) / N =$$
$$K / N + (0 :_{M/N} I).$$

(d) Let I be an ideal of R. Since N is a copure submodule of M,

$$(0:_{M/N} I) = (N:_M I)/N = ((0:_M I) + N)/N.$$

Now since K/N is a copure submodule of M/N, it follows that

$$(K:_M I)/N = (K/N:_{M/N} I) = K/N + (0:_{M/N} I) =$$
$$K/N + ((0:_M I) + N)/N = (K + (0:_M I))/N.$$

Thus $(K:_M I) = K + (0:_M I)$ as desired.
(e) This follows from part (c) and (d) and the proof is completed.

Theorem 2.10. Let

$$0 \longrightarrow N \xrightarrow{\psi} L \xrightarrow{\varphi} K \longrightarrow 0$$

be an exact sequence of R -modules and R -homomorphisms. Then the following assertions are equivalent.

(a) $\psi(N)$ is a copure submodule of L.

(b) For every ideal I of R the following sequence is exact.

$$0 \longrightarrow Hom_R(R/I, N) \xrightarrow{\bar{\psi}} Hom_R(R/I, L) \xrightarrow{\bar{\varphi}} Hom_R(R/I, K) \longrightarrow 0.$$

Proof. $(a) \Rightarrow (b)$ Let I be an ideal of R. Since $Hom_R(R/I,-)$ is a left exact functor, it is enough to show that $\bar{\phi} : Hom_R(R/I, L) \longrightarrow Hom_R(R/I, K)$ is epic. To see this let $f: R/I \longrightarrow K$ be an R -homomorphism. Then since ϕ is epic, there exists $x \in L$ such that $\phi(x) = f(1+I)$. Thus $Ix \subseteq Ker(\phi) = Im(\psi)$. Therefore by assumption, $x\ \psi(n) + y$ for some $y \in (0:_L I)$ and $n \subseteq N$.Thus we can define $g : R/I \longrightarrow L$ given by $r + I \mapsto ry$ for each $r \in R$.Therefore, $\bar{\phi}(g) = \phi g = f$ as desired. $(b) \Rightarrow (a)$. Let I be an ideal of R. Clearly $\psi(N) + (0:_L I) \subseteq (\psi(N):_L I)$. Now let $x \in (\psi(N):_L I)$. If $r_1 - r_2 \in I$, then $r_1\phi(x) = r_2\phi(x)$. Hence we can define an R -homomorphism $f: R/I \longrightarrow K$ given by $r + I \mapsto r\phi(x)$, where $r \in R$. By assumption, $\bar{\phi}$: $Hom_R(R/I, L) \longrightarrow Hom_R(R/I, K)$ is epic. Thus there exists $g \in Hom_R(R/I, L)$ such that $\bar{\phi}(g) = \phi g = f$. This implies that $g(1 + I) - x \in \text{Im}(\psi)$. Hence $\psi(n) = g(1 + I) - x$ for some $n \in N$. It follows that $x \in \psi(N) + (0:_L I)$ as desired.

Proposition 2.11. Let M be an R -module.

(a) If M is a multiplication module and N is a small copure submodule of M, then $N = 0$.

(b) If M is a comultiplication module and N is a large pure submodule of M, then $N = M$.

(c) If N and K are submodules of M such that $N \cap K$ and $N + K$ are copure submodules of M. Then N is a copure submodule of M.

(d) If $\{M_\lambda\}_\Lambda$ is a family of submodules of M with copure submodules $N_\lambda \subseteq M_\lambda$, then $\sum_{\lambda\in\Lambda} N_\lambda$ is a copure submodule of $\sum_{\lambda\in\Lambda} M_\lambda$.

Proof. (a) Since N is copure,

$$M = (N:_M (N:_R M)) = N + (0:_M (N:_R M)).$$

Thus $M = (0:_M (N:_R M))$ because N is small. Now since M is a multiplication module, $N = 0$.

(b) Since N is pure,

$$0 = Ann_R(N)N = Ann_R(N)M \cap N.$$

Thus $Ann_R(N)M = 0$ because N is large. Now since M is a comultiplication module, $N = M$.

(c) Let I be an ideal of R. Clearly $(N:_M I) \supseteq N + (0:_M I)$. Now let $m \in (N:_M I)$. Then $Im \subseteq N + K$. Since $K + N$ is copure $m = x + y + t$ for some $x \in N$, $y \in K$ and $t \in (0:_M I)$. Thus $mI = xI + yI$. This implies that $yI \subseteq N \cap K$. Since $N \cap K$ is copure, $y = x' + t'$ for some $x' \in N \cap K$ and $t' \in (0:_M I)$. It follows that $m \in N + (0:_M I)$ as desired.

(d) This is straightforward.

Theorem 2.12. Let R be a principal ideal domain and let M be an R -module.

(a) Every submodule of M is a pure submodule of M if and only if it is a copure submodule of M.

(b) If M is a second module, then every pure submodule of M is a second submodule of M.

(c) If M is a prime module, then every copure submodule of M is a prime submodule of M.

Proof. (a) First suppose that N is a pure submodule of M and $r \in R$. Let $m \in M$ and $rm \in N$. Then $rm = rn$, where $n \in N$. Thus $m = (m - n) + n \in (0:_M r) + N$. This shows that N is copure because the reverse inclusion is clear. Now suppose that N is a copure submodule of M and $r \in R$. Let $m \in M$ and $rm \in N$. Then $m = n_1 + m_1$, where $n_1 \in N$ and $rm_1 = 0$. Thus $rm = rn_1 \in rN$. This shows that N is pure because the reverse inclusion is clear.

(b) Let N be a pure submodule of M and r $\in$ R. Then $rN = rM \cap N$. Since M is a second module, $rM = M$ or $rM = 0$. Therefore, $rN = M \cap N = N$ or $rN = 0 \cap N = 0$ as desired.

(c) Let N be a copure submodule of M and $rm \in N$, where $r \in R$ and $m \in M$. Since N is a copure, $(N:_M r) = N + (0:_M r)$. But $(0:_M r) = 0$ or $r \in Ann_R(M)$ because M is a prime module. Therefore, $rm \in N$ implies that $m \in (N:_M r) = N$ or $r \in (N:_R M)$ as desired.

Theorem 2.13. Let M be a strong comultiplication R-module.

(a) N is a copure submodule of M if and only if $Ann_R(N)$ is a pure ideal of R.

(b) An ideal I of R is pure if and only if $(0:_M I)$ is a copure submodule of M.

(c) If N is a copure submodule of M, then for every non-empty collection $\{I_\lambda\}_{\lambda \in \Lambda}$ of ideals of R, we have

$$\sum_{\lambda \in \Lambda} (N :_M I_\lambda) = (N :_M \bigcap_{\lambda \in \Lambda} I_\lambda).$$

(d) If N is a copure submodule of M, then $Ann_R(N)$ is the intersection of all ideals I of R such that $N=(N:_M I)$.

(e) If N is a copure submodule of M, then $(N:_R M)=Ann_R Ann_R(N)$

Proof. (a) Let N be a copure submodule of M and let I be an ideal of R. Then since M is comultiplication R -module,

$$(0:_M Ann_R(N)I) = (N:_M I) = (0:_M Ann_R(N)\cap I).$$

It follows that $Ann_R(N)I = Ann_R(N) \cap I$ because M is a strong comultiplication module. Therefore, $Ann_R(N)$ is a pure ideal of R. Conversely, assume that N is a submodule of M such that $Ann_R(N)$ is a pure ideal of R. Then for each ideal I of R, we have

$$(N:_M I) = (0:_M Ann_R(N)I) = (0:_M Ann_R(N)\cap I)$$
$$= N + (0:_M I)$$

as desired.

(b) Let I be a pure ideal of R. Since M satisfis the DAC conditions, $I = Ann_R(0:_M I)$Thus the result follows by part (a).

(c) Let $\{I_\lambda\}_{\lambda\in\Lambda}$ be any collection of ideals of R. Then

$$Ann_R(\sum_{\lambda\in\Lambda}(N:_M I_\lambda)) = \bigcap_{\lambda\in\Lambda} Ann_R(N:_M I_\lambda) =$$
$$\bigcap_{\lambda\in\Lambda} Ann_R((0:_M Ann_R(N)):_M I_\lambda) =$$
$$\bigcap_{\lambda\in\Lambda} Ann_R(N)I_\lambda = Ann_R(N)(\bigcap_{\lambda\in\Lambda} I_\lambda)$$

as desired.

(d) Let S be the collection of all ideals I of R with the property that $N = (N{:}M\ I)$. Then by part (c),

$$N = \sum_{I\in S}(N:_M I) = (N:_M \bigcap_{I\in S} I).$$

Thus

$$Ann_R(N) = Ann_R(N:_M \bigcap_{I\in S} I) = Ann_R(0:_M Ann_R(N)(\bigcap_{I\in S} I))$$
$$= Ann_R(N)(\bigcap_{I\in S} I).$$

Therefore, $Ann_R(N) \subseteq \bigcap_{I\in S} I$. On the other hand, since N is pure and M is a comultiplication R -module, $(N{:}MAnn_R(N))=N$. Thus $\bigcap_{I\in S} I \subseteq Ann_R(N)$.

(e) Since N is copure,

$$M = (N:_M (N:_R M)) = N + (0:_M (N:_R M)).$$

Thus

$$0 = Ann_R(M) = Ann_R(N)\cap Ann_R(0:_M (N:_R M))$$
$$= Ann_R(N)\cap(N:_R M).$$

Hence $Ann_R(N)(N:_R M) = 0$. Thus

$$(N :_R M) = Ann_R Ann_R(N).$$

Conversely, if $r \in Ann_R Ann_R(N)$, then $Ann_R(N)rM = 0$. Hence $rM \subseteq (0:_M Ann_R(N))=N$. Therefore, $R \in (N:_R M)$ as desired.

The following example shows that in Theorem 2.13 (e) the condition M is a strong comultiplication module cannot be omitted.

Example 2.14. The $\square$ -module $M = \square_{p^\infty} \oplus \square_{p^\infty}$ is not a strong comultiplication $\square$ -module. We have $N = 0 \oplus \square_{p^\infty}$ is a copure submodule of M. But $(N:_\square M) \neq Ann_\square Ann_\square (N)$.

Theorem 2.15. Let M be an R -module.

(a) If M is a comultiplication module and Soc(M) is a pure submodule of M, then $M=soc(M)$. In particular, if R is a local ring, then M is simple. (Here $Soc(M)$ denotes the sum of all minimal submodules of M.)

(b) If M is a multiplication module and $Rad(M)$ is a copure submodule of M, then $Rad(M)=0$. In particular, if R is a local ring, then M is simple. (Here $Rad(M)$ denotes the intersection of all maximal submodules of M.)

Proof. (a) Set $I=Ann_R(Soc(M))$. Since $Soc(M)$ is pure, $IM \cap Soc(M) = ISoc(M) = 0$. Now if $IM \neq 0$, then by Remark 1.2, there exists a minimal submodule K of M such that $K \subseteq IM$. Thus $K = K \cap Soc(M) = 0$, which is a contradiction. Therefore $IM = 0$. Hence $I \subseteq Ann_R(M)$. Thus $M = Soc(M)$ because M is a comultiplication R -module. The last assertion follows from this and Remark 1.2.

(b) Set $I = (Rad(M):_R M)$. Since $Rad(M)$ is copure,

$$(Rad(M) :_M I) = Rad(M) + (0 :_M I).$$

This implies that $M = Rad(M)+(0:_M I)$ Now if $(0:_M I) \neq M$, by (El-Bast and Smith (1988), 2.5), there exists a maximal submodule K of M such that $(0:_M I) \subseteq K$. Thus $M = Rad(M) + K = K$, which is a contradiction. Thus $(0:_M I) = M$. It follows that $Rad(M) = IM = 0$ because M is a multiplication R -module. The last assertion follows from this and (El-Bast and Smith, 1988, 2.5).

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