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Contributed Paper

On Graded Prime Submodules

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ABSTRACT

Let G be a monoid with identity e, and let R be a G-graded commutative ring. Here we study the graded prime submodules of a G-graded R-module. A number of results concerning of these class of submodules are given.

Keywords: graded prime submodules, graded rings.

1. INTRODUCTION

Several authors have extended the notion of prime ideals to modules (see [1] and [2], for example). Let R be a G-graded commutative ring and M a graded R-module. In this paper we introduce the concepts of graded prime submodules of M and give some of their basic properties. However, the prime and graded prime are different concepts.

Before we state some results, let us introduce some notations and terminologies. Let G be an arbitrary monoid with identity e. By a G-graded commutative ring we mean a commutative ring R with non-zero identity together with a direct sum decomposition (as an additive group) $R=\oplus_{g\in G}R_g$ with the property that $R_gR_h \subseteq R_{gh}$ for all $g,h\in G$. We denote this by $G(R)$. Also, we write $h(R)=\bigcup_{g\in G}R_g$. The summands R_g are called homogeneous components and elements of these summands are called homogeneous elements. If $a\in R$, then a can be written uniquely as $\sum_{g\in G}a_g$ where a_g is the component of a in R_g. In this case, R_e is a subring of R and $1_R\in R_e$.

Let R be a graded ring and M an R-module. We say that M is a G-graded R-module if there exists a family of subgroups $\{M_g\}_{g\in G}$ of M such that $M=\oplus_{g\in G}M_g$ (as abelian groups), and $R_gM_h\subseteq M_{gh}$ for all $g,h\in G$, here R_gM_h denotes the additive subgroup of M consisting of all finite sums of elements r_gs_h with $r_g\in R_g$ and $s_h\in M_h$. Also, we write $h(M)=\bigcup_{g\in G}M_g$. If $M=\oplus_{g\in G}M_g$ is a graded module, then, M_g is an R_e-module for all $g\in G$. Let $M=\oplus_{g\in G}M_g$ be a graded R-module and N a submodule of M. For $g\in G$, let $N_g=N\bigcap M_g$. Then N is a graded submodule of M if $N=\oplus_{g\in G}N_g$. In this case, N_g is called the g-component of N for $g\in G$. Moreover, M/N becomes a G-graded module with g-component $(M/N)_g$ = $(M_g+N)/N$ for $g\in G$. Clearly, 0 is a graded submodule of M. An ideal I of $G(R)$ is said to be graded prime ideal if $I\neq R$ and whenever $ab\in I$, we have $a\in I$ or $b\in I$, where $a,b\in h(R)$.

2. THE RESULTS

Our starting point is the following lemma:

Lemma 2.1. *Let R be a G-graded ring, and M,N be graded R-modules. Then* $(N:_RM)=\{r\in R: rM\subseteq N\}$ *is a graded ideal of* $G(R)$.

Proof. Since $\oplus_{h\in G}(N:_RM)_h\subseteq(N:_RM)$ is trivial, we will prove the reverse inclusion. Let $a=\sum_{h\in G}a_h\in(N:_RM)$. It is enough to show that

$a_h M \subseteq N$ for all $h \in G$. Without loss of generality we may assume that $a = \sum_{i=1}^{m} a_{h_i}$ where $a_{h_i} \neq 0$ for all $i = 1,2,...,m$ and $a_h = 0$ for all $h \notin \{h_1,....,h_m\}$. As $a \in (N :_R M)$, we obtain $\sum_{i=1}^{m} a_{h_i} M \subseteq N$. It suffices to show that for each i, $a_{h_i} m \in N$ for any $m \in M$. Since M is a graded module, we can assume that $m = \sum_{j=1}^{n} m_{g_j}$ with $m_{g_j} \neq 0$ for all j. Now we show that $a_{h_i} m_{g_j} \in N$ for all j. Since for each j, $am_{g_j} \in N$ and N is a graded module, we obtain $a_{h_i} m_{g_j} \in N \bigcap M_{h_i g_j} \subseteq N$. Thus $a_{h_i} M \subseteq N$ for all $i = 1,2,...,m$, as required. •

Definition 2.2. Let R be a G-graded ring, M a graded R-module, N a graded submodule of M and $g \in G$.

(i) We say that M_g is a g-torsion-free R_e-module whenever $a \in R_e$ and $m \in M_g$ with $am = 0$ implies that either $m = 0$ or $a = 0$.

(ii) We say that M is a graded torsion-free R-module whenever $a \in h(R)$ and $m \in M$ with $am = 0$ implies that either $m = 0$ or $a = 0$.

(iii) We say that N_g is a g-pure submodule of the R_e-module M_g if for each $a \in R_e$, $aN_g = N_g \bigcap aM_g$.

(iv) We say that N is a graded pure submodule of M if for each $a \in h(R)$, $aN = N \bigcap aM$.

(v) We say that N_g is a g-prime submodule of the R_e-module if $N_g \neq M_g$; and whenever $a \in R_e$ and $m \in M_g$ with $am \in N_g$, then either $m \in N_g$ or $a \in (N_g :_{R_e} M_g)$.

(vi) We say that N is a graded prime submodule of M if $N \neq M$ and whenever $a \in h(R)$ and $m \in h(M)$ with $am \in N$, then either $m \in N$ or $a \in (N :_R M)$.

Lemma 2.3. *Let R be a G-graded ring, M a graded R-module and N a graded submodule of M.*

(i) If N is a graded prime submodule of M, then N_g is a g-prime submodule of M_g for every $g \in G$.

(ii) If M is a graded torsion-free R-module, then M_g is a g-torsion-free R_e-module for every $g \in G$.

(iii) If N is a graded pure submodule of a graded torsion-free R-module M, then N_g is a g-pure submodule of M_g for every $g \in G$.

Proof. (i) Suppose that N is a graded prime submodule of M. For $g \in G$, assume that $am \in N_g \subseteq N$ where $a \in R_e$ and $m \in M_g$. Since N is graded prime it gives either $m \in N$ or $a \in (N :_R M)$. If $m \in N$, then $m \in N_g$. If $a \in (N :_R M)$, then $aM_g \subseteq aM \subseteq N$. Hence $a \in (N_g :_{R_e} M_g)$. So N_g is a prime submodule of M_g.

(ii) This part is obvious.

(iii) Assume that N is a graded pure submodule of M and let $a \in R_e$ and $g \in G$. Since $aN_g \subseteq N_g \bigcap aM_g$ is trivial, we will prove the reverse inclusion. Take any $am \in N_g \bigcap aM_g$ where $m \in M_g$. We can assume that $am \neq 0$. Then $am \in N \bigcap aM = aN$ since N is a graded pure submodule. Therefore, $am = at$ for some $t \in N_g$; hence $m = t$ since M_g is g-torsion-free by (ii). Thus $am \in aN_g$, as required. •

Proposition 2.4. *Let R be a G-graded ring, M a graded torsion-free R-module and N a proper graded submodule of M. Then N is a graded pure submodule of M if and only if, it is graded prime in M with $(N :_R M) = 0$.*

Proof. Assume that N is a graded pure submodule of M and let $rm \in N$ with $r \notin (N :_R M)$, where $r \in h(R)$ and $m \in h(M)$. Then $rm \in N \bigcap rM = rN$, so $rm = rn$ for some homogeneous element n of N. It follows that $m = n \in N$ since M is graded torsion-free. Suppose that $a \in (N :_R M)$ with $a \neq 0$. Without loss of generality assume $a = \sum_{i=1}^{n} a_{g_i}$ where $a_{g_i} \neq 0$ for all $i = 1,2,...,n$ and $a_{g=0}$ for all $g \notin \{g_1,...,g_n\}$. As $N \neq M$, there is a homogeneous element m_s of M such that $m_s \notin N$ and $a_{g_i} m_s \in N$ for all $i = 1,2,...,n$ since N is a graded submodule. Since for every i, $a_{g_i} m_s \in N \bigcap a_{g_i} M = a_{g_i} N$, there exists a homogeneous element b of N such that $a_{g_i} m_s = a_{g_i} b$. Thus $m_s = b \in N$ since M is graded torsion-free, which is a contradiction. So $(N :_R M)=0$.

Conversely, assume that N is graded prime in M with $(N :_R M)=0$ and let $a \in h(R)$. It is enough to show that $aM \bigcap N \subseteq aN$. Let $ax \in aM \bigcap N$ where $x \in M$. We can assume that $ax \neq 0$. There are non-zero homogeneous elements $x_{h_1},...,x_{h_t}$ of M such that $ax_{h_1},...,ax_{h_t} \in N$ since N is graded submodule. So N graded prime and $a \neq 0$ gives $x_{h_1},...,x_{h_t} \in N$. Hence $x \in N$, which is required. •

Let R be a G-graded ring and M a graded R-module. We say that R is a graded integral domain whenever $a,b \in h(R)$ with $ab = 0$ implies that either $a = 0$ or $b = 0$. If R is a graded ring and M is a graded R-module, the subset $T(M)$ of M is defined by $T(M) = \{m \in M : rm = 0$ for some $0 \neq r \in h(R)\}$

Clearly, R is an integral domain if and only if R is a graded integral domain, so if R is a graded integral domain, then $T(M)$ is a submodule of M.

Proposition 2.5. *Let* R *be a G-graded ring, M a graded* R*-module and P a graded ideal of G(*R*). Then the following hold:*

(i) If R *is a graded integral domain, then T(M) is a graded submodule of M.*

(ii) If R *is a graded integral domain and T(M) ≠ M, then T(M) is a graded prime submodule of M.*

(iii) Let R *be an overring of S such that S is a G-graded ring. Then every graded prime ideal P of* R *is a graded prime submodule of S-module* R *with* $(P :_S R) = P \cap S$.

(iv) R/P *is a graded integral domain if and only if P is a graded prime ideal of G(*R*).*

Proof.

(i) It is enough to show that $T(M) = \oplus_{g\in G}(T(M) \cap M_g)$. Clearly, $\oplus_{g\in G}(T(M) \cap M_g) \subseteq T(M)$. Let $m = \sum_{g\in G} m_g \in T(M)$. Our goal is to show that $m_g \in T(M)$ for all $g \in G$. Without loss of generality assume $m = \sum_{i=1}^{n} m_{g_i}$ where $m_{g_i} \neq 0$ for all $i = 1,...,n$ and $m_g = 0$ for all $g \notin \{g_1,...,g_n\}$. Since $m \in T(M)$, there exists a non-zero element $r \in h(R)$ such that $rm = 0$, so we get $rm_{g_1} = ... = rm_{g_n} = 0$. Hence $m_{g_i} \in T(M)$ for all i, as needed.

(ii) Let $am \in T(M)$ with $a \notin (T(M) :_R M)$, where $a \in h(R)$ and $m \in h(M)$. Then $a \in R_g$ and $m \in M_h$ for some $g,h \in G$. Since $am \in T(M)$, there exists a non-zero element b of $h(R)$, say $b \in R_t$, such that $abm = 0$. If $am = 0$, then $m \in T(M)$. So suppose that $am \neq 0$. As R is a graded integral domain, we get $0 \neq ab \in R_{gt} \subseteq h(R)$. Hence $m \in T(M)$. Thus $T(M)$ is graded prime.

(iii) Let $ab \in P$ where $a \in h(S)$ and $b \in h(R)$. Then either $a \in P$ or $b \in P$ since P is a graded prime ideal of $G(R)$. If $a \in P$, then $a \in (P :_R R)$. Otherwise, $b \in P$. Hence P is a prime submodule. Finally, the equality $(P :_S R) = P \cap S$ is clear.

(iv) The proof is completely straightforward. •

Lemma 2.6. *Let* R *be a G-graded ring, M a graded* R*-module, N a graded submodule of M and g ∈ G. Then the following assertions are equivalent.*

(i) N_g *is a prime submodule of* M_g*;*

(ii) If whenever $IB \subseteq N_g$ *with I an ideal of* R_e *and B a submodule of* M_g *implies that* $I \subseteq (N_g :_{R_e} M_g)$ *or* $B \subseteq N_g$.

Proof. (i) ⇒ (ii) Suppose that N_g is a prime submodule of M_g. Let $IB \subseteq N_g$ with $x \in B - N_g$. We want to prove that $I \subseteq (N_g :_{R_e} M_g)$. Let $a \in I$. Then $ax \in N_g$, so $a \in (N_g :_{R_e} M_g)$ since is prime.

(ii) ⇒ (i) Suppose that $cy \in N_g$ where $c \in R_e$ and $y \in M_g$. Take $I = R_e c$ and $B = R_e y$. Then $IB \subseteq N_g$, so either $B \subseteq N_g$ or $I \subseteq (N_g :_{R_e} M_g)$ by (ii). Hence either $y \in N_g$ or $c \in (N_g :_{R_e} M_g)$. So N_g is prime. •

Proposition 2.7. *Let* R *be a G-graded ring, M a graded* R*-module, N a graded prime submodule of M and g ∈ G. Then the following hold:*

(i) $(N_g :_{R_e} M_g)$ *is a prime ideal of* R_e.

(ii) $(N :_R M)$ *is a graded prime ideal of G(*R*).*

Proof. (i) By Lemma 2.3, N_g is a prime submodule of M_g, so $(N_g :_{R_e} M_g) \neq R_e$. Let $ab \in (N_g :_{R_e} M_g)$ where $a,b \in R_e$. Then $abM_g \subseteq N_g$. If $bt \in N_g$ for every $t \in M_g$, then $b \in (N_g :_{R_e} M_g)$. So suppose that there is an element $n \in M_g$ such that $bn \notin N_g$. As $abn \in N_g$ and $bn \notin N_g$, we get $a \in (N_g :_{R_e} M_g)$, as needed.

(ii) As N is a graded prime submodule of M, we get $(N :_R M) \neq R$. Let $cd \in (N :_R M)$

where $c,d \in h(R)$. Then $cdM \subseteq N$. If $dM \subseteq N$, then $d \in (N :_R M)$. So suppose that there exists $m \in M$ such that $dm \notin N$. As M is a graded R-module, there is an element $h \in G$ such that $dm_h \notin N$. Since $cdm \in N$ and N is a graded submodule, we have $cdm_h \in N$. Since N is graded prime gives $c \in (N :_R M)$ since $dm_h \notin N$, as required. □

Lemma 2.8. *Let R be a G-graded ring and M a graded R-module. Assume that N and K are graded submodules of M with $K \subseteq N$. Then N is a graded prime submodule of M if and only if N/K is a graded prime submodule of the R-module M/K.*

Proof. Let N be a graded prime submodule of M. Then $N/K \neq M/K$. To show that N/K is a prime submodule of M/K, let $a(m+K) \in N/K$ where $a \in h(R)$ and $m + K \in h(M/K)$, so $m \in h(M)$ and $am \in N$. Since, N is graded prime it gives either $m + K \in (M/K)$ or $a \in (N :_R M) = (N/K :_R M/K)$. Similarly, we can prove that if N/K is graded prime, then N is graded prime. □

Theorem 2.9. *Let R be a G-graded ring and M a graded R-module. Assume that A and B are graded submodules of M with $A + B \neq M$. Then $A + B$ is a graded prime submodule of M.*

Proof. Since $(A + B)/B \cong B/(A \cap B)$, we obtain $A + B$ is a graded prime submodule of M by Lemma 2.8. □

Theorem 2.10. *Let R be a G-graded ring, M a graded R-module and N a graded prime submodule of N with $(N :_R M) = P$. Then there is a one-to-one correspondence between graded prime submodules of the R/P-module M/N and the graded prime submodules of M containing N.*

Proof. Let K be a graded prime submodule of M containing N. Since $K \neq M$ and $P = (N :_R M) \subseteq (K :_R M)$, we get that K/N is a proper R/P-submodule of M/N. Let $(a+P)(m+N) = am + N \in K/N$ for $a \in h(R)$ and $m \in h(M)$. Then K being graded prime gives either $m \in M$ or $aM \subseteq K$. Hence either $m + N \in K/N$ or $(a+P)(M/N) \subseteq K/N$. Therefore, K/N is a graded prime submodule of M/N. Conversely, let K/N be a graded prime submodule of M/N. To show that K is a graded prime submodule of M, we suppose that $bt \in K$ where $b \in h(R)$ and $t \in h(M)$. Then $(b+P)(t+N) = bt+N \in K/N$. So K/N being graded prime gives either $t \in K$ or $bM \subseteq K$, as required. □

Theorem 2.11. *Let R be a G-graded ring, M a graded R-module, N a graded submodule of M and $g \in G$. Then the following hold:*

(i) N_g is a prime submodule of M_g if and only if $(N_g :_{R_e} M_g) = P_g$ is a prime ideal of R_e and M_g/N_g is a g-torsion-free R_e/P_g-module.

(ii) N is a graded prime submodule of M if and only if $(N :_R M) = P$ is a graded prime ideal of G(R) and M/N is a graded torsion-free R/P-module.

Proof. (i) First suppose that N_g is a prime submodule of M_g. Then by Proposition 2.7 (i), P_g is a prime ideal of R_e and M_g/N_g is an R_e/P_g-module. If $(a+P_g)(m+N_g) = N_g$ where $a \in R_e$ and $m \in M_g$, then $am \in N_g$. So either $m \in N_g$ or $a \in P_g$. Hence $m+N_g = N_g$ or $a+P_g = P_g$. Therefore, M_g/N_g is a g-torsion-free R_e/P_g-module. Conversely, let P_g be a prime ideal of R_e and let M_g/N_g be a g-torsion-free R_e/P_g-module. Since $P_g = (N_g :_{R_e} M_g) \neq R_e$, $N_g \neq M_g$. To show that N_g is a prime submodule of M_g, assume that $bt \in N_g$ for $b \in R_e$, $t \in M_g$. So $(b+P_g)(t+N_g) = bt+N_g = N_g$. Hence either $b \in P_g$ or $t \in N_g$ and the proof is complete.

(ii) Let N be a graded prime submodule of M. Then by Proposition 2.7(ii), P is a graded prime ideal of $G(R)$ and M/N is an R/P-module. Suppose that $(p+P)(n+N) = N$ where $p+P \in h(R/P)$ and $n+N \in h(M/N)$. Then $pn \in N$ for some $p \in h(R)$, $n \in h(M)$. Therefore, N being graded prime gives either $p+P = P$ or $n+N = N$. Hence M/N is graded torsion-free. Conversely, assume that P is a graded prime ideal of $G(R)$ and let M/N be a graded torsion-free R/P-module. Clearly, $N \neq M$. To see that N is graded prime, assume that $am \in$

N where $a \in h(R)$ and $m \in h(M)$. Then $(a+P)(m+N) = N$. So either $a \in P$ or $m \in N$. Thus N is graded prime. •

REFERENCES

[1] Chin-Pi Lu, Prime submodules of modules, *Comm. Math. Univ. Sancti Pauli*, 1984; **33**: 61-69.

[2] Chin-Pi Lu, Spectra of modules, *Comm. in Algebra*, 1995; **23**: 3741-3752.

[3] Nastasescu C., and Van Oystaeyen F., *Graded Ring Theory*, Mathematical Library 28, North Holand, Amsterdam, 1982.

[4] Refai M., and Al-Zoubi K., On Graded Primary Ideals, *Turkish J. Mathematics*, 2004; **28**: 217-229.