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The Truncated Power Lomax Distribution: Properties and Applications

Sirinapa ARYUYUEN[1,*] and Winai BODHISUWAN[2]

[1]*Department of Mathematics and Computer Science, Faculty of Science and Technology, Rajamangala University of Technology Thanyaburi, Pathum Thani 12110, Thailand*
[2]*Department of Statistics, Faculty of Science, Kasetsart University, Bangkok 10900, Thailand*

([*]Corresponding author's e-mail: sirinapa_a@rmutt.ac.th)

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Abstract

A new truncated distribution, called the truncated power Lomax (TPL) distribution, is proposed. This is a truncated version of the power Lomax distribution. The TPL distribution has increasing and decreasing shapes of the hazard function. Some statistical properties, such as moments, survival, hazard, and quantile functions, are discussed. The maximum likelihood estimation (MLE) is constructed for estimating the unknown parameters of the TPL distribution. Moreover, the distribution has been fitted with real data sets to illustrate the usefulness of the proposed distribution. From the results of the example applications, the TPL distribution provides a consistently better fit than the other distributions, i.e., power Lomax and Lomax.

Keywords: Truncated distribution, power Lomax, hazard function, truncated power Lomax, MLE

Introduction

Data analysis is undertaken in various fields, including engineering, medicine, finance, and demographics, where such types of truncated data arise in practical statistics. It is used in cases where the ability to record exists, or even when occurrences are limited to values which lie above or below a given threshold or within a specified range. Truncated distributions are quite effective for using data analysis [1,2]. In 1994, the truncated normal distribution was introduced by Johnson *et al*. [2], which is a probability distribution derived from that of a normal random variable by bounding it from either below or above (or both). Many researchers have therefore been attracted to the problems of analyzing such truncated data encountered in various disciplines and have proposed truncated versions of usual statistical distributions [2-6].

Truncated distribution has been derived from that of a parent distribution, such as normal and exponential distributions, by bounding the random variable from either below or above (or both). The power Lomax distribution is a parent distribution, which was introduced by Rady *et al.* in 2016. This distribution accommodates both the inverted bathtub and decreasing hazard rates. Rady *et al.* [7] presented the PL distribution to analyze data, in that case, for the remission time of bladder cancer. The results showed that it could offer a better fit than a set of extensions of the Lomax distribution when the Lomax distribution is introduced as a heavy-tail probability function to use data analysis in many fields, such as in business, economics, actuarial science, queuing theory, and internet traffic modeling [8-10]. From the above commentary and monitoring, and the wide applicability of the truncated distributions, this work has proposed a new truncated distribution for data analysis when datasets have values that are outside of a usual range. This work also proposes a new truncated distribution by using the PL distribution of the parent distribution.

The rest of the paper has been organized into the following sections. In the material and methods section, the methods of the parent and truncated distributions are introduced. Next, a new truncated distribution is proposed, called the truncated power Lomax distribution. Some statistical properties, such as moments, survival, hazard, and quantile functions, are discussed. In addition, the method of the maximum likelihood is applied to obtain estimates of the TPL parameters. Moreover, some applications are discussed as to how they fit when using different distributions, and their applicability is compared. Finally, some conclusions are presented.

Methods

The power Lomax distribution

The power Lomax (PL) distribution proposed by Rady *et al.* [7], which is an extension of the Lomax distribution [8], is proposed by considering the power transformation $X = T^{1/\lambda}$, where the random variable T follows Lomax distribution, with the cumulative distribution function (cdf) as follows;

$$\mathrm{G}(t) = 1 - \left(1 + \frac{t}{\beta}\right)^{\alpha}; \qquad t > 0, \tag{1}$$

where $\alpha > 0$ and $\beta > 0$ are the shape and scale parameters, respectively. This associates the probability density function (pdf), given by

$$\mathrm{g}(t) = \frac{\alpha}{\beta}\left(1 + \frac{t}{\beta}\right)^{-(\alpha+1)}. \tag{2}$$

We have a random variable $X = T^{1/\lambda}$, distributed as the PL distribution with positive parameters α, β, and λ. The pdf and cdf of X are defined as, respectively,

$$\mathrm{g}(x) = \alpha\lambda\beta^{\alpha}x^{\lambda-1}(\beta + x^{\lambda})^{-(\alpha+1)} \text{ and } \mathrm{G}(x) = 1 - \beta^{\alpha}(x^{\lambda} + \beta)^{-\alpha} \;;\; x > 0. \tag{3}$$

The pdf in Eq. (3) is unimodal if $\alpha > 0, \beta > 0, \lambda > 1$, and it is decreasing if $\alpha > 0$, $\beta > 0, 0 < \lambda \leq 1$.

Truncated distributions

Suppose X is a random variable that is distributed according to some pdf $g(x)$, with cdf, $G(x)$, both of which have infinite support. A random variable X lies within the interval $X \in [a,b]$, $-\infty < a \leq x \leq b < \infty$. Then, the conditional on $a \leq x \leq b$ has a truncated distribution. Pdf for $a \leq x \leq b$ is given by [2];

$$f(x \mid a \leq X \leq b) = \frac{g(x)}{G(b) - G(a)} \tag{4}$$

where $g(x) = f(x)$ for all $a \leq x \leq b$ and $g(x) = 0$ everywhere else. Notice that $f(x \mid a \leq X \leq b)$ has the same support as $g(x)$. Notice that, in fact, $f(x \mid a \leq X \leq b)$ is a distribution;

$$\begin{aligned}\int_a^b f(x \mid a \leq X \leq b)\,dx &= \frac{1}{G(b) - G(a)}\int_a^b g(x)dx = \frac{1}{G(b) - G(a)}\left[G(x)\right]_{x=a}^{b} \\ &= \frac{1}{G(b) - G(a)}\left[G(b) - G(a)\right] = 1\end{aligned} \tag{5}$$

Truncated distribution need not have parts removed from the top and bottom. A truncated distribution, where just the bottom at a of the distribution has been removed, is as follows;

$$f(x \mid X \geq a) = \frac{g(x)}{1 - G(a)} \tag{6}$$

where $g(x) = f(x)$ for all $x \geq a$, and $g(x) = 0$ everywhere else, and $f(x \mid X \geq a)$ is called the left truncated (L-T) distribution at a. A truncated distribution where the top at b of the distribution has been removed is as follows;

$$f(x \mid X \leq b) = \frac{g(x)}{G(b)} \tag{7}$$

where $g(x) = f(x)$ for all $x \leq b$, and $g(x) = 0$ everywhere else, and $f(x \mid X \leq b)$ is called the right truncated (R-T) distribution at b.

Results and discussion

In this section, the truncated distribution of PL distribution, and its statistical properties, are discussed. In addition, the parameter estimation is shown for the unknown parameters of the distribution. Moreover, the application study is illustrated.

Truncated power Lomax distribution

Theorem 1: Let X be distributed as the PL distribution with the positive parameters α, β, and λ. When a random variable X lies within the interval $X \in [a,b]$, $0 < a \leq x \leq b < \infty$. The conditional of random variable X on $a \leq x \leq b$ is distributed as the truncated power Lomax (TPL) distribution, with pdf and cdf as follows;

$$\mathrm{f}(x \mid a \leq x \leq b) = \frac{\alpha\lambda x^{\lambda-1}(x^{\lambda}+\beta)^{-(\alpha+1)}}{(a^{\lambda}+\beta)^{-\alpha}-(b^{\lambda}+\beta)^{-\alpha}} \text{ and } \mathrm{F}(x \mid a \leq x \leq b) = \frac{(a^{\lambda}+\beta)^{-\alpha}-(x^{\lambda}+\beta)^{-\alpha}}{(a^{\lambda}+\beta)^{-\alpha}-(b^{\lambda}+\beta)^{-\alpha}}. \tag{8}$$

Suppose expression $X \sim \mathrm{TPL}(\alpha,\beta,\lambda,a,b)$, which is called X, is distributed as the TPL distribution with parameters α, β, and λ on interval $[a,b]$.

Proof. Let X be distributed as the PL distribution, with the pdf and cdf in Eq. (3). When a random variable X lies within the interval $X \in [a,b]$, $0 < a \leq x \leq b < \infty$. The pdf for the conditional of random variable X on $a \leq x \leq b$ is obtained by replacing $g(x)$ and $G(x)$ in Eq. (3) into Eq. (4), i.e.;

$$\begin{aligned}\mathrm{f}(x \mid a \leq x \leq b) &= \frac{\alpha\lambda\beta^{\alpha}x^{\lambda-1}(\beta+x^{\lambda})^{-(\alpha+1)}}{\left[1-\beta^{\alpha}(b^{\lambda}+\beta)^{-\alpha}\right]-\left[1-\beta^{\alpha}(a^{\lambda}+\beta)^{-\alpha}\right]} \\ &= \frac{\alpha\lambda x^{\lambda-1}(\beta+x^{\lambda})^{-(\alpha+1)}}{(a^{\lambda}+\beta)^{-\alpha}-(b^{\lambda}+\beta)^{-\alpha}};\ a \leq x \leq b, \end{aligned} \tag{9}$$

and $\mathrm{f}(x) = 0$ everywhere else, where $\mathrm{f}(x) \geq 0$ for all x and $\int_{-\infty}^{\infty} \mathrm{f}(x)dx = 1$, i.e.;

$$\int_a^b \frac{\alpha\lambda x^{\lambda-1}(\beta+x^\lambda)^{-(\alpha+1)}}{(a^\lambda+\beta)^{-\alpha}-(b^\lambda+\beta)^{-\alpha}}dx = \frac{1}{\beta^\alpha\left[(a^\lambda+\beta)^{-\alpha}-(b^\lambda+\beta)^{-\alpha}\right]}\times\int_a^b \alpha\lambda\beta^\alpha x^{\lambda-1}(\beta+x^\lambda)^{-(\alpha+1)}dx$$
$$= \frac{1}{\beta^\alpha\left[(a^\lambda+\beta)^{-\alpha}-(b^\lambda+\beta)^{-\alpha}\right]}\times\left[1-\beta^\alpha(x^\lambda+\beta)^{-\alpha}\right]_{x=a}^{b}$$
$$= \frac{1}{\beta^\alpha\left[(a^\lambda+\beta)^{-\alpha}-(b^\lambda+\beta)^{-\alpha}\right]}\times\left[\beta^\alpha(a^\lambda+\beta)^{-\alpha}-\beta^\alpha(b^\lambda+\beta)^{-\alpha}\right]$$
$$=1. \tag{10}$$

The cdf of X is;

$$F(x\mid a\le x\le b) = \frac{1}{\beta^\alpha\left[(a^\lambda+\beta)^{-\alpha}-(b^\lambda+\beta)^{-\alpha}\right]}\times\int_a^x \alpha\lambda\beta^\alpha s^{\lambda-1}(\beta+s^\lambda)^{-(\alpha+1)}ds$$
$$= \frac{1}{\beta^\alpha\left[(a^\lambda+\beta)^{-\alpha}-(b^\lambda+\beta)^{-\alpha}\right]}\times\left\{\left[1-\beta^\alpha(x^\lambda+\beta)^{-\alpha}\right]-\left[1-\beta^\alpha(a^\lambda+\beta)^{-\alpha}\right]\right\}$$
$$= \frac{(a^\lambda+\beta)^{-\alpha}-(x^\lambda+\beta)^{-\alpha}}{(a^\lambda+\beta)^{-\alpha}-(b^\lambda+\beta)^{-\alpha}}. \tag{11}$$

***Corollary 1**:* Let X be distributed as the PL distribution with the positive parameters α, β, and λ. When a random variable X lies within the interval $X\in[a,\infty)$, $0<a\le x<\infty$. The conditional of random variable X on $a\le x<\infty$ is distributed as the left truncated power Lomax (L-TPL) distribution, with pdf and cdf, respectively, as follows;

$$f(x\mid x\ge a) = \frac{\alpha\lambda x^{\lambda-1}(x^\lambda+\beta)^{-(\alpha+1)}}{(a^\lambda+\beta)^{-\alpha}} \quad\text{and}\quad F(x\mid x\ge a) = 1-\frac{(x^\lambda+\beta)^{-\alpha}}{(a^\lambda+\beta)^{-\alpha}}. \tag{12}$$

Proof. Let X be distributed as the PL distribution, with the pdf and cdf in Eq. (3). When a random variable X lies within the interval $X\in[a,\infty)$, $0<a\le x<\infty$. The pdf for the conditional of random variable X on $a\le x<\infty$ is obtained by replacing $g(x)$ and $G(x)$ in Eq. (3) into Eq. (6), i.e.;

$$f(x\mid x\ge a) = \frac{\alpha\lambda\beta^\alpha x^{\lambda-1}(\beta+x^\lambda)^{-(\alpha+1)}}{1-\left[1-\beta^\alpha(a^\lambda+\beta)^{-\alpha}\right]}$$
$$= \frac{\alpha\lambda x^{\lambda-1}(\beta+x^\lambda)^{-(\alpha+1)}}{(a^\lambda+\beta)^{-\alpha}}; \quad a\le x<\infty, \tag{13}$$

and $f(x)=0$ everywhere else, where $f(x)\ge 0$ for all x and $\int_{-\infty}^{\infty} f(x)dx=1$, i.e.;

$$\int_a^\infty \frac{\alpha\lambda x^{\lambda-1}(x^\lambda+\beta)^{-(\alpha+1)}}{(a^\lambda+\beta)^{-\alpha}}dx = \frac{1}{1-\left[1-\beta^\alpha(a^\lambda+\beta)^{-\alpha}\right]}\times\int_a^\infty \alpha\lambda\beta^\alpha x^{\lambda-1}(\beta+x^\lambda)^{-(\alpha+1)}dx$$
$$= \frac{1}{1-\left[1-\beta^\alpha(a^\lambda+\beta)^{-\alpha}\right]}\times\left[1-\beta^\alpha(x^\lambda+\beta)^{-\alpha}\right]_{x=a}^{\infty}$$
$$= \frac{1}{1-\left[1-\beta^\alpha(a^\lambda+\beta)^{-\alpha}\right]}\times\left\{1-\left[1-\beta^\alpha(a^\lambda+\beta)^{-\alpha}\right]\right\}$$
$$=1. \tag{14}$$

The cdf of X is;

$$\mathrm{F}(x \mid x \geq a) = \frac{1}{\beta^{\alpha}\left[(a^{\lambda}+\beta)^{-\alpha}\right]} \times \int_{a}^{x} \alpha\beta^{\alpha}\lambda s^{\lambda-1}(\beta+s^{\lambda})^{-(\alpha+1)}\,ds$$

$$= \frac{1}{\beta^{\alpha}\left[(a^{\lambda}+\beta)^{-\alpha}\right]} \times \left\{\left[1-\beta^{\alpha}(x^{\lambda}+\beta)^{-\alpha}\right]-\left[1-\beta^{\alpha}(a^{\lambda}+\beta)^{-\alpha}\right]\right\}$$

$$= \frac{(a^{\lambda}+\beta)^{-\alpha}-(x^{\lambda}+\beta)^{-\alpha}}{(a^{\lambda}+\beta)^{-\alpha}}$$

$$= 1-\frac{(x^{\lambda}+\beta)^{-\alpha}}{(a^{\lambda}+\beta)^{-\alpha}}. \tag{15}$$

Corollary 2: Let X be distributed as the PL distribution with the positive parameters α, β, and λ. When a random variable X lies within the interval $X \in [0,b]$, $0 < x \leq b < \infty$. The conditional of random variable X on $0 \leq x \leq b$ is distributed as the right truncated power Lomax (R-TPL) distribution, with pdf and cdf as follows;

$$\mathrm{f}(x \mid x \leq b) = \frac{\alpha\lambda\beta^{\alpha}x^{\lambda-1}(x^{\lambda}+\beta)^{-(\alpha+1)}}{1-\beta^{\alpha}(b^{\lambda}+\beta)^{-\alpha}} \quad \text{and} \quad \mathrm{F}(x \mid x \leq b) = \frac{\beta^{\alpha}(x^{\lambda}+\beta)^{-\alpha}}{1-\beta^{\alpha}(b^{\lambda}+\beta)^{-\alpha}}. \tag{16}$$

Proof. Let X be distributed as the PL distribution with the pdf and cdf in Eq. (3). When a random variable X lies within the interval $X \in [0,b]$, $0 \leq x \leq b < \infty$. The pdf for the conditional of random variable X on $0 \leq x \leq b$ is obtained by replacing $g(x)$ and $G(x)$ in Eq. (3) into Eq. (7), i.e.;

$$\mathrm{f}(x \mid x \leq b) = \frac{\alpha\lambda\beta^{\alpha}x^{\lambda-1}(\beta+x^{\lambda})^{-(\alpha+1)}}{1-\beta^{\alpha}(b^{\lambda}+\beta)^{-\alpha}}. \tag{17}$$

and $\mathrm{f}(x)=0$ everywhere else, where $\mathrm{f}(x) \geq 0$ for all x and $\int_{-\infty}^{\infty}\mathrm{f}(x)dx = 1$, i.e.;

$$\int_{0}^{b}\frac{\alpha\lambda\beta^{\alpha}x^{\lambda-1}(\beta+x^{\lambda})^{-(\alpha+1)}}{1-\beta^{\alpha}(b^{\lambda}+\beta)^{-\alpha}}dx = \frac{1}{1-\beta^{\alpha}(b^{\lambda}+\beta)^{-\alpha}} \times \int_{0}^{b}\alpha\lambda\beta^{\alpha}x^{\lambda-1}(\beta+x^{\lambda})^{-(\alpha+1)}dx$$

$$= \frac{1}{1-\beta^{\alpha}(b^{\lambda}+\beta)^{-\alpha}} \times \left[1-\beta^{\alpha}(x^{\lambda}+\beta)^{-\alpha}\right]_{x=0}^{b}$$

$$= \frac{1}{1-\beta^{\alpha}(b^{\lambda}+\beta)^{-\alpha}} \times \left\{\left[1-\beta^{\alpha}(b^{\lambda}+\beta)^{-\alpha}\right]-0\right\}$$

$$= 1. \tag{18}$$

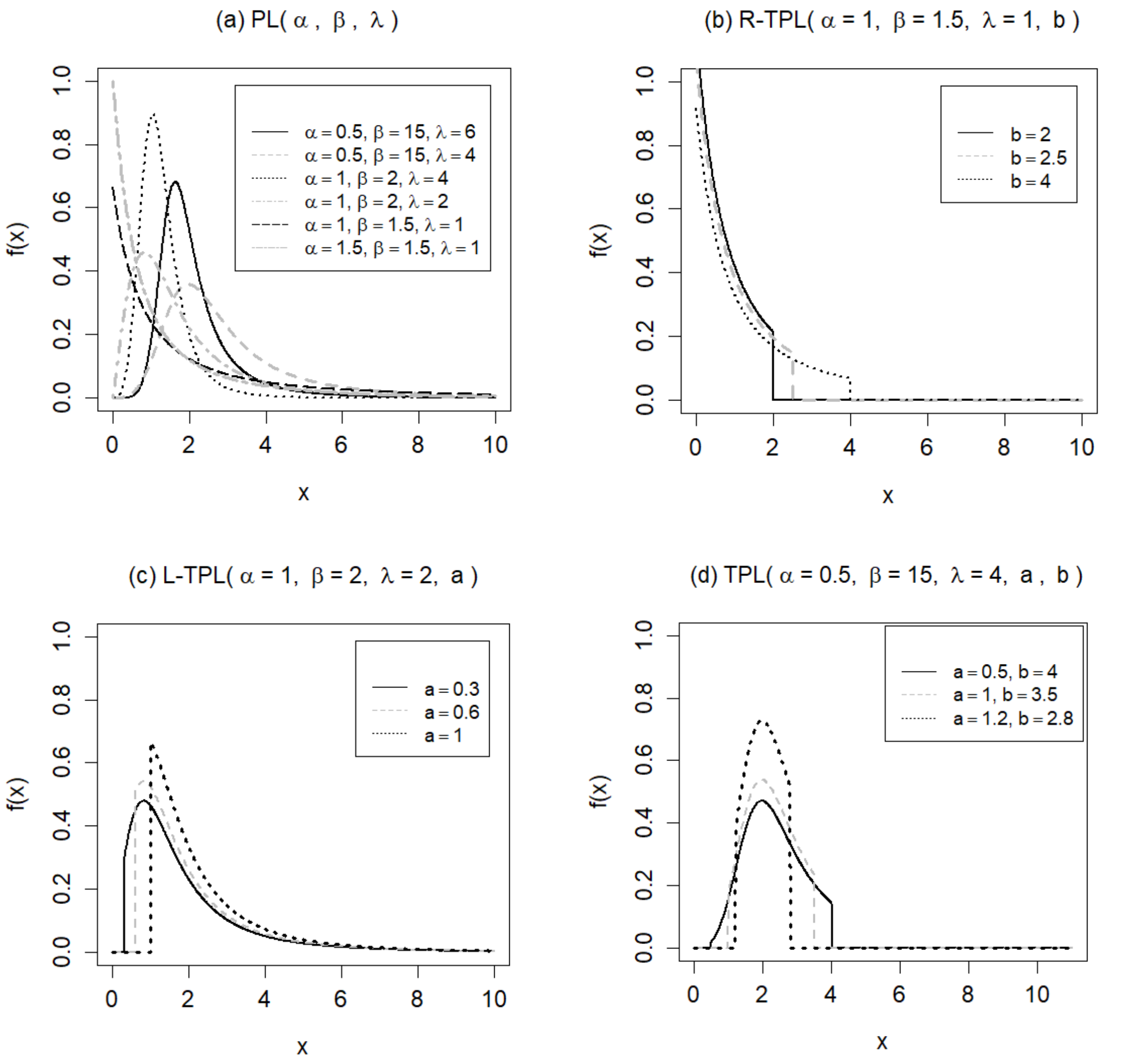

Figure 1 Some pdf plots of the distributions: (a) PL, (b) R-TPL, (c) L-TPL, and (d) TPL.

The cdf of X is;

$$\begin{aligned}
\mathrm{F}(x \mid x \le b) &= \frac{1}{1-\beta^{\alpha}(b^{\lambda}+\beta)^{-\alpha}} \int_0^x \alpha\lambda\beta^{\alpha} s^{\lambda-1}(\beta+s^{\lambda})^{-(\alpha+1)}\,ds \\
&= \frac{1}{1-\beta^{\alpha}(b^{\lambda}+\beta)^{-\alpha}} \times \left\{\left[1-\beta^{\alpha}(x^{\lambda}+\beta)^{-\alpha}\right]-\left[1-\beta^{\alpha}(0^{\lambda}+\beta)^{-\alpha}\right]\right\} \\
&= \frac{\beta^{\alpha}(x^{\lambda}+\beta)^{-\alpha}}{1-\beta^{\alpha}(b^{\lambda}+\beta)^{-\alpha}}. \qquad (19)
\end{aligned}$$

Figure 1 shows some pdf plots of the distributions. **Figure 1(a)** shows plots of the PL pdf in Eq. (3) with some parameters α and β. For $0 \le x \le b < \infty$, we have the R-TPL distribution at b, and its plot of the pdf in Eq. (16) is illustrated in **Figure 1(b)**. The pdf in Eq. (12) of the L-TPL distribution at a for $0 < a \le x < \infty$ is presented in **Figure 1(c)**. For a random variable X on interval $[a,b]$, some plots of the pdf in Eq. (8) are shown in **Figure 1(d)**. Accordingly, the pdf of the TPL distribution is; a) decreasing if $a > 0,\ \beta > 0$ and $0 < \lambda \le 1$, and b) unimodal if $a > 0,\ \beta > 0$ and $\lambda > 1$.

Moments

Some characteristics of a distribution can be studied through moments (e.g., mean, variance, skewness, and kurtosis).

Theorem 2: Let X be distributed as the TPL distribution with pdf and cdf in Eq. (8); we have the kth moment about the origin of X, that is, $\mu_k' = \mathrm{E}(X^k)$,

$$\mu_k' = \frac{\beta^{-(\alpha-k/\lambda)}\Gamma(\alpha-k/\lambda)\Gamma(k/\lambda+1)}{\Gamma(\alpha)\left[(a^\lambda+\beta)^{-\alpha}-(b^\lambda+\beta)^{-\alpha}\right]},\ k=1,2,3,... \tag{20}$$

where $\Gamma(s)$ is a gamma function, i.e., $\Gamma(s)=\int_0^\infty t^{s-1}e^{-t}dt.$

Proof. From the pdf for the conditional of X on $0 < a \le x \le b < \infty$ in Eq. (8), we have;

$$\begin{aligned}\mu_k' = \mathrm{E}(X^k) &= \int_0^\infty x^k \frac{\alpha\lambda x^{\lambda-1}(x^\lambda+\beta)^{-(\alpha+1)}}{(a^\lambda+\beta)^{-\alpha}-(b^\lambda+\beta)^{-\alpha}}dx \\ &= \frac{1}{\beta^\alpha\left[(a^\lambda+\beta)^{-\alpha}-(b^\lambda+\beta)^{-\alpha}\right]}\times\int_0^\infty x^k\alpha\lambda\beta^\alpha x^{\lambda-1}(x^\lambda+\beta)^{-(\alpha+1)}\,dx \\ &= \frac{1}{\beta^\alpha\left[(a^\lambda+\beta)^{-\alpha}-(b^\lambda+\beta)^{-\alpha}\right]}\times\frac{\alpha\beta^{k/\lambda}\,\Gamma(\alpha-k/\lambda)\Gamma((k+\lambda)/\lambda)}{\Gamma(1+\alpha)} \\ &= \frac{\beta^{-(\alpha-k/\lambda)}\Gamma(\alpha-k/\lambda)\Gamma(k/\lambda+1)}{\Gamma(\alpha)\left[(a^\lambda+\beta)^{-\alpha}-(b^\lambda+\beta)^{-\alpha}\right]}.\end{aligned} \tag{21}$$

Straightforwardly, as in Eq. (20), the first four moments are, respectively;

$$\mu_1' = \frac{\beta^{-(\alpha-1/\lambda)}\Gamma(\alpha-1/\lambda)\Gamma(1/\lambda+1)}{\Gamma(\alpha)\left[(a^\lambda+\beta)^{-\alpha}-(b^\lambda+\beta)^{-\alpha}\right]},\quad \mu_2' = \frac{\beta^{-(\alpha-2/\lambda)}\Gamma(\alpha-2/\lambda)\Gamma(2/\lambda+1)}{\Gamma(\alpha)\left[(a^\lambda+\beta)^{-\alpha}-(b^\lambda+\beta)^{-\alpha}\right]}, \tag{22}$$

$$\mu_3' = \frac{\beta^{-(\alpha-3/\lambda)}\Gamma(\alpha-3/\lambda)\Gamma(3/\lambda+1)}{\Gamma(\alpha)\left[(a^\lambda+\beta)^{-\alpha}-(b^\lambda+\beta)^{-\alpha}\right]},\quad \mu_4' = \frac{\beta^{-(\alpha-4/\lambda)}\Gamma(\alpha-4/\lambda)\Gamma(4/\lambda+1)}{\Gamma(\alpha)\left[(a^\lambda+\beta)^{-\alpha}-(b^\lambda+\beta)^{-\alpha}\right]}. \tag{23}$$

The mean, variance, skewness, and kurtosis of the TPL distribution are, respectively;

$$\mathrm{E}(X) = \frac{\beta^{-(\alpha-1/\lambda)}\Gamma(\alpha-1/\lambda)\Gamma(1/\lambda+1)}{\Gamma(\alpha)\left[(a^\lambda+\beta)^{-\alpha}-(b^\lambda+\beta)^{-\alpha}\right]}, \tag{24}$$

$$\mathrm{Var}(X)=\mathrm{E}(X^2)-\left[\mathrm{E}(X)\right]^2$$
$$=\frac{\beta^{-(\alpha-2/\lambda)}\Gamma(\alpha-2/\lambda)\Gamma(2/\lambda+1)}{\Gamma(\alpha)\left[(a^\lambda+\beta)^{-\alpha}-(b^\lambda+\beta)^{-\alpha}\right]}-\frac{\beta^{-2(\alpha-1/\lambda)}\Gamma^2(\alpha-1/\lambda)\Gamma^2(1/\lambda+1)}{\Gamma^2(\alpha)\left[(a^\lambda+\beta)^{-\alpha}-(b^\lambda+\beta)^{-\alpha}\right]^2}, \tag{25}$$

$$\mathrm{Skewness}(X)=\left\{\mathrm{E}(X^3)-3\mathrm{E}(X^2)\mathrm{E}(X)+2\left[\mathrm{E}(X)\right]^3\right\}\times\left[\mathrm{Var}(X)\right]^{-3/2}$$
$$=\left\{\frac{\beta^{-(\alpha-3/\lambda)}\Gamma(\alpha-3/\lambda)\Gamma(3/\lambda+1)}{\Gamma(\alpha)\left[(a^\lambda+\beta)^{-\alpha}-(b^\lambda+\beta)^{-\alpha}\right]}-\frac{3\Gamma(\alpha-2/\lambda)\Gamma(2/\lambda+1)\Gamma(\alpha-1/\lambda)\Gamma(1/\lambda+1)}{\beta^{(2\alpha-3/\lambda)}\Gamma^2(\alpha)\left[(a^\lambda+\beta)^{-\alpha}-(b^\lambda+\beta)^{-\alpha}\right]^2}\right.$$
$$\left.+\frac{2\beta^{-3(\alpha-1/\lambda)}\Gamma^3(\alpha-1/\lambda)\Gamma^3(1/\lambda+1)}{\Gamma^3(\alpha)\left[(a^\lambda+\beta)^{-\alpha}-(b^\lambda+\beta)^{-\alpha}\right]^3}\right\}\left\{\frac{\beta^{-(\alpha-2/\lambda)}\Gamma(\alpha-2/\lambda)\Gamma(2/\lambda+1)}{\Gamma(\alpha)\left[(a^\lambda+\beta)^{-\alpha}-(b^\lambda+\beta)^{-\alpha}\right]}\right.$$
$$\left.-\frac{\beta^{-2(\alpha-1/\lambda)}\Gamma^2(\alpha-1/\lambda)\Gamma^2(1/\lambda+1)}{\Gamma^2(\alpha)\left[(a^\lambda+\beta)^{-\alpha}-(b^\lambda+\beta)^{-\alpha}\right]^2}\right\}^{-3/2}, \tag{26}$$

$$\mathrm{Kurtosis}(X)=\left\{\mathrm{E}(X^4)-4\mathrm{E}(X^3)\mathrm{E}(X)+6\mathrm{E}(X^2)\left(\mathrm{E}(X)\right)^2-3\left[\mathrm{E}(X)\right]^4\right\}\times\left[\mathrm{Var}(X)\right]^{-2}$$
$$=\left\{\frac{\beta^{-(\alpha-4/\lambda)}\Gamma(\alpha-4/\lambda)\Gamma(4/\lambda+1)}{\Gamma(\alpha)\left[(a^\lambda+\beta)^{-\alpha}-(b^\lambda+\beta)^{-\alpha}\right]}-\frac{4\Gamma(\alpha-3/\lambda)\Gamma(3/\lambda+1)\Gamma(\alpha-1/\lambda)\Gamma(1/\lambda+1)}{\beta^{(2\alpha-4/\lambda)}\Gamma^2(\alpha)\left[(a^\lambda+\beta)^{-\alpha}-(b^\lambda+\beta)^{-\alpha}\right]^2}\right.$$
$$\left.+\frac{6\Gamma(\alpha-2/\lambda)\Gamma(2/\lambda+1)\Gamma^2(\alpha-1/\lambda)\Gamma^2(1/\lambda+1)}{\beta^{(3\alpha-4/\lambda)}\Gamma^3(\alpha)\left[(a^\lambda+\beta)^{-\alpha}-(b^\lambda+\beta)^{-\alpha}\right]^3}-\frac{3\beta^{-4(\alpha-1/\lambda)}\Gamma^4(\alpha-1/\lambda)\Gamma^4(1/\lambda+1)}{\Gamma^4(\alpha)\left[(a^\lambda+\beta)^{-\alpha}-(b^\lambda+\beta)^{-\alpha}\right]^4}\right\}$$
$$\times\left\{\frac{\beta^{-(\alpha-2/\lambda)}\Gamma(\alpha-2/\lambda)\Gamma(2/\lambda+1)}{\Gamma(\alpha)\left[(a^\lambda+\beta)^{-\alpha}-(b^\lambda+\beta)^{-\alpha}\right]}-\frac{\beta^{-2(\alpha-1/\lambda)}\Gamma^2(\alpha-1/\lambda)\Gamma^2(1/\lambda+1)}{\Gamma^2(\alpha)\left[(a^\lambda+\beta)^{-\alpha}-(b^\lambda+\beta)^{-\alpha}\right]^2}\right\}^{-2}. \tag{27}$$

Survival and hazard functions

Survival function is a probability that a subject survives longer than time x, i.e., $\mathrm{S}(x)=\mathrm{P}(\mathrm{X}>x)$ $=1-\mathrm{F}(x)$. Let X be distributed as the TPL distribution with the cdf $\mathrm{F}(x)$ as in Eq. (8). When $\mathrm{S}(x)=1-\mathrm{F}(x)$ is replaced by $\mathrm{F}(x)$, we have the survival function given by;

$$\mathrm{S}(x\,|\,a\le x\le b)=\frac{(x^\lambda+\beta)^{-\alpha}-(b^\lambda+\beta)^{-\alpha}}{(a^\lambda+\beta)^{-\alpha}-(b^\lambda+\beta)^{-\alpha}},\qquad 0<a\le x\le b<\infty, \tag{28}$$

where $\alpha>0,\beta>0$, and $\lambda>0$. Consider the ratio of $\mathrm{f}(x)$ to $\mathrm{S}(x)$, i.e., $\mathrm{H}(x)=\mathrm{f}(x)/\mathrm{S}(x)$, which is called the hazard function. From replacement of the pdf in Eq. (8), and the survival function in Eq. (28) as in function $\mathrm{H}(x)$, we obtain the hazard function of X, i.e.;

$$\mathrm{H}(x\,|\,a\le x\le b)=\frac{\alpha\lambda x^{\lambda-1}(x^\lambda+\beta)^{-(\alpha+1)}}{(x^\lambda+\beta)^{-\alpha}-(b^\lambda+\beta)^{-\alpha}},\qquad 0<a\le x\le b<\infty. \tag{29}$$

Some plots of the hazard function of the TPL distribution in Eq. (29) are shown in **Figure 2(d)**. In addition, some plots of the hazard function of PL distribution are presented in **Figure 2(a)**,, and some plots of the R-TPL and L-TPL distribution are shown in **Figures 2(b)** and **2(c)**, respectively.

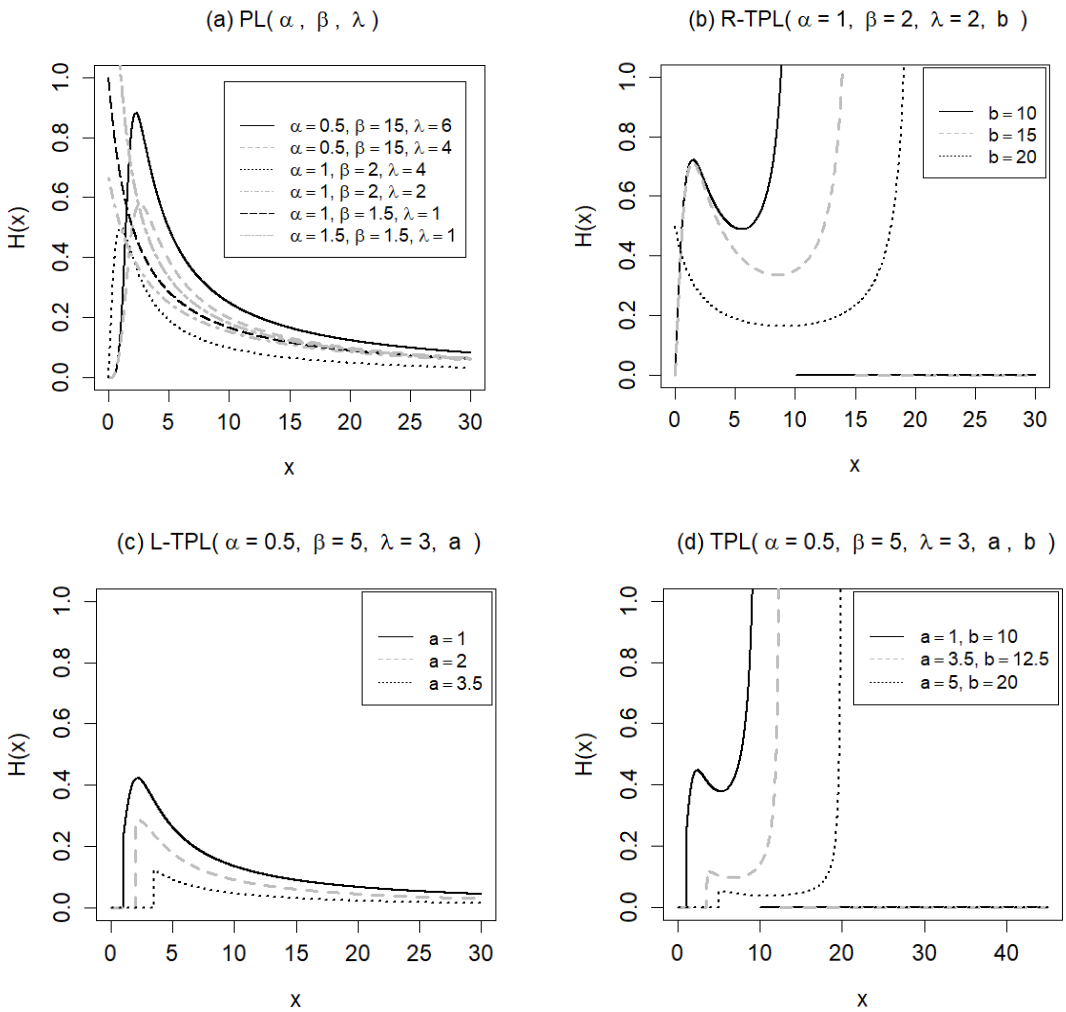

Figure 2 Some plots of the hazard function of distributions: (a) PL, (b) R-TPL, (c) L-TPL, and (d) TPL.

Quantile function

One specific property distribution, the quantile function, is in widespread use in general statistics, and is often found represented in specifying the location of the data. It is also called the percent-point function, or inverse cumulative distribution function.

Theorem 3: Let X be distributed as the TPL distribution with the cdf in Eq. (8); we have the quantile function of X, that is,

$$\mathrm{Q}(u)=\left\{\left[(a^{\lambda}+\beta)^{-\alpha}-u\left((a^{\lambda}+\beta)^{-\alpha}-(b^{\lambda}+\beta)^{-\alpha}\right)\right]^{-1/\alpha}-\beta\right\}^{1/\lambda}. \quad (30)$$

Proof. From the cdf $\mathrm{F}(x)$ in Eq. (8), we have the quantile function of X, i.e., $\mathrm{Q}(u)=\mathrm{F}^{-1}(x)$. The inverse transformation technique is used to generate a random variate for the TPL distribution by setting $x=\mathrm{F}^{-1}(u)$, or $x=Q(u)$, where u is a random variable which has distributed the uniform distribution on (0,1). Suppose $u=\mathrm{F}(x)$. We have;

$$u=\frac{(a^{\lambda}+\beta)^{-\alpha}-(x^{\lambda}+\beta)^{-\alpha}}{(a^{\lambda}+\beta)^{-\alpha}-(b^{\lambda}+\beta)^{-\alpha}}$$

$$(x^{\lambda}+\beta)^{-\alpha}=(a^{\lambda}+\beta)^{-\alpha}-u\left((a^{\lambda}+\beta)^{-\alpha}-(b^{\lambda}+\beta)^{-\alpha}\right)$$

$$x^{\lambda}+\beta=\left[(a^{\lambda}+\beta)^{-\alpha}-u\left((a^{\lambda}+\beta)^{-\alpha}-(b^{\lambda}+\beta)^{-\alpha}\right)\right]^{-1/\alpha}$$

$$x=\left\{\left[(a^{\lambda}+\beta)^{-\alpha}-u\left((a^{\lambda}+\beta)^{-\alpha}-(b^{\lambda}+\beta)^{-\alpha}\right)\right]^{-1/\alpha}-\beta\right\}^{1/\lambda}=Q(u) \quad (31)$$

Generation of TPL random variate

Simulating the TPL random variable is straightforward. The inverse transformation technique is used to generate a random variate for the TPL distribution by setting $x_i=\mathrm{F}^{-1}(u_i)$, where u_i is a random variable of the uniform distribution on (0,1). Then, $X_i, i=1,2,...,n,$ can be generated as follows:

1) Generate u_i, $i=1,2,...,n$ from uniform distribution on (0,1).

2) Set, $x_i=\left\{\left[(a^{\lambda}+\beta)^{-\alpha}-u_i\left((a^{\lambda}+\beta)^{-\alpha}-(b^{\lambda}+\beta)^{-\alpha}\right)\right]^{-1/\alpha}-\beta\right\}^{1/\lambda}$.

Parameter estimation

The procedure to estimate the parameters of the TPL distribution based on the random sample $\tilde{x}=(x_1,x_2,...,x_n)$ of sample size n by using the maximum likelihood estimation (MLE) is discussed here. If $X_i, i=1,2,...,n,$ is independent and identically distributed, the log-likelihood function of $X\sim$TPL $(\alpha,\beta,\lambda,a,b),$ on the observed sample $\tilde{x}$ is given by $\log L(\alpha,\beta,\lambda,a,b\mid\tilde{x})=\ell(\Theta\mid\tilde{x}),$

$$\ell(\Theta\mid\tilde{x})=n\log(\alpha)+n\log(\lambda)-n\log[(a^{\lambda}+\beta)^{-\alpha}-(b^{\lambda}+\beta)^{-\alpha}]$$
$$+(\lambda-1)\sum_{i=1}^{n}\log x_i-(\alpha+1)\sum_{i=1}^{n}\log(x_i^{\lambda}+\beta) \quad (32)$$

If $X\sim$TPL $(\alpha,\beta,\lambda,a,b),$ when $\hat{a}=\min(x_i)$ and $\hat{b}=\max(x_i)$ then the MLE of parameters α, β, and λ are the solutions of the simultaneous in Eqs. (33) - (35), respectively;

$$\frac{\partial\ell(\Theta\mid\tilde{x})}{\partial\alpha}=\frac{n}{\alpha}-\frac{n[(a^{\lambda}+\beta)^{-\alpha}\log(a^{\lambda}+\beta)-(b^{\lambda}+\beta)^{-\alpha}\log(b^{\lambda}+\beta)]}{(a^{\lambda}+\beta)^{-\alpha}-(b^{\lambda}+\beta)^{-\alpha}}-\sum_{i=1}^{n}\log(x_i^{\lambda}+\beta), \quad (33)$$

$$\frac{\partial\ell(\Theta\mid\tilde{x})}{\partial\beta}=-\frac{n[(a^{\lambda}+\beta)^{-\alpha}\log(a^{\lambda}+\beta)-(b^{\lambda}+\beta)^{-\alpha}\log(b^{\lambda}+\beta)]}{(a^{\lambda}+\beta)^{-\alpha}-(b^{\lambda}+\beta)^{-\alpha}}-(\alpha+1)\sum_{i=1}^{n}\frac{1}{(x_i^{\lambda}+\beta)}, \quad (34)$$

$$\frac{\partial \ell(\Theta \mid \tilde{x})}{\partial \lambda} = \frac{n}{\lambda} - \frac{n[(a^{\lambda}+\beta)^{-\alpha}\log(a^{\lambda}+\beta) - (b^{\lambda}+\beta)^{-\alpha}\log(b^{\lambda}+\beta)]}{(a^{\lambda}+\beta)^{-\alpha} - (b^{\lambda}+\beta)^{-\alpha}}$$
$$+\sum_{i=1}^{n}\log x_i - (\alpha+1)\sum_{i=1}^{n}\frac{x_i^{\lambda}\log(x_i)}{(x_i^{\lambda}+\beta)}. \tag{35}$$

The expression of these differential Eqs. (33) - (35) are not in closed form. Therefore, the parameter estimates, such as $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\lambda}$, can be obtained by using the numerical optimization with the *nlm* function in the R language [12].

Applications

The usefulness of the TPL distribution is illustrated by having been fitted to real datasets, by using MLE to estimate the parameters, and by comparing the proposed TPL distribution with the Lomax (L) and PL distributions. The distribution selection is carried out is using the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC), which are given respectively as follows: $\text{AIC} = -2\log L(\hat{\theta}) + 2k$ and $\text{BIC} = -2\log L(\hat{\theta}) + k\log(n)$, where $\log L(\hat{\theta})$ denotes the log-likelihood function evaluated at the maximum likelihood estimates of $\hat{\theta}$, k is the number of parameters of any distribution, and n is the sample size. In addition, we used Kolmogorov–Smirnov test with the value of D_n, which is given as $\text{D}_n = \sup_x |F_n(x) - F_0(x)|$, where $F_n(x)$ and $F_0(x)$ are the empirical distribution function and the theoretical cumulative distribution of the distribution being tested, respectively. The distribution with minimum AIC (or BIC or D_n) value is chosen as the best distribution to fit the data. The results of the application studies are shown in **Tables 1 - 3**. The first data set, an uncensored data set corresponding to failure times for a particular windshield model, included 88 observations that were classified as failed times of windshields [13], which appears in Ramos *et al.* [14]. In addition, the failure times of 67 truncated aircraft windshield data was shown to the sample data set (see [15]). Finally, the cancer patient data, which is an uncensored data set corresponding to remission times (in months) of a random sample of 128 bladder cancer patients, was given in Lee and Wang in 2003 [16,17]. These example data are considered to fit the data by using the TPL, PL, and Lomax distributions.

First data set: Failure times for a particular windshield: 0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779,1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.82,3, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663.

Second data set: Failure times of 67 truncated aircraft windshields: 1.866, 2.385, 3.443, 1.876, 2.481, 3.467, 1.899, 2.610, 3.478, 1.911, 2.625, 3.578, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779,1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.82,3, 3.000, 1.281, 2.085, 2.890, 1.303, 2.089, 2.902, 1.432, 2.097, 2.934, 1.480, 2.135, 2.962, 1.505, 2.154, 2.964, 1.506, 2.190, 3.000, 1.568, 2.194, 3.103, 1.615, 2.223, 3.114, 1.619, 2.224, 3.117, 1.652, 2.229, 3.166, 1.652, 2.300, 3.344, 1.757, 2.324, 3.376.

Third data set: Remission times (in months): 0.08, 2.09, 3.48, 4.87, 6.94 , 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46 , 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

Table 1 MLE estimates of distributions of failure times for a particular windshield.

Distributions	Estimates					-logL	AIC	BIC	D_n (p-value)
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$				
TPL(α,β,λ,a,b)	6.0296	75.0378	1.9645	0.0400	4.6630	124.89	259.78	271.99	0.0682 (0.8236)
PL(α,β,λ)	6.3196	70.3703	2.9645	-	-	150.80	307.6	314.93	0.2353 (0.0002)
L(α,β)	17,050	39,818	-	-	-	165.37	334.74	339.63	7.3641 (<0.0001)

Table 2 MLE estimates of distributions of failure times of 67 truncated aircraft windshields.

Distributions	Estimates					-logL	AIC	BIC	D_n (p-value)
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$				
TPL(α,β,λ,a,b)	1.9994	31.0496	2.8581	1.0700	3.9240	69.03	148.06	159.16	0.1466 (0.1076)
PL(α,β,λ)	1.2063	35.0496	3.3349	-	-	109.94	225.88	232.54	0.2180 (0.0031)
L(α,β)	66,957	159,034	-	-	-	124.96	253.92	258.36	5.2178 (<0.0001)

Table 3 MLE estimates of distributions of remission times (in months).

Distributions	Estimates					-logL	AIC	BIC	D_n (p-value)
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$				
TPL(α,β,λ,a,b)	1.7230	28.9151	1.4476	0.0800	79.0500	408.99	827.98	842.24	0.0374 (0.9941)
PL(α,β,λ)	2.0701	34.8626	1.4276	-	-	409.74	825.48	834.04	0.0351 (0.9975)
L(α,β)	13.936	121.003	-	-	-	413.83	831.66	837.36	0.0967 (0.4865)

For the 3 data-sets, we use nlm() function in R with the starting parameter values, i.e., $(\alpha_0,\beta_0,\lambda_0) = (1,5,1)$, to estimate the parameter $\alpha,\beta,\lambda.$ When data set is fitted by the distributions, i.e., TPL, PL, and Lomax, The results in **Table 1** indicate that AIC, BIC, and D_n have the smallest values for data about failure times for a particular windshield under the TPL distribution model with regard to the PL and Lomax distributions, and **Figure 3(a)** shows the estimated pdf for the fitted distribution by using the TPL when closed to the histogram of the data. In addition, the results in **Table 2** and the estimated pdf for the fitted distributions as in **Figure 3(b)** show that the TPL distribution provides a better fit to the data about the failure times of 67 truncated aircraft windshields than the PL and Lomax distributions. Finally, the results in **Table 3** show that the TPL and PL distributions have similar values of AIC, BIC, and D_n, and **Figure 3(c)** gives the estimated pdf for the fitted distribution by using the TPL when closed to the estimated pdf of the PL distribution. These results show that the TPL distribution is an alternative to fit these data sets, and therefore could be chosen as the best distribution for the first two example data when compared with PL and Lomax distributions. However, the TPL distribution provides fitting of the remission times close to the PL distribution.

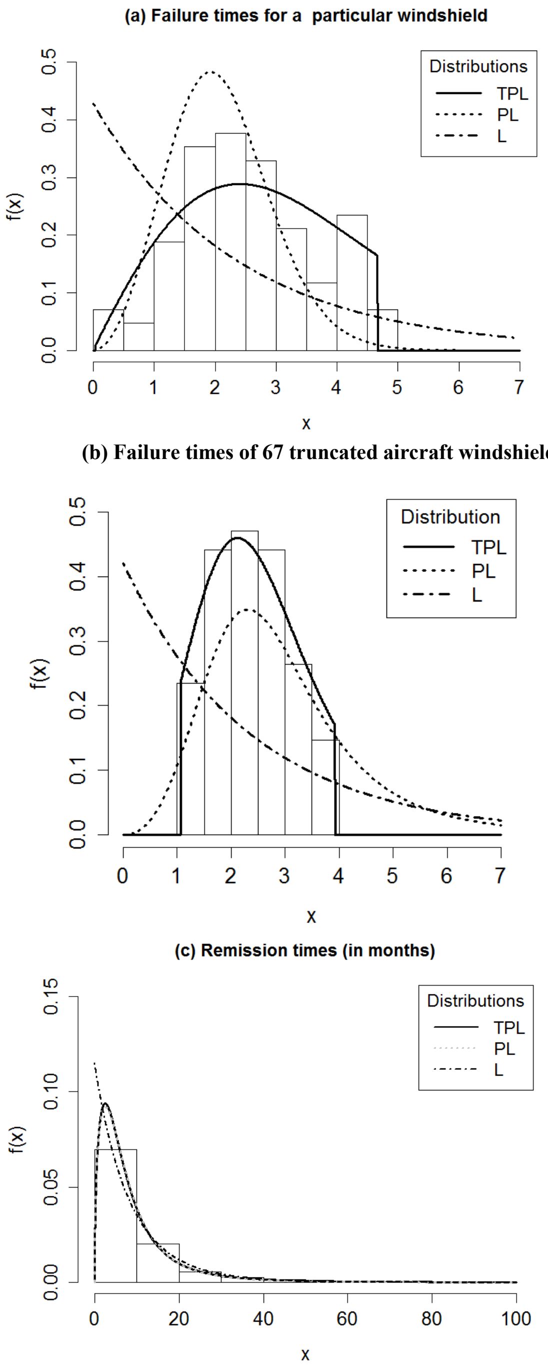

Figure 3 The estimated pdf plots of the TPL, PL, and L distributions for the data sets.

Conclusions

In this paper, we proposed a new truncated distribution, called the TPL distribution. Some statistical properties, i.e., moments, survival, hazard, and quantile functions, were also discussed. Unknown parameters of TPL distribution were estimated by the maximum likelihood estimation. The example data of 3 real data sets have been considered to show the usefulness of the proposed distribution. From the first 2 examples, i.e., the failure times for a particular windshield and the failure times of 67 truncated aircraft windshields, the TPL distribution provided a better fit than the Lomax and power Lomax distribution. We hope that the proposed distribution will attract wider application in many areas, such as engineering, economics, medicine, finance, demographics, etc. A suitable situation to use the TPL distribution would be such as analysis of data which are lifetime data on any interval values in cases where the ability to record exists, or even when occurrences are limited to values which lie above or below a given threshold or within a specified range.

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