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Contributed Paper

On Quasi-Primary Submodules

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ABSTRACT

Let R be a commutative ring with non-zero identity. We define a proper submodule *N*of an R-module *M* to be quasi-primary if $rad(N :_R M) = P$ is a prime ideal of R. In this case we also say that N is a *P*-quasi-primary submodule of *M*. A number of results concerning quasi-primary submodules are given. For example, we show that over a Prüfer domain of finite character R, every non-zero R-submodule of a module M is the intersection of finite number of quasi-primary submodules with incomparable radicals.

Keywords and phrases: quasi-primary, multiplication, secondary.

1. INTRODUCTION

In this paper all rings are commutative rings with non-zero identity and all modules are unital. Quasi-primary ideals in a commutative ring have been introduced and studied by L. Fuchs in [8] (also see [9]). An ideal *I* of *R* is called quasi-primary if its radical (we will denote it by *rad*(*I*)) is a prime ideal. Here we study quasi-primary submodules of a module. The primary and quasi-primary submodules are different concepts. In fact, every primary submodule is quasi-primary, but a quasi-primary submodule need not be primary (see Example 2.2). Various properties of quasi-primary submodules of a module are considered. For example, in Theorem 2.10, we show that if N is a quasi-primary R-submodule of a representable module *M*, then M/N is secondary. In Theorem 3.4, we show that over a Prüfer domain of finite character R, if M is a finitely generated multiplication *R*-module, then every non-zero submodule of *M* is the product of finite number of pairwise comaximal quasi-primary submodules of *M*. We also prove, in Theorem 3.5, if is a faithful multiplication module over a commutative ring *R*, then every quasi-primary submodule of *M* contained in a prime submodule of *M*.

Now we define the concepts that we will use. If R is a ring and N is a submodule of an *R*-module *M*, the ideal $\{r \in R : rM \subseteq N\}$ will be denoted by $(N :_R M)$. Then $(0 :_R M)$ is the annihilator of *M*, *ann*(*M*). An *R*-module *M* is called a multiplication module if for each submodule *N* of *M*, *N*=*IM* for some ideal *I* of *R*. In this case we can take $I = (N :_R M)$. For an *R*-module *M*, we define the ideal $\theta(M) = \sum_{m \in M} (Rm :_R M)$. So if N is a submodule of *M*, then $M = \theta(M)M$ and $N = \theta(M)N$ (see [1]).

A proper submodule N of M is primary (resp. prime) if for any $r \in R$ and $m \in M$ such that $rm \in N$, either $m \in N$ or $r^n M \subseteq N$ for some *n* (resp. either $m \in N$ or $rM \subseteq N$). It is easy to show that if N is a primary submodule of M (resp. N is a prime submodule of M)

then the $nilrad(M/N)=P$ (resp. the annihilator P' of the module M/N) is a prime ideal of R, and N is said to be P-primary submodule (resp. P'-prime submodule) of M. So every prime is primary.

An R-module $M \neq 0$ is called a secondary module provided that for every element $r \in R$, the R-endomorphism of M produced by multiplication by r is either surjective or nilpotent. This implies that $nilrad(M)=P$ is a prime ideal of R, and M is said to be P-secondary. A secondary representation for an R-module M is an expression for M as a finite sum of secondary modules. If such a representation exists, we will say that M is representable. So whenever an R-module M has a secondary representation, then the set of attached primes of M, which is uniquely determined, is denoted by $att(M)$ (see [11]).

If R is a commutative ring and S is a multiplicative subset of R then we denote by R_S the localization of R with respect to S.

2. Quasi-primary Submodules

We first recall that the definition of quasi-primary submodules of a module over a commutative ring R. A proper submodule N of an R-module M is said to be quasi-primary if $rad(N :_R M) = P$ is a prime ideal of R. In this case we also say that N is a P-quasi-primary submodule of M. Clearly, every primary submodule (so prime submodule) of a module is quasi-primary, but a quasi-primary submodule need not be primary (see Example 2.2). Our starting point is the following lemma:

Lemma 2.1 Every proper submodule of a secondary module over a commutative ring R is quasi-primary.

Proof. Assume that M is a P-secondary R-module and let N be a non-zero submodule of M. Then M/N is P-secondary by [11, Theorem 2.4]; hence $rad(N :_R M) = P$, as required. •

Let R be a commutative ring, M an R-module and M an R-submodule of M. An element $r \in R$ is called prime to N if $rm \in N$ ($m \in M$) implies that $m \in N$. In this case, $(N :_M r) = \{m \in M : rm \in N\} = N$. A submodule N of M is called primal if the elements of R that are not prime to N form an ideal; this ideal is always a prime ideal, called the adjoint ideal P of N. In this case we also say that N is a P-primal submodule [6].

Example 2.2 Let R be a local Dedekind domain with maximal ideal $P = RP$. Set $E = E(R/P)$ and $A_n = (0 :_E P^n)$ $(n \geq 1)$. Then E is a 0-secondary R-module, so A_n is -quasi-primary by Lemma 2.1 for every positive integer n, but it is not primary and primal ([7, p. 324] and [6]).

Lemma 2.3 Let M be a module over a commutative ring R. If N is both P-primal submodule and P-quasi-primary submodule of M, then N is primary.

Proof. Assume that $rm \in N$ and $m \notin N$ where $r \in R$, $m \in M$, so r is not prime to $P = rad(N :_R M)$, as required. •

Theorem 2.4 Let R be an integral domain, M an R-module and N a 0-primal submodule of M. Then N is 0-quasi-primary. In particular, N is 0-primary.

Proof. Let $a \in (N :_R M)$. Then there is an element $m \in (N :_M a)$ such that $m \notin N$ since N is primal, so a is not prime to N; hence $a = 0$. Therefore, $(N :_R M) = 0$. It follows that $rad(N :_R M) = 0$ is prime, so N is 0-quasi-primary. Finally, N is 0-primary by Lemma 2.3.

Theorem 2.5 Assume that M is a module over a commutative ring R and let N, K be P-quasi-primary and Q-quasi-primary submodules of M respectively.

Then $K \bigcap N$ is quasi-primary if and only if either $P \subseteq Q$ or $Q \subseteq P$.

Proof. Suppose first that $K \bigcap N$ is a P'-quasi-primary submodule of M. As $K \bigcap N \subseteq N, K$, we get $P \bigcap Q \subseteq P'$. Let $r \in P'$. Then $r^n M \subseteq N \bigcap K$ for some positive integer n, $r \in P \bigcap Q$; hence $P' = P \bigcap Q$. It follows that either $P \subseteq Q$ or $Q \subseteq P$. Conversely, suppose that $P \subseteq Q$ and $ab \in rad(N \bigcap K :_R M) = A$ where $a, b \in R$. Then $(ab)^m M \subseteq N \bigcap K$ for some positive integer m, so either $a \in P \subseteq Q$ or

$b \in P \subseteq Q$; hence either $a \in A$ or $b \in A$. The case $Q \subseteq P$ is similar. •

Let $N, K, N_1, ..., N_m$ be submodules of an R-module M. We say that N and K are comaximal (resp. incomparable radicals) when $N + K = M$ (resp. when $rad(N :_R M)$ and $rad(K :_R M)$ are not comparable); also we say that the submodules $N_1, ..., N_m$ are pairwise comaximal if and only if $N_i + N_j = M$ whenever $1 \le i, j \le m$ and $i \ne j$.

Proposition 2.6 Let M be a module over a commutative ring R. Then the following hold:

(i) If R is a valuation domain, then every proper submodule of M is quasi-primary.

(ii) If R is a Prüfer domain and N, K are quasi-primary submodules of M with incomparable radicals, then $M = N + K$.

Proof. (i) Let N be a proper submodule of M. Then $(N :_R M)$ is a quasi-primary ideal of R by [9, Lemma 5.4], as required.

(ii) By hypothesis, $(N :_R M)$ and $(K :_R M)$ are quasi-primary ideals of R with incomparable radicals, so $(N :_R M) + (K :_R M) = R$ by [9, Lemma 5.5]. It follows that $M = ((N :_R M) + (K :_R M))M = (N :_R M)M + (K :_R M)M \subseteq N + K$, as needed. •

Let M be an R-module. The idealization of R and M is the commutative ring with identity $R(M) = R \oplus M$ with addition $(r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2)$ and multiplication $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$. Note that $0 \oplus M$ is an ideal of $R(M)$ satisfying $(0 \oplus M)^2 = 0$ and that the structure of $0 \oplus M$ as $R(M)$-module (i.e., an ideal of $R(M)$ is essentially the same as the R-module structure of M. Moreover, an ideal J of $R(M)$ is a prime ideal if and only if $J = P \oplus M$ for some prime ideal P of R. A good reference for the basic facts about idealization is [10, sec. 25].

Proposition 2.7 Let I be a quasi-primary ideal of a commutative ring R and let M be an R-module. Then $I \oplus N$ is a quasi-primary ideal of $R(M)$ for every submodule N of M satisfying $IM \subseteq N$.

Proof. Assume that I is a P-quasi-primary ideal of R. Then by [10, Theorem 25.1 (5)], $rad(I \oplus N) = rad(I) \oplus M = P \oplus M$ is a prime ideal of $R \oplus M$. Thus $I \oplus N$ is a $(P \oplus M)$-quasi-primary ideal of $R(M)$. •

Theorem 2.8 Let P be a prime ideal of a commutative ring R, M an R-module and K an R-submodule M. Then there exists a bijective correspondence between the P-quasi-primary R-submodules of M/K and the P-quasi-primary R-submodules of M containing K.

Proof. Let N be a submodule of M with $K \subseteq N$. Since $rad(N :_R M) = rad(N/K :_R M/K)$, we get N is a P-quasi-primary submodule of M if and only if N/K is a P-quasi-primary submodule of M/K. •

We continue our program of studying quasi-primary submodules of a fraction module. Let R be a commutative ring, M an R-module and S a multiplicatively closed set in R. If B is a submodule of M_S, define $B \bigcap M = v^{-1}(B)$ where $v: M \to M_S$ is the natural homomorphism. Clearly, $B \bigcap M$ is a submodule of M.

Theorem 2.9 Let be a multiplicatively closed subset of a commutative ring R, P a prime ideal of R and M an R-module. Then the following hold:

(i) If B is a Q-quasi-primary submodule of M_S, then $B \bigcap M$ is a $(Q \bigcap R)$-quasi-primary submodule of M.

(ii) If M is finitely generated, $P \bigcap S = \phi$ and N is a P-quasi-primary submodule of M, then M_S is a P_S-quasi-primary submodule of M_S.

Proof. (i) As $Q \bigcap R$ is a prime ideal of R, it suffices to show that $Q \bigcap R = rad(B \bigcap M :_R M)$. If $r^t \in (B \bigcap M :_R M)$ for some positive integer t, then for every $m/s \in M_S$ we have $r^t m \in B \bigcap M$, so $(r^t m)/1 \in B$; hence we have $(r/1)^t (m/s) \in B$. Therefore, $r/1 \in rad(B :_{R_S} M_S) = Q$, so $r \in Q \bigcap R$. Conversely, if $r \in Q \bigcap R$, then $r/1 \in Q$, so for every $m \in M$, $(r/1)^k (m/1) \in B$ for some positive integer k. This implies that $r^k m \in B \bigcap M$; hence $r \in rad(B \bigcap M :_R M)$, as required.

(ii) By [12, Theorem 5.32], P_S is a prime ideal of R_S and by [12, Lemma 9.12 and Lemma

5.31] we have $rad(N_S :_{R_s} M_S) = rad(N :_R M)_S = (rad(N :_R M))_S = P_S$, as needed. •

Theorem 2.10 is a generalization of [4, Theorem 2.2]

Theorem 2.10 Assume that M is a module over a commutative ring R and let $M = \sum_{i=1}^{n} M_i$ be a minimal secondary representation of M with $Att(M) = \{P_1, P_2, ..., P_n\}$. Let be a P-quasi-primary submodule of M. Then the following hold:

(i) $N = N \bigcap M_i + M_1 + ... + M_{i-1} + M_{i+1} + ... + M_n$ for some I $(1 \le i \le n)$.

(ii) M/N is P-secondary.

Proof. We claim that for each $1 \le i \le n$, either $M_i \subseteq N$ or $P \subseteq P_i$. Otherwise, $M_i \not\subset N$ and $P \not\subset P_i$ for some i. Take $a \in P - P_i$. Then there exists a positive integer k such that $M_i = a^k M_i \subseteq a^k M \subseteq N$ which is a contradiction. It follows from Lemma 2.1 that if $M_i \not\subset N$ for some i, then $N \bigcap M_i$ is a P_i-quasi-primary submodule of M_i (otherwise $(N \bigcap M_i :_R M_i) = R$). Moreover, we have:

$$P = rad(N :_R M) = rad(N :_R \sum_{i=1}^{n} M_i) = rad(\bigcap (N :_R M_i))$$

$$= \bigcap_{i=1}^{n} rad(N :_R M_i) = \bigcap_{i=1}^{n} rad(N \bigcap M_i :_R M_i)$$

By the above considerations, if $M_i \not\subset N$ and $M_j \not\subset N$ for some $i \neq j$, then $P = P_i \bigcap P_j$; hence $P = P_i = P_j$ which is a contradiction. Therefore, without loss of generality, we can assume that $M_i \not\subset N$ for some i, so $P = P_i$ and $M_j \subseteq N$ $(j = 1,2,...,i-1,i+1,...n)$. Then $M_1 + M_2 + ... + M_{i-1} + M_{i+1} + ... + M_n \subseteq N$ and

$$N = N \bigcap M = M_1 + M_2 + ... + M_{i-1} + M_{i+1} + ... + M_n + N \bigcap M_i$$

(ii) Since $M = N + M_i$, we have $M/N \cong M_i/(M_i \bigcap N)$, as needed. •

Corollary 2.11 Let be a commutative ring M, a representable R-module and N a P-primary R-submodule of M. Then the following hold:

(i) N is representable.

(ii) M / N is P-secondary.

Proof. This follows from [4, Lemma 2.1] and Theorem 2.10. •

3. MULTIPLICATION MODULES

The aim of this section is to study some properties of quasi-primary submodules of a multiplication module.

Let M be an R-module and N be a submodule of M such that $N = IM$ for some ideal I of R. Then we say that I is a presentation ideal of N. It possible that for a submodule N no such presentation exist. For example, assume that M is a vector space over an arbitrary field F with $\dim_F M \ge 2$ and let N be a proper subspace of M such that $N \neq 0$. Then M is of finite length (so M is artinian and noetherian), but M is not multiplication and N has not any presentation. Clearly, every submodule of M has a presentation ideal if and only if M is a multiplication module.

Let N and K be submodules of a multiplication R-module M with $N = I_1 M$ and $K = I_2 M$ for some ideals I_1 and I_2 of R. The product of N and K is denoted by NK and defined by $NK = I_1 I_2 M$. Then by [2, theorem 3.4], the product of N and K is independent of presentation of N and K. Moreover, for $a, b \in M$, by ab, we mean the product of Ra and Rb. Clearly, NK is a submodule of M and $NK \subseteq N \bigcap K$.

Lemma 3.1 Let R be a commutative ring, M a multiplication R-module and $N = IM$ an R-submodule of M. Then N is a P-quasi-primary submodule of M if and only if I is a P-quasi-primary ideal of R.

Proof. The proof is clear since $rad(N :_R M) = rad(I) = P$. •

Theorem 3.2 Let R be a commutative ring, M a multiplication R-module and $N_1, N_2, ..., N_k$ be R-submodules of M. Then the following hold:

(i) If $N_1, N_2, ..., N_k$ are P-quasi-primary submodules of M, then $N_1 N_2 ... N_k$ is a P-quasi-primary submodule of M.

(ii) If $N_1 N_2 ... N_k$ is a P-quasi-primary submodule of M, then N_i is P-quasi-primary for some $1 \le i \le k$.

Proof. (i) There are ideals $I_1, I_2, ..., I_k$ of R such that $N_i = I_i M$ for every $1 \le i \le k$, so $N_1 N_2 ... N_k = I_1 I_2 ... I_k M$. By Lemma 3.1, every I_i

is P-quasi-primary, so $I_1I_2...I_k$ is a P-quasi-primary ideal of R. Now the assertion follows from Lemma 3.1.

(ii) Let $N_1N_2...N_k = I_1I_2...I_kM$ be P-quasi-primary where $N_i = I_iM$ for every $1 \le i \le k$. Then $I_1I_2...I_k$ is a P-quasi-primary ideal of R by Lemma 3.1. As $rad(I_1I_2...I_k) = rad(I_1) \cap rad(I_2) \cap ... \cap rad(I_k) = P$, we get $rad(I_j) = P$ for some j; hence N_j is P-quasi-primary by Lemma 3.1. •

Lemma 3.3 Let R be a commutative ring, M a non-zero finitely generated multiplication R-module and $N_1, N_2,..., N_k$ pairwise comaximal R-submodules of M. Then the following hold:

(i) $N_1N_2 = N_1 \bigcap N_2$.

(ii) $N_1 \bigcap N_2 \bigcap ... \bigcap N_{k-1}$ and N_k are comaximal.

(iii) $N_1N_2...N_k = N_1 \bigcap N_2 \bigcap ... \bigcap N_k$..

Proof. (i) Clearly, $N_1N_2 \subseteq N_1 \bigcap N_2$. By hypothesis, $N_1 + N_2 = M$, so by [3, Theorem 1.6] we have

$N_1 \bigcap N_2 = (N_1 \bigcap N_2)M = (N_1 \bigcap N_2)(N_1 + N_2) = (N_1 \bigcap N_2)N_1 + (N_1 \bigcap N_2)N_2 \subseteq N_1N_2$ and the proof is complete.

(ii) Set $N = \bigcap_{i=1}^{k-1} N_i$. We claim that $N + N_k = M$. Otherwise, there exists a maximal submodule K of M such that $N + N_k \subseteq K$ by [3, Theorem 2.5], so $N_k \subseteq K$ and $N \subseteq K$; by [5, Lemma 2.2], there is a positive integer j with $1 \le j \le k-1$ such that $N_j \subseteq K$ (since every maximal submodule is prime), so $M = N_j + N_k \subseteq K$ which is a contradiction.

(iii) This follows from (i) and (ii) by induction on n. •

An integral domain R is said to be finite character if every non-zero element is contained but in a finite number of maximal ideals.

Theorem 3.4 Let M be a multiplication module over a Prüfer domain of finite character. Then the following hold:

(i) Every non-zero submodule of M is the intersection of finite number of quasi-primary submodules with incomparable radicals.

(ii) If M is finitely generated, then every non-zero submodule of M is the product of finite number of pairwise comaximal quasi-primary submodules of M.

Proof. (i) Let N be a non-zero submodule of M. Then there exists anon-zero ideal I of R such that $N = IM$. By [9, Theorem 5.6], we can write $I = I_1 \bigcap I_2 \bigcap ... \bigcap I_k$ where I_i is P_i-quasi-primary for every $1 \le i \le k$ and where the ideals P_i are incomparable. Therefore, $N = (I_1 \bigcap I_2 \bigcap ... \bigcap I_k)M = I_1M \bigcap I_2M \bigcap ... \bigcap I_kM$ by [3, Theorem 1.6]. Now the result follows from Lemma 3.1.

(ii) By Proposition 2.6, quasi-primary submodules whose radicals are incomparable are comaximal. For comaximal submodules of a finitely generated multiplication module, intersection equals product by Lemma 3.3. Now the assertion follows from (i). •

Theorem 3.5 Let M be a faithful multiplication module over a commutative ring R. Then every quasi-primary submodule of M is contained in a prime submodule of M.

Proof. Let N be a P-quasi-primary submodule of M. Then we can write $N = (N :_R M)M \subseteq PM$. By [3, Corollary 2.11], it is enough to show that $PM \ne M$. First, we show that $\theta(M) \not\subseteq P$. Assume that $\theta(M) \subseteq P$ and let $m \in M$. Then we have $Rm = \theta(M)Rm \subseteq PRm = Pm \subseteq Rm$, so $m = pm$ for some $p \in P$. As N is P-quasi-primary, there exists a positive integer t such that $p^tM \subseteq N$; hence $m = p^tm \in N$. It follows that $M = N$ which is a contradiction. Next, suppose that $PM = M = RM$. Then by [1, Theorem 1.5], $R \bigcap \theta(M) = \theta(M) = P \bigcap \theta(M)$; hence $\theta(M) \subseteq P$, which is a contradiction, as required. •

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