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Speiser's Theorem on the Road†

Janyarak TONGSOMPORN[1,*] and Jörn STEUDING[2]

[1]School of Science, Walailak University, Nakhon Si Thammarat 80160 Thailand
[2]Department of Mathematics, University of Würzburg, Würzburg 97074 Germany

(*Corresponding author's e-mail: tjanyarak@gmail.com)

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Abstract

In this note, we discuss the Gauss-Lucas theorem (for the zeros of the derivative of a polynomial) and Speiser's equivalent for the Riemann hypothesis (about the location of zeros of the Riemann zeta-function). We indicate similarities between these results and present there analogues in the context of elliptic curves, regular graphs, and finite Euler products.

Keywords: Riemann zeta-function, elliptic curve, Riemann hypothesis, Speiser's Theorem, regular graphs, Gauss-Lucas theorem

Mathematics Subject Classification: 11M06, 30C10, 11G07, 05C50

Introduction

The Riemann zeta-function is defined as the infinite product

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} \tag{1}$$

taken over all prime numbers p. This so-called Euler product converges absolutely for $\mathrm{Re}\, s > 1$, and eversince its first appearance in Leonhard Euler's early works from 1737, the zeta-function plays a central role in number theory. In 1859, Bernhard Riemann studied the analytic properties of ζ as a function of a complex variable in detail which led to the first proof of the celebrated prime number theorem in 1896 by Jacques Hadamard and (independently) Charles Jean de la Vallée-Poussin. As a matter of fact, the location of zeros of the zeta-function is a crucial point here and that explains the relevance of the yet unsolved *R*iemann hypothesis on the non-vanishing of the Riemann zeta-function $\zeta(s)$ in the half-plane $\mathrm{Re}\, s > 1/2$ (a careful claim in Riemann's paper without a proof and one of the six open millennium problems). In view of the functional equation,

$$\pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma(\tfrac{1-s}{2})\zeta(1-s), \tag{2}$$

this open problem can also be formulated as

Riemann hypothesis. *All non-real zeros of $\zeta(s)$ lie on the so-called critical line $1/2 + i\mathbb{R}$.*

The non-real zeros of the Riemann zeta-function are called *nontrivial* in order to disinguish them from the *trivial* ones which are located on the negative real axis at $s = -2n$ for $n \in \mathbb{N}$ and compensate the poles of the Gamma-function in the functional equation. For this and more we refer to Titchmarsh [1].

In 1934, Andreas Speiser [2] proved an interesting equivalence for the Riemann hypothesis. Studying the Riemann surface of the logarithm of ζ, he came up with the following result:

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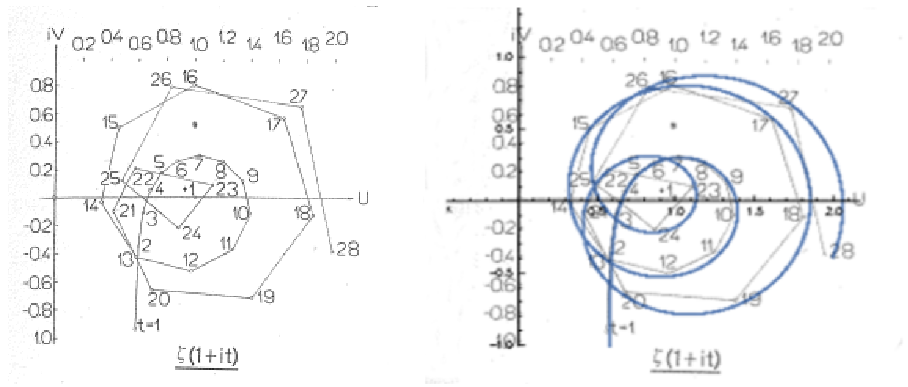

Figure 1 The figure on the left is from Utzinger’s PhD thesis. In the figure on the right we have added the values of $\zeta(1+it)$ (in dark blue) computed by a computer algebra package for this range of t.

Speiser’s theorem. *The Riemann hypothesis is true if, and only if, the first derivative* ζ' *of the Riemann zeta-function is non-vanishing in the strip* $0 < \mathrm{Re}\, s < 1/2$.

In his short and only paper on the zeta-function of only seven and a half pages from 1934 we count altogether just three formulas and read “Für einen Teil der folgenden Beweise vergleiche man die Dissertation des Herrn A. Utzinger (Über die reellen Züge der Riemannschen Zetafunktion, Zürich 1934)”.[‡] Speiser is well-known for his contributions to group theory and philosophy, however, in the 1930s he was investigating Riemann surfaces and besides his theoretical results as, e.g. [3], Speiser was also interested in their applications to special functions. His observation was supported by the PhD thesis [4] of his student Albert Utzinger who provided remarkable calculations of special values of the zeta-function which come very close to modern computations (see **Figure 1**). According to Juan Arias de Reyna [5], “[Speiser’s] methods are between the proved and the acceptable” and we do agree on his judgement. For a detailed and rigorous reasoning along Speiser’s work and further impressive illustrations of the mysterious value-distribution of the Riemann zeta-function we refer to the “X-Ray” [5] of Arias de Reyna.

In the 1960s, with the power of the upcoming computers, Robert Spira [6] delivered further numerical material about ζ and its derivatives (see **Figure 2**); moreover, he rediscovered one implication of Speiser’s theorem.

In 1974, Norman Levinson and Hugh Montgomery [7] found a quantitative version of Speiser’s theorem which turned out to be very influential for further research in that direction. Their approach omits Riemann surfaces completely and relies on Littlewood’s lemma. From the many papers on that topic we just mention here only that Rasa Šleževičienė [8] verified the analogue for elements of the Selberg class, which is widely expected to contain all relevant number-theoretical zeta- and L-functions (having an Euler product representation); prototypical examples besides Riemann’s ζ are Dirichlet L-functions $L(s,\chi)$ to residue class characters. In the context of the Extended Selberg class, Ramūnas Garunkštis and Raivydas Šimėnas [9] showed that an analogue of Speiser’s theorem can even be true if the zeta-function does not fulfill the analogue of the Riemann hypothesis (or does not have an Euler product); for example, the so-called Davenport–Heilbronn zeta-function. This function is defined as a linear combination of Dirichlet

[‡]Our translation: “For part of the proofs one shall compare the dissertation of Mr. A. Utzinger (Über die reellen Züge der Riemannschen Zetafunktion, Zürich 1934).

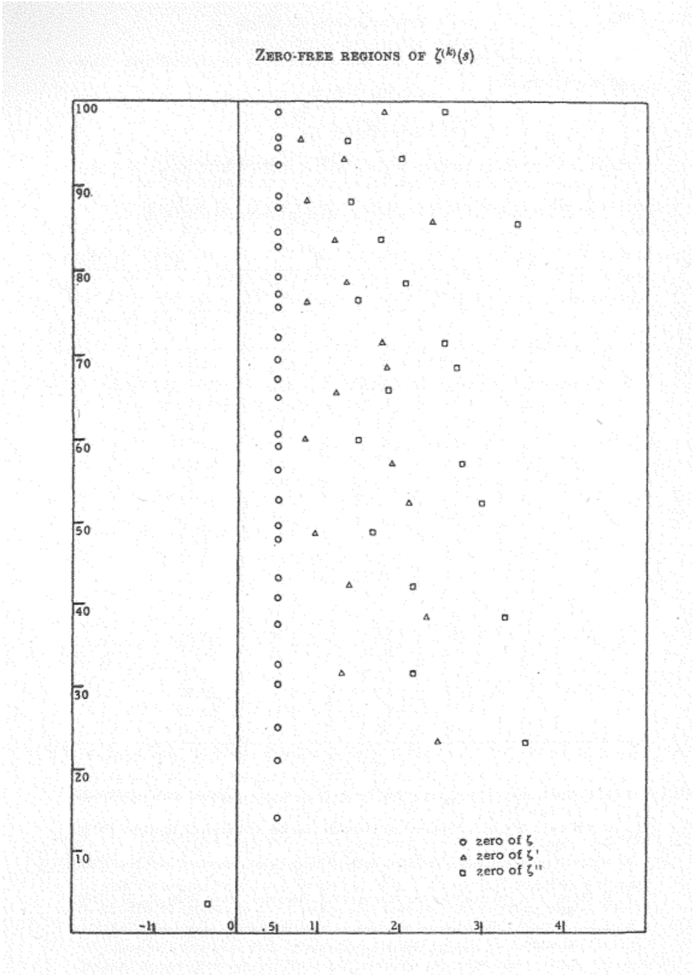

Figure 2 This illustration is due to Spira [6] and indicates the location of the low-lying zeros of ζ and its first and second derivative.

L-functions, namely

$$F(s) := \frac{1-\mathrm{i}\alpha}{2} L(s,\chi) + \frac{1+\mathrm{i}\alpha}{2} L(s,\overline{\chi}) \qquad \text{with} \quad \alpha = \frac{\sqrt{10-2\sqrt{5}}-2}{\sqrt{5}-1}, \tag{3}$$

where χ denotes the character mod 5 satisfying $\chi(2) = i$, and $\overline{\chi}$ is its conjugate (or inverse). It appears that $F(s)$ has infinitely many zeros in the open left half $0 < \mathrm{Re}\, s < 1/2$ of the critical strip and their number equals asymptotically the number of zeros of its derivative in the same region.

In this article we have a closer look on further analogues of Speiser's theorem with respect to rather different zeta-functions (namely those associated with elliptic curves and regular graphs) and here we shall meet further examples of zeta-functions for which Speiser's theorem holds true (with a Riemann hypothesis in the background and without). Although this observation is definitely not very deep, it seems that this relation is not known within the community. Finally, we indicate how Speiser's observation is related to the Gauss–Lucas theorem about the location of zeros of a polynomial and its derivative.

Results and discussion

Elliptic Curves

For simplicity, we may assume that we are given an elliptic curve $E = E(\mathbb{Q})$ over the rationals in Weierstrass form, i.e., E consists of all rational points (x, y) satisfying the cubic equation

$$Y^2 = X^3 + aX + b \qquad \text{with} \quad a, b \in \mathbb{Q} \quad \text{and} \quad -\Delta := a^3 + 27b^2 \neq 0; \tag{4}$$

the latter condition on the non-vanishing of the discriminant implies that E is smooth. It has been observed that by the tangent-chord method there is an addition law on such elliptic curves (see **Figure 3**) which provides a group structure, where a point at infinity serves as neutral element. This has first been noticed by Henri Poincaré in 1901 for the field of complex numbers in place of $\mathbb{Q}$, however, we may exchange the rationals by any field (still taking care of the non-vanishing discriminant).

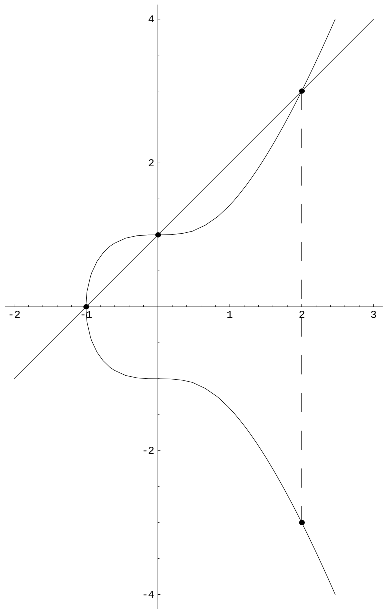

Figure 3 The addition law by the tangent–chord method: $(-1, 0) \oplus (0, 1) = (2, -3)$ on the elliptic curve $Y^2 = X^3 + 1$ (over the field of real numbers).

In the case of a finite field $\mathbb{F}_q$, of course, there are only finitely many points on the elliptic curve, so its group order $\sharp E(\mathbb{F}_q)$ is finite too. If $q = p$ is a prime number, then the equation can be rewritten as a congruence, namely,

$$Y^2 \equiv X^3 + aX + b \bmod p, \tag{5}$$

where the coefficients a and b are adjusted to their residues $\bmod\, p$. For the order of this group one may expect that $x^3 + ax + b$ is with probability $1/2$ a non-zero square (according to the equal number of quadratic residues and non-residues modulo p), each of which giving two solutions to the congruence plus one for the point at infinity. And indeed, in 1933/34, Helmut Hasse [10] proved for an arbitrary finite field the

Riemann hypothesis for elliptic curves. *If $N_q = \sharp E(\mathbb{F}_q)$ denotes the order of $E(\mathbb{F}_q)$, then*

$$|q + 1 - N_q| \leq 2\sqrt{q}. \tag{6}$$

The name refers to the classical Riemann hypothesis. The reason behind is that the generating function for this elliptic curve takes the form

$$\begin{aligned} Z_E(T) &:= \exp\left(\sum_{m\geq 1}\frac{N_{q^m}}{m}T^m\right) \\ &= \frac{(1-\alpha T)(1-\beta T)}{(1-T)(1-qT)} = \frac{qT^2-aT+1}{(1-T)(1-qT)}, \end{aligned} \tag{7}$$

where $a = q + 1 - N_q$ is the sum of the reciprocals of the two roots α, β of the numerator polynomial $P = qT^2 - aT + 1$ (not to be confused with the coefficient a in the cubic equation above) and

$$N_{q^m} = \sharp E(\mathbb{F}_{q^m}) = q^m + 1 - \alpha^m - \beta^m. \tag{8}$$

In view of Hasse's bound the pair of zeros $t_{1,2} = \frac{1}{2q}(a \pm \sqrt{a^2 - 4q})$ of P are complex conjugates or a double zero. Hence, the derivative P' has a single root at their real midpoint $t_1 + t_2 = \frac{a}{2q}$. Setting $T = q^{-s}$ one observes that that $\zeta_E(s) = Z_E(q^{-s})$ vanishes if, and only if, $q^s = \alpha$ or $q^s = \beta$. Notice that by $T = q^{-s}$ a circle in the T-plane is mapped to a vertical line in the s-plane as well as disks correspond to right half-planes. Moreover, one zero of $Z_E(T)$ corresponds to infinitely many zeros of $\zeta_E(s)$ since switching by $T = q^{-s}$ from the T- to the s-plane implies that with $\rho_0 = \beta_0 + i\gamma_0$ also $\rho_n = \beta_0 + i\frac{2\pi}{\log q}n$ for every $n \in \mathbb{Z}$ is a zero. In view of $\alpha\beta = q$ and $\beta = \overline{\alpha}$ it follows from Hasse's inquality that $|\alpha| = |\beta| = q^{1/2}$, hence the analogue of the Riemann hypothesis is true for $\zeta_E(s)$. There is a functional equation for this zeta-function too, namley $\zeta_E(1-s) = \zeta_E(s)$, so the complex zeros lie all on the axis of symmetry $1/2 + i\mathbb{R}$. For details we refer to Washington [11].

Since $\zeta_E(s) = -\log q Z_E(q^{-s})$ we have $\zeta_E(s)' = 0$ if, and only if, $q^{-s} = \frac{a}{2q}$ which, by Hasse's inequality, is equivalent to

$$2q^{1-\sigma} \leq 2\sqrt{q}, \tag{9}$$

where σ denotes the real part of s. Hence, $\sigma \geq 1/2$ so that the analogue of Speiser's theorem for elliptic curves holds too.

It should be mentioned that Andre Weil generalized Hasse's result significantly. He proposed the famous *Weil conjectures* for extending those results to varieties over finite fields which had been proven by Pierre Deligne, Alexander Grothendieck and others in the 1960s and 70s. Another important issue of Hasse's bound is that the global L-function associated to an elliptic curve $E(\mathbb{Q})$ is defined by a convergent Euler product

$$L_E(s) = \prod_{p \text{ bad}}(1 - a_p p^{-s})^{-1} \cdot \prod_{p \text{ good}}(1 - a_p p^{-s} + p^{1-2s})^{-1} \quad ; \tag{10}$$

here the primes in the Euler product are distinguished according to the reduction of $E(\mathbb{Q})$ modulo the primes p. Notice that the Euler factors for the good primes equal the local zeta-functions ζ_E for $E = E(\mathbb{F}_p)$. There are only finitely many primes with bad reduction, while for all others one has $a_p := p + 1 - N_p$ which gives $|a_p| \leq 2\sqrt{p}$ by Hasse's theorem, hence the product converges for $\mathrm{Re}\, s > 3/2$. The interesting features of $L_E(s)$, however, show up only to the left of this abscissa of convergence by analytic continuation. Since $L_E(s)$ is an element of the Selberg class, it follows from the work of Šleževičienė [8] that also for the global zeta-function $L_E(s)$ the analogue of Speiser's theorem holds true. There are further interesting things to study as, for example, the Birch and Swinnerton-Dyer conjecture, another open millennium problem and another story.

Regular Graphs

A graph $G = (V, E)$ with vertex set V and edge set E which is free of loops and without any multi-edge is called k-regular if the degree at each vertex equals k. In this case $\lambda_1 = k$ is a simple eigenvalue of the associated adjacency matrix of G and all other eigenvalues are of absolute value less than or equal to k. If G is bipartite, then the spectrum is symmetric with respect to the origin and, in particular, $-k$ is a simple eigenvalue too; otherwise, if G is not bipartite, all other eigenvalues are of absolute value strictly less than k. Since G is undirected, the adjacency matrix is symmetric and thus all eigenvalues are real. Therefore, the multiset of eigenvalues, or the spectrum spec(G) of G for short, is contained in the interval $[-k, k]$. A k-regular graph G is called a *Ramanujan graph* if all other eigenvalues λ of G satisfy the inequality

$$\lambda(G) := \max\{|\lambda| \,:\, \lambda \in \mathrm{spec}(G), \lambda \neq \pm k\} \leq 2\sqrt{k-1}. \tag{11}$$

This notion is due to Alexander Lubotzky, Ralph Phillips and Peter Sarnak [12]. In view of the relation to a certain number theoretical feature studied by Srinivasa Ramanujan [13] one hundred years ago, they have coined this name, and they have provided non-trivial families of such Ramanujan graphs. Trivial families of Ramanujan graphs are, for instance the complete (bipartite) graphs, Paley graphs, etc. First non-trivial examples were given by Grigori Margulis [14] and Alexander Lubotzky et al. [12]; another rich family had been found by Arias de Reyna [15] by a construction that relies on the Riemann hypothesis for curves over finite fields and its generalizations. The graph-theoretical relevance of Ramanujan graphs is indicated by a theorem due to Noga Alon & Ravi Boppana [16] which claims that for every k-regular graph G on n vertices

$$\lambda(G) \geq 2\sqrt{k-1} - o(1), \tag{12}$$

where $o(1)$ is a small positive quantity that tends to zero for fixed k as n tends to infinity. So Ramanujan graphs have, in some sense, the largest possible gap in the spectrum and this makes them optimal examples for so-called expanders. In graph theory, an *expander* is a highly connected sparse graph which, on first glance may appear contradictory, however, they do exist and have plenty of applications, e.g., in the design of explicit efficient communication networks. For the formal definition we refer to Sarnak's short survey [17].

There is also a zeta-function associated with graphs and an analogue of the Riemann hypothesis. Harold Stark and Audrey Terras [18] came up with the following

Ramanujan Criterion. *A regular graph is Ramanujan if, and only if, the associated Ihara zeta-function satisfies the analogue of the Riemann hypothesis for graphs.*

So the graph-theoretical Riemann hypothesis is true for some graphs and false for others. Given a $(q+1)$-regular graph G, the analogue of the Riemann hypothesis for G is that all zeros of $\zeta_G(q^{-s})^{-1}$ in $0 < \mathrm{Re}\, s < 1$ satisfy $\mathrm{Re}\, s = 1/2$. For example, the Ihara zeta-function associated with the complete graph K_4 on four vertices is given by

$$\zeta_{K_4}(u)^{-1}(u) = (1-u^2)^2(1-3u+2u^2)(1+u+2u^2)^3 \tag{13}$$

and satisfies the analogue of the Riemann hypothesis; however, the zeta-function for the complete graph on four vertices minus one edge, $K_4 - e$, can be computed as

$$\zeta_{K_4-e}(u)^{-1}(u) = (1-u^2)(1-u)(1+u^2)(1+u+2u^2)(1-u^2-2u^3) \tag{14}$$

does not fulfill the Riemann hypothesis. So K_4 is Ramanujan and $K_4 - e$ is not. The same holds true for K_n and $K_n - e$ in general but the formulas expand.

The Ihara zeta-function had been introduced by Yasutaka Ihara [19] in the 1960s in the context of discrete subgroups of the 2×2 p-adic special linear group. Twenty years later, Toshikazu Sunada [20] defined their counterparts in graph theory by

$$\zeta_G(u) = \prod_{[P]} (1 - u^{\nu[P]})^{-1}, \tag{15}$$

where the product is taken over all equivalence classes $[P]$ of primitive (closed) paths P in a regular graph G of length $\nu[P]$. This definition reminds us of the Euler product for the classical Riemann zeta-function. The so-called three-term determinant formula, however, is more useful in many applications: if $G = (V, E)$, then

$$\zeta_G(u)^{-1} = (1 - u^2)^{r-1} \det(I - Au + Qu^2), \tag{16}$$

where A is the adjacency matrix of G and $Q = (Q_{ij})$ is the diagonal matrix with diagonal entries $Q_{jj} = \deg(v_j) - 1$ and the v_j's are the vertices of G; the quantity r equals $\sharp E - \sharp V + 1$ and is the rank of the fundamental group of G relative to one fixed vertex. For this and more we refer to the monograph [21] by Terras and Stark.

Define

$$\xi_G(u) = \zeta_G(u)^{-1}(1 - u^2)^{1-r}. \tag{17}$$

Then ξ_G has zeros where ζ_G has poles except the trivial ones at $u = \pm 1$. In view of the three-term determinant formula we have

$$\xi_G(q^{-s}) = \prod_{\lambda \in \mathrm{spec}(G)} (1 - \lambda q^{-s} + q^{1-2s}). \tag{18}$$

Each factor is of the form

$$1 - \lambda q^{-s} + q^{1-2s} = (1 - \alpha q^{-s})(1 - \beta q^{-s}), \tag{19}$$

where α and β are the reciprocals of zeros of ζ_G^{-1} satisfying $\alpha + \beta = \lambda$ and $\alpha\beta = q$, as in the case of the zeta-function of an elliptic curve!

Therefore, it is no surprise that the analogue of Speiser's theorem holds in the context of graph zeta-functions too. Different to the case of zeta-functions to elliptic curves is that for some graphs Speiser's theorem yields zeros off the critical line and for others not (Ramanujan graphs). This leads to the following

Speiser–type criterion for Ramanujan graphs. *A $q+1$-regular graph G is Ramanujan if, and only if, the first derivative of the associated Ihara zeta-function $\zeta_G(q^{-s})$ has no pole in $0 < \mathrm{Re}\, s < 1/2$.*

If we translate the non-vanishing condition for the derivative, we get back the definition of a Ramanujan graph! In fact, one can prove this criterion straightforward by the explicit form of the Ihara zeta-function and the interested reader is invited to do so (this is essentially what Stark and Terras [18] did). Before we give a slightly different proof of the Speiser–type criterion in the following section, however, let us stress another similarity to other zeta-functions, namely that (18) implies

$$q^{sn}\xi_G(q^{-s}) = \prod_{\lambda} (q^s - \lambda + q^{1-s}) \tag{20}$$

which is obviously invariant under $s \mapsto 1 - s$. Hence, there is again a functional equation which we may rewrite as

$$\xi_G(1/(qu)) = (qu^2)^{-n}\xi_G(u). \tag{21}$$

Zeros of a Polynomial and its Derivative

The well-known Gauss-Lucas theorem provides a geometric relation for the zeros of a polynomial and of its derivative. Its first appearance is implicit in some work [22] of Carl Friedrich Gauss from 1816. He considered the zeros of the derivative as the equilibria when unit-masses are attached to the zeros of the polynomial with a force proportional to the distances. The first rigorous proof is due to the French engineer Félix Lucas [23] in 1880 (probably not being aware of Gauss' earlier work).

Gauss-Lucas theorem. *Given a polynomial $P \in \mathbb{C}[X]$, then the zeros of P' lie in the convex hull of the zeros of P.*

Here is the simple proof: Starting with the fundamental theorem of algebra and the factorization of complex polynomials, we have

$$P(z) = a \prod_{\rho} (z - \rho) \tag{22}$$

with some leading coefficient $a \in \mathbb{C}$ and roots ρ of P. Thus, we deduce by logarithmic differentiation that

$$\frac{P'}{P}(z) = (\log P(z))' = \sum_{\rho} \frac{1}{z - \rho}. \tag{23}$$

Now supppose that $P'(\omega) = 0 \neq P(\omega)$ with an ω outside the convex hull of the zeros ρ of P. Then one can find a straight line through ω that does not intersect the convex hull of the P-zeros ρ. So the numbers $\omega - \rho$ lie in one half-plane determined by this line and the same holds for their reciprocals (as follows from a simple computation). This implies

$$\frac{P'}{P}(\omega) = \sum_{\rho} \frac{1}{\omega - \rho} \neq 0, \tag{24}$$

the desired contradiction. For more details we refer to Prasolov [24]. For various generalizations to entire functions we refer to the early survey article by Edward Burr Van Vleck [25].

Next we shall use this result to prove the *Ramanujan criterion* from the previous section.

Recall that G is a Ramanujan graph if, and only if, $\zeta_G(q^{-s})$ has all its poles in $0 < \mathrm{Re}\, s < 1$ on the line $\mathrm{Re}\, s = 1/2$. The latter statement, however, claims that the polynomial $\xi_G(u)$ has all its zeros inside the unit disk on the circle $|u| = q^{-1/2}$. By the theorem of Gauss–Lucas, $\xi_G'(u)$ has all its zeros in $|u| \leq q^{-1/2}$ or $\xi_G(q^{-s})$ has all its zeros in the half-plane $\mathrm{Re}\, s \geq 1/2$. In view of

$$\frac{\xi_G'}{\xi_G}(u) = -2(1-r)\frac{u}{1-u^2} - \frac{\zeta_G'}{\zeta_G}(u) \tag{25}$$

and the differentiation rule

$$\big(\log f(q^{-s})\big)' = -\log q \cdot \frac{f'}{f}(q^{-s}) \tag{26}$$

it follows that $\zeta_G'(q^{-s})$ has no poles in $0 < \mathrm{Re}\, s < 1/2$.

For the converse, assume $\zeta_G'(q^{-s})$ has a pole $\rho = \beta + i\gamma$ with $0 < \beta < 1/2$, then it follows from (25) and (26) that $\xi_G'(q^{-\rho}) = 0$ too. This proves the criterion.

Notice that any logarithmic derivative f'/f has simple poles at the zeros and poles of f. Of course, in view of

$$(\log(ab))' = \frac{a'}{a} + \frac{b'}{b} \tag{27}$$

we could have argued also with each factor $1 - \lambda q^{-s} + q^{1-2s}$ individually.

Back to Euler Products

Let $\mathcal{P}$ be a finite set of prime numbers. We consider the finite polynomial Euler product

$$f(s) = \prod_{q\in\mathcal{P}} P_q(q^{-s}), \qquad \text{where} \quad P_q \in \mathbb{C}[X] \tag{28}$$

is some non-zero polynomial (which explains the attribute "polynomial" for the Euler product above). Then, according to (26) above,

$$\frac{f'}{f}(s) = -\sum_{q\in\mathcal{P}} \frac{P_q'}{P_q}(X)\log q \qquad \text{with} \quad X = q^{-s}. \tag{29}$$

This is similar to the situation in the proof of the Gauss–Lucas theorem. Actually, we can apply the Gauss–Lucas theorem for each Euler factor $P_q(X)$ individually and obtain by the same reasoning

A Speiser Theorem for Finite Euler Products. *If f is a finite polynomial Euler product of the form (28), then the zeros of f' lie in the convex hull of the zeros of f.*

This result may serve as a toy model for the observation Spira [6] made about the location of ζ- and ζ'-zeros (illustrated by **Figure 2** above) although the Riemann zeta-function is not a finite polynomial Euler product at all.

An expression similar to the partial fraction decomposition (29) holds for entire functions of finite order too; in fact one can derive from the Hadamard product, for example

$$f(s) = \exp(a+bs)\prod_{\rho}\left(1-\frac{s}{\rho}\right), \tag{30}$$

by logarithmic differentiation the partial fraction decomposition

$$\frac{f'}{f}(s) = b + \sum_{\rho}\frac{1}{s-\rho}. \tag{31}$$

But in general we are dealing here with an infinite product and an infinite series. Nevertheless, this logarithmic derivative replaces the logarithm in Speiser's work or its quantitative version due to Levinson and Montgomery (mentioned in the introduction) and it often plays a crucial role in any deeper discussion of the location of zeros of f and its derivative.

There is some interesting difference with the Riemann zeta-function and the so-called completed zeta-function, i.e.,

$$\xi(s) = \tfrac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})\zeta(s), \tag{32}$$

where all trivial (real) zeros of ζ are removed by the Gamma-factor from the functional equation and which was already studied by Riemann. Assuming the Riemann hypothesis, Brian Conrey [26] showed that all the zeros of all the derivatives $\xi^{(m)}(s)$ are on the critical line. Moreover, unconditionally, the proportion of zeros of $\xi^{(m)}(s)$ on the critical line equals $1+O(m^{-2})$. This is not unrelated to investigations on real entire functions of genre 0 and 1 due to Edmond Laguerre, Geogre Pólya, Issai Schur, Joseph Walsh, and others; for more on this we refer once more to Van Vleck [25]. For more recent work see [27] by Thomas Craven & George Csordas as well as Eduardo Duenez et al. [28] for the situation of characteristic polynomials in random matrix theory and its conjectural link to the Riemann zeta-function.

A Final Observation

We begin with a simple modification of the Gauss–Lucas theorem. Given $P \in \mathbb{C}[X]$, define $Q = P - a$, where a is an arbitrary complex number. Then

$$\frac{Q'}{Q} = \frac{P'}{P-a}, \tag{33}$$

hence the the zeros of P' lie in the convex hull of the zeros of $P - a$. Since this reasoning holds for all complex a, we see that *the zeros of P' lie inside the intersection of all convex hulls of zeros of $P - a$ as a ranges trough $\mathbb{C}$.* This can directly be applied to the *polynomial* zeta-functions associated with curves, graphs, and matrices, but what can be said about the classical Riemann zeta-function?

The investigation of the location of the roots of the equation

$$\zeta(s) = a, \tag{34}$$

where a is some arbitrary but fixed complex number, started with Edmund Landau and his address [29] at the ICM 1912 in Cambridge (UK). Every solution of the latter equation is called an a-point. Landau [30][§] showed that there are approximately as many a-points as zeros, a result which is not too surprising for anyone who is familiar with Nevanlinna theory; moreover, he showed under assumption of the Riemann hypothesis that the a-points are clustered around the critical line. Later Levinson [31] gave an unconditional proof and a quantitative version of this clustering. Atle Selberg [32] showed that at least one half of the a-points lie to the left of the critical line; moreover, he conjectured that about 75 percent of the a-points lie to the left of the critical line while about 25 percent are located to its right. And this matches very well the observation from above that there will always be an a-point to the left of the zeros of the derivative.

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