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Some Basic Properties of Prime and Left Prime Ideals in Γ-Left Almost Rings

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Abstract

The purpose of this paper is to introduce the notion of prime and left prime ideals in Γ-LA-rings. Some characterizations of prime, left prime, and weakly left ideals are obtained. Moreover, we investigate relationships between prime and left prime ideals in Γ-LA-rings. Finally, we obtain the necessary and sufficient conditions of a prime to be a left prime in Γ-LA-rings.

Keywords: Γ-LA-ring, prime ideal, left prime ideal, weakly left prime, left (right) ideal

Introduction

The concepts of a Γ-ring were first introduced by Nobusawa [1] in 1964. His concept is more general than a ring. Nowadays, his Γ-ring is called a Γ-ring in the sense of Nobusawa. In 1966, Barnes [2] gave a definition of a Γ-ring which is more general. He introduced the notation of Γ-homomorphisms, prime, and primary ideals in Γ-rings.

Let S be a non empty set. If there exists a mapping $S \times S \to S$ written (a,b) by ab, S is called a left almost semigroup (LA-semigroup) if S satisfies the identity: $(ab)c = (cb)a$, for all $a,b,c \in S$. The concepts of an LA-semigroup was first introduced by Naseeruddin [3] in 1970. The fundamentals of this non associative algebraic structure were the first discovered by Kazim and Naseeruddin [4]. This structure is closely related with a commutative semigroup because, if an LA-semigroup contains a right identity, then it becomes a commutative monoid [5]. A left identity in an LA-semigroup is unique. An ideal in LA-semigroups has been discussed by Mushtaq and Yousuf [5,6]. In 1981, the notion of Γ-semigroups was introduced by Sen [7]. Let S and Γ be any non empty set. If there exists a mapping $S \times \Gamma \times S \to S$ written (a,α,c) by $a\alpha c, S$ is called a Γ-semigroup if S satisfies the identity, $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a,b,c \in S$ and $\alpha,\beta \in \Gamma$. Γ-LA-semigroups are analogous to Γ-semigroups. A groupoid S is called an Γ-LA-semigroup if it satisfies the left Γ-invertive law: $(a\gamma b)\delta c = (c\gamma b)\delta a$, for all $a,b,c,d \in S$ and $\gamma,\delta \in \Gamma$ [8]. This structure is also known as Abel-Grassmann's groupoid (AG-groupoid).

In 2006, Yusuf [9] introduced the concept of a left almost ring (LA-ring), where a non empty set R with two binary operations "$+$" and "$\cdot$" is called a left almost ring if $(R,+)$ is an LA-group, $(R,\cdot)$ is an LA-semigroup, and distributive laws of "$\cdot$" over "$+$" holds. Further in [10], Shah and Rehman generalize the notions of commutative semigroup rings into LA-semigroup LA-rings. In 2011, Shah *et al.*, [10] generalized the notion of an LA-ring into an nLA-ring. A near left almost ring (nLA-ring) N is an

LA-group under "$+$", an LA-semigroup under "$\cdot$" and left distributive property of "$\cdot$" over "$+$" holds. In [11], Shah *et al*., asserted that with a commutative ring $(R,+,\cdot)$, we can always obtain an LA-ring $(R,\oplus,\cdot)$ by defining, for $a,b,c \in R, a \oplus b = b - a$ and $a \cdot b$ is the same as in the ring. The development of Γ-LA-ring has since been greatly inspired by the results of the research [12-20]. In this paper, we study left (right) ideals, primes, and left prime ideals in Γ-LA-rings. Some characterizations of primes and left prime ideals are obtained. Moreover, we investigate relationships between primes and left prime ideals in Γ-LA-rings.

Preliminaries

In this section, we refer to [21] for some elementary aspects and quote a few definitions and essential examples to set up this study. For more details, we refer to the papers in the references.

Definition 1.1 [21] Let $(R,+)$ and $(\Gamma,+)$ be 2 LA-groups, R is called a Γ-left almost ring (an Γ-LA-ring) if there exists a mapping $R \times \Gamma \times R \to R$ by $(a,\alpha,b) \mapsto a\alpha b$ for all $a,b \in R$ and $\alpha \in \Gamma$ satisfying the following conditions**;**

1. $a\alpha(b+c) = a\alpha b + a\alpha c$
2. $(a+b)\alpha c = a\alpha c + b\alpha c$
3. $a(\alpha+\beta)b = a\alpha b + a\beta b$
4. $(a\alpha b)\beta c = (c\alpha b)\beta a,$

for all $a,b,c \in R$ and $\alpha,\beta \in \Gamma.$

Example 1.2 Let R be an arbitrary LA-ring and Γ any non empty set. Define a mapping $R \times \Gamma \times R \to R$ by $x\gamma y = xy$ for all $x,y \in R$ and $\gamma \in \Gamma.$ It is easy to see that R is an Γ-LA-ring.

Example 1.3 Let $R = \left\{\begin{bmatrix} x & 0 \end{bmatrix} : x \in \mathbb{Z}\right\}$ and $\Gamma = \left\{\begin{bmatrix} x \\ y \end{bmatrix} : x,y \in \mathbb{Z}\right\}.$ It is easy to see that R is an Γ-LA-ring. But R is not an LA-ring.

Lemma 1.4 [21] If R is an Γ-LA-ring with a left identity, then $a\gamma b = a\beta b$ for all $a,b \in R$ and $\gamma,\beta \in \Gamma.$

Proof. The proof is available in [21].

Lemma 1.5 [21] Let R be an Γ-LA-ring with a left identity $e.$ Then $R\Gamma R = R$ and $R = e\Gamma R = R\Gamma e.$

Proof. The proof is available in [21].

Definition 1.6 [21] A non empty subset I of an Γ-LA-ring R is a subring of R if under the binary operations in $R,$ forming an Γ-LA-ring.

Definition 1.7 [21] A subring I of an Γ-LA-ring R is called a left (right) ideal of R if $R\Gamma I \subseteq I$ $(I\Gamma R \subseteq I)$ and is called ideal if it is a left as well as right ideal.

Lemma 1.8 [21] Let I be a proper left (right) ideal of an Γ-LA-ring R with left identity e. Then, $I = R$ if and only if $e \in I$.
Proof. The proof is available in [21].

Lemma 1.9 [21] If R is an Γ-LA-ring with a left identity, then every right ideal is a left ideal.
Proof. The proof is available in [21].

Theorem 1.10 Let I and J be 2 ideals in Γ-LA-ring R. Then, $I \cap J$ is an ideal of an Γ-LA-ring.
Proof. Since I and J are subgroups of the Γ-LA-ring R under addition, thus, $I \cap J$ is an additive subgroup of R. Let $r \in R$ and $a \in I \cap J$. Then, $a \in I$ and $a \in J$. Thus, $r\alpha a, a\alpha r \in I$ (since I is an ideal of R) and $r\alpha a, a\alpha r \in J$. Hence, $r\alpha a, a\alpha r \in I \cap J$. That is, $I \cap J$ is an ideal of a Γ-LA-ring.

Corollary 1.11 Let $\{A_i \mid i \in I\}$ be a family of ideals in an Γ-LA-ring R. Then, $\bigcap_{i \in I} A_i$ is an ideal of R.
Proof. This follows from Theorem 1.10.

Lemma 1.12 Let R be an Γ-LA-ring with a left identity, and I, J are left ideals of R. Then, $I\Gamma J$ is a left ideal of R, where $I\Gamma J = \{a_1\alpha_1 b_1 + \ldots + a_n\alpha_n b_n \mid a_i \in I, b_i \in J, \alpha_i \in \Gamma, i = 1, \ldots, n\}$.

Proof. Since $0 \in I, 0 \in J$ and $\alpha \in \Gamma$, we have $0 = \sum_{i=1}^{n} 0\alpha_i 0 \in I\Gamma J$ so $I\Gamma J \neq \varnothing$. Let I and J be ideals of R. Suppose that $x, y \in I\Gamma J$, then, $x = a_1\alpha_1 b_1 + \ldots + a_n\alpha_n b_n, y = c_1\beta_1 d_1 + \ldots + c_n\beta_n d_n$ where the a_i and c_i are in I and the b_i and d_i are in $J, \alpha_i, \beta_i \in \Gamma$. From this, we obtain $x - y = a_1\alpha_1 b_1 + \ldots + a_n\alpha_n b_n - c_1\beta_1 d_1 - \ldots - c_n\beta_n d_n \in I\Gamma J$.
Now, we have $R\Gamma(I\Gamma J) = I\Gamma(R\Gamma J) \subseteq I\Gamma J$. Therefore, $I\Gamma J$ is a left ideal of R.

Lemma 1.13 Let R be an Γ-LA-ring with a left identity and let I be a left ideal of R. Then, $I\Gamma I$ is an ideal of R.
Proof. By Lemma 1.12, we have $I\Gamma I$ as a left ideal of R. Now, consider $(I\Gamma I)\Gamma R = (R\Gamma I)\Gamma I \subseteq I\Gamma I$. Therefore, $I\Gamma I$ is an ideal of R.

Lemma 1.14 [21] If I is a left ideal of an Γ-LA-ring R with a left identity, and if for any $a \in R, \gamma \in \Gamma$, then $a\gamma I$ is a left ideal of R.
Proof. The proof is available in [21].
Lemma 1.15 [21] Let R be an Γ-LA-ring with a left identity, and $a \in R, \gamma \in \Gamma$. Then, $R\gamma a$ is a left ideal of R.
Proof. The proof is available in [21].

Lemma 1.16 [21] If I is an ideal of an Γ-LA-ring R with a left identity, and if for any $a \in R, \gamma \in \Gamma$, then $a^2\gamma I$ is an ideal of R.
Proof. The proof is available in [21].

Lemma 1.17 [21] Let R be an Γ-LA-ring with a left identity, and $a \in R, \gamma \in \Gamma$. Then, $R\gamma a^2$ is an ideal of R.
Proof. The proof is available in [21].

Lemma 1.18 [21] Let R be an Γ-LA-ring with a left identity, and let A, B be 2 left ideals of R. Then, $(A:\Gamma:B)$ is a left ideal in R, where $(A:\Gamma:B) = \{r \in R : B\Gamma r \subseteq A\}$.
Proof. The proof is available in [21].

Corollary 1.19 [21] Let R be an Γ-LA-ring with a left identity, and let A be left ideals of R. Then, $(A:\gamma:b)$ is a left ideal in R, where $(A:\gamma:b) = \{r \in R : b\Gamma r \in A\}$.
Proof The proof is available in [21].

Remark. [21] Let R be an Γ-LA-ring with left identity e.

1. If A is a left ideal of R, then $A \subseteq (A:\gamma:r)$.
2. Let A be a proper left (right) ideal of R. By Lemma 1.8, we have $e \notin (A:\gamma:r)$, where $r \in R - A$.
3 If A, B, C are left ideals of R, then $(A:\Gamma:C) \subseteq (A:\Gamma:B)$, where $B \subseteq C$.

Lemma 1.20 Let A be an ideal of an Γ-LA-ring R. Then, R/A is an Γ-LA-ring under the following operations: for all $(a+A)+(b+A) = (a+b)+A$ and $(a+A)\gamma(b+A) = a\gamma b + A$.
Proof. We leave the straightforward proof for the reader.

Lemma 1.21 Let R be an Γ-LA-ring. If A and B are 2 left ideals of R, then $(A+B)/A \approx A/(A \cap B)$.
Proof. We leave the straightforward proof for the reader.

Prime and left prime ideals in Γ-LA-rings

We start with the following theorem that gives a relation between the prime and left prime ideal in an Γ-LA-ring. Our starting points are the following definitions;

Definition 2.1 Let R be an Γ-LA-ring. An ideal P is called prime if $A\Gamma B \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$, where A, B are ideals of R.

Definition 2.2 Let R be an Γ-LA-ring. A left ideal P is called left prime if $A\Gamma B \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P,$ where A, B are left ideals of $R.$

Remark. It is easy to see that every left prime ideal is prime.

Lemma 2.3 Let P be a left ideal of an Γ-LA-ring with left identity $R.$ Then, P is left prime of R if and only if $a\gamma(R\alpha b) \subseteq P$ implies that $a \in P$ or $b \in P,$ where $\gamma, \alpha \in \Gamma$ and $a, b \in R.$

Proof. Let P be a left prime ideal of an Γ-LA-ring with left identity $R.$ Now, suppose that $a\gamma(R\alpha b) \subseteq P.$ Then, by the definition of left ideal, we get $R(a\gamma(R\alpha b)) \subseteq R\beta P \subseteq P$, that is;

$$\begin{aligned} R\beta(a\gamma(R\alpha b)) &= (R\delta R)\beta(a\gamma(R\alpha b)) \\ &= (R\delta a)\beta(R\gamma(R\alpha b)) \\ &= (R\delta a)\beta((R\delta R)\gamma(R\alpha b)) \\ &= (R\delta a)\beta((b\delta R)\gamma(R\alpha R)) \\ &= (R\delta a)\beta((b\delta R)\gamma R) \\ &= (R\delta a)\beta((R\delta R)\gamma b) \\ &= (R\delta a)\beta(R\gamma b) \end{aligned}$$

for all $\gamma, \alpha, \beta, \delta \in \Gamma.$ This implies that $(R\delta a)\beta(R\gamma b) \subseteq P$, so that $a = e\delta a \in R\delta a \subseteq P$ or $b = e\gamma b \in R\gamma b.$ Conversely, assume that if $a\gamma(R\alpha b) \subseteq P$ implies that $a \in P$ or $b \in P,$ where $\gamma, \alpha \in \Gamma$ and $a, b \in R.$ Now, suppose that $A\Gamma B \subseteq P,$ where A and B are left ideals of R , such that $A \nsubseteq P.$ Then, there exists $x \notin A$, such that $x \in P.$ Now,

$$x\gamma(R\alpha y) \subseteq A\Gamma(R\Gamma B) \subseteq A\Gamma B \subseteq P,$$

for all $y \in B.$ So, by hypothesis, $y \in P,$ for all $y \in B$ implies that $B \subseteq P.$ Hence, P is a left prime ideal in $R.$

Theorem 2.4 Let P be a left ideal of an Γ-LA-ring with left identity $R.$ Then, P is a left prime ideal of R if and only if $(R\gamma a)\beta(R\alpha b) \subseteq P$ implies that $a \in P$ or $b \in P,$ where $\gamma, \alpha, \beta \in \Gamma$ and $a, b \in R.$

Proof. Let P be a left prime of an Γ-LA-ring with left identity $R.$ Suppose that $(R\gamma a)\beta(R\alpha b) \subseteq P.$ Then, by the definition of left ideal, we get;

$$\begin{aligned} (R\gamma a)\beta(R\alpha b) &= (R\gamma R)\beta(a\alpha b) \\ &= R\beta(a\alpha b) \\ &= a\beta(R\alpha b) \end{aligned}$$

that is, $a\beta(R\alpha b) \subseteq P.$ By Lemma 2.3, we have $a \in P$ or $b \in P.$ Conversely, assume that if $(R\gamma a)\beta(R\alpha b) \subseteq P$ implies that $a \in P$ or $b \in P,$ where $\gamma, \alpha, \beta \in \Gamma$ and $a, b \in R.$ Let

$a\gamma(R\alpha b) \subseteq P.$ Now, consider $a\gamma(R\alpha b) = (R\gamma a)\beta(R\alpha b).$ By using the given assumption, if $a\gamma(R\alpha b) \subseteq P,$ then $a \in P$ or $b \in P.$ Then, by Lemma 2.3, we have P as a left prime ideal in $R.$

Corollary 2.5 Let P be a prime ideal of an Γ-LA-ring with left identity $R.$ If $(R\gamma a^2)\beta(R\alpha b^2) \subseteq P,$ then $a^2 \in P$ or $b^2 \in P,$ where $\gamma, \alpha, \beta \in \Gamma$ and $a, b \in R.$
Proof. This follows from Theorem 2.4.

Theorem 2.6 Let P be a left ideal of an Γ-LA-ring with left identity $R.$ Then, P is a left prime ideal of R if and only if $a\gamma b \in P$ implies that $a \in P$ or $b \in P,$ where $\gamma \in \Gamma$ and $a, b \in R.$
Proof. Let P be a left ideal of an Γ-LA-ring with left identity $R.$ Suppose that $a\gamma b \in P.$ Then, by the definition of left ideal, we get;

$$\begin{aligned}(R\gamma a)\beta(R\alpha b) &= (R\gamma R)\beta(a\alpha b)\\ &= R\beta(a\alpha b)\\ &\subseteq R\beta P\\ &\subseteq P.\end{aligned}$$

By Theorem 2.4, we have $a \in P$ or $b \in P.$ Conversely, the proof is easy.

Corollary 2.7 Let P be a prime of an Γ-LA-ring with left identity $R.$ If $a^2\gamma b^2 \in P,$ then $a^2 \in P$ or $b^2 \in P,$ where $\gamma \in \Gamma$ and $a, b \in R.$
Proof. This follows from Theorem 2.6.

Theorem 2.8 Let R be an Γ-LA-ring with a left identity. If A is a left prime ideal of $R,$ then $(A:\gamma:r)$ is a left prime ideal in $R,$ where $\gamma \in \Gamma$ and $r \in R.$
Proof. Assume that A is a left prime ideal of $R.$ By Corollary 1.19, we have $(A:\gamma:r)$ as a left ideal in $R.$ Let $a\alpha b \in (A:\gamma:r).$ Suppose that $b \notin (A:\gamma:r).$ Since $a\alpha b \in (A:\gamma:r),$ we have $r\gamma(a\alpha b) \in A$, so that $a\gamma(r\alpha b) \in A.$ By Theorem 2.6, we have $a \in A$ or $r\alpha b \in A.$ Therefore, $a \in (A:\gamma:r)$ and, hence, $(A:\gamma:r)$ is a left prime ideal in $R.$

Corollary 2.9 Let R be an Γ-LA-ring with a left identity. If A is a left prime ideal of $R,$ then $(A:\Gamma:B)$ is a left prime ideal in $R,$ where $B \subseteq R.$
Proof. This follows from Theorem 2.8.

Definition 2.10 An Γ-LA-ring R is called an Γ-LA-3-band if its every element satisfies $(a\alpha a)\beta a = a\alpha(a\beta a) = a.$
Proposition 2.11 Every left identity in an Γ-LA-3-band is a right identity.
Proof. Let e be a left identity and a be any element in an Γ-LA-3-band $R.$ Then,

$$\begin{aligned} a\alpha e &= (a\delta(a\beta a))\alpha e \\ &= (e\delta(a\beta a))\alpha a \\ &= (a\beta a)\alpha a \\ &= a \end{aligned}$$

for all $\alpha, \beta, \delta \in \Gamma$. Hence, e is a right identity

Lemma 2.12 If an Γ-LA-3-band R has a left identity, then every left ideal is an ideal.
Proof. Let A be a left ideal of an Γ-LA-3-band R. Then,

$$\begin{aligned} a\gamma s &= ((a\alpha a)\beta a)\gamma s \\ &= (s\beta a)\gamma(a\alpha a) \\ &\in (R\Gamma A)\Gamma(A\Gamma A) \\ &\subseteq A\Gamma A \\ &\subseteq A \end{aligned}$$

for all $\gamma, \alpha, \beta \in \Gamma, a \in A$, and $s \in R$. Hence, A is an ideal of R.

Theorem 2.13 Let R be an Γ-LA-3-band with a left identity. Then, P is a left prime ideal in R if and only if P is a prime ideal in R.
Proof The proof is straightforward.

Definition 2.14 Let R be an Γ-LA-ring. A left ideal P is called a weakly left prime if $\{0\} \neq AB \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$, where A and B are 2 left ideals of R.

Remark. It is easy to see that every left prime ideal is weakly left prime.

Theorem 2.15 Let P be a left ideal of an Γ-LA-ring with left identity R. Then, P is a weakly left prime ideal of R if and only if $\{0\} \neq a\Gamma(R\Gamma b) \subseteq P$ implies that $a \in P$ or $b \in P$, where $a, b \in R$.
Proof. This follows from Lemma 2.3.

Lemma 2.16 Let P be a left ideal of an Γ-LA-ring with left identity R. Then, P is a weakly left prime ideal of R if and only if $\{0\} \neq (R\Gamma a)\Gamma(R\Gamma b) \subseteq P$ implies that $a \in P$ or $b \in P$, where $a, b \in R$.
Proof. This follows from Theorem 2.4.

Theorem 2.17 Let P be a left ideal of an Γ-LA-ring with left identity R. Then, P is a weakly left prime ideal of R if and only if $0 \neq a\gamma b \in P$ implies that $a \in P$ or $b \in P$, where $a, b \in R$ and $\gamma \in \Gamma$.
Proof. This follows from Theorem 2.6.
Lemma 2.18 Let R be an Γ-LA-ring with a left identity and let P be a weakly left prime ideal but not a left prime ideal of R. If $a\gamma b = 0$ for some $a, b \notin P$, then $a\gamma P = P\gamma b = 0$, where $\gamma \in \Gamma$.
Proof. Let $p \in P$ such that $p\gamma b \neq 0$. Then,

$$\begin{aligned} 0 &\neq a\gamma p \\ &= 0 + a\gamma p \\ &= a\gamma b + a\gamma p \\ &= a\gamma(b+p) \in P. \end{aligned}$$

Since P is a weakly left prime ideal of $R,$ we have $a+p \in P$ or $b \in P,$ that is, $a \in P$ or $b \in P.$ This is a contradiction. Therefore, $a\gamma P = 0.$ Similarly, we can show that $P\gamma b = 0.$

Theorem 2.19 Let R be an Γ-LA-ring with a left identity. If P is a weakly left prime ideal that is not left prime, then $P^2 = \{0\}$.

Proof. Let P be a weakly left prime ideal of $R.$ We can show that $P^2 = \{0\}.$ Suppose that $P^2 \neq \{0\}.$ We will show that P is a left prime ideal in $R.$ Let $a\gamma b \in P,$ where $a,b \in R$ and $\gamma \in \Gamma.$ If $a\gamma b \neq 0,$ then $a \in P$ or $b \in P,$ since P is a weakly left prime ideal of $R.$ Now, suppose that $a\gamma b = 0.$ If $a\gamma P \neq 0,$ then there is an element p' of P, such that $a\gamma p' \neq 0$, so that;

$$\begin{aligned} 0 &\neq a\gamma p' \\ &= a\gamma p' + 0 \\ &= a\gamma p' + a\gamma b \\ &= a\gamma(p'+b) \in P \end{aligned}$$

And, hence, P as a weakly left prime ideal gives either $a \in P$ or $p'+b \in P.$ As $p' \in P$, we have $a \in P$ or $b \in P.$ So, we can assume that $a\gamma P = 0.$ Similarly, we can assume that $a\gamma P = P\gamma b = 0.$ Since $P^2 \neq 0,$ there exists $p_1, p_2 \in P$, such that $p_1 p_2 \neq 0.$ Then, $0 \neq (a+p_1)\gamma(b+p_2) \in P$, so $a+p_1 \in P$ or $b+p_2 \in P,$ and hence $a \in P$ or $b \in P.$ Thus, P is a left prime ideal of $R.$ Clearly, $\{0\} \subseteq P^2.$ Hence, $P^2 = \{0\}$ as required.

Corollary 2.20 Let R be an Γ-LA-ring with a left identity and $P^2 \neq \{0\}.$ Then, P is left prime ideal of R if and only if P is a weakly left prime ideal of $R.$
Proof. This follows from Theorem 2.19.

Lemma 2.21 Let R be an Γ-LA-ring with a left identity. If P is a weakly left prime ideal of $R,$ then $(P:\gamma:a) = P \cup (0:\gamma:a),$ where $a \in R-P$ and $\gamma \in \Gamma.$

Proof. Let P be a weakly left prime ideal of $R.$ Clearly, $P \cup (0:\gamma:a) \subseteq (P:\gamma:a).$ For the other inclusion, suppose that $x \in (P:\gamma:a)$, so that $a\gamma x \in P.$ If $a\gamma x \neq 0$ and P is a weakly left prime ideal of $R,$ then $x \in P.$ If $a\gamma x = 0,$ then $x \in (0:\gamma:a).$ So, we have the equality.

Theorem 2.22 Let R be an Γ-LA-ring with a left identity and $a,b \in R-P.$ If $(P:\gamma:a) = P$ or $(0:\gamma:a) = P,$ then P is a weakly left prime ideal of $R,$ where $a \in R-P$ and $\gamma \in \Gamma.$

Proof. Let $a \in R-P$ and $\gamma \in \Gamma$, such that $(P:\gamma:a)=P$ or $(0:\gamma:a)=P$. Suppose that $0 \neq x\alpha y \in P$, where $x, y \in R$ and $\alpha \in \Gamma$. Then, $x \in (P:\alpha:y)$ by Lemma 2.21, we have $(P:\alpha:y)=P \cup (0:\alpha:y)$. Thus, $x \in P$ or $x \in (0:\alpha:y)$, hence, $x \in P$ since $x\alpha y \neq 0$, as required.

Theorem 2.23 Let $R = R_1 \times R_2$, where each R_i is an Γ-LA-ring with a left identity. If $P \times R_2$ is a weakly left prime ideal of R, then P is a weakly left prime ideal of R_1.

Proof. Suppose that $P \times R_2$ is a weakly left prime ideal of R. Let $a_1, a_2 \in R_1$ and $\gamma \in \Gamma$, such that $0 \neq a_1\gamma a_2 \in P$. Then, $0 \neq (a_1\gamma b_1, a_2\gamma b_2) = (a_1, a_2)\gamma(b_1, b_2) \in P \times R_2$. Since $P \times R_2$ is weakly left prime, we have $(a_1, a_2) \in P \times R_2$ or $(b_1, b_2) \in P \times R_2$. It follows that $a_1 \in P$ or $b_1 \in P$. By the definition of a weakly left prime ideal, we have P as a weakly left prime ideal of R_1.

Theorem 2.24 Let $R = R_1 \times R_2$, where each R_i is an Γ-LA-ring with a left identity. Then, P is a left prime ideal of R_1 if and only if $P \times R_2$ is a left prime ideal of R.

Proof. Suppose that P is a left prime ideal of R_1. Let $(a_1, a_2)\gamma(b_1, b_2) = (a_1\gamma b_1, a_2\gamma b_2) \in P \times R_2$, where $(a_1, a_2), (b_1, b_2) \in R, \gamma \in \Gamma$, so that $a_1 \in P$ or $b_1 \in P$, since P is left prime. It follows that $(a_1, a_2) \in P \times R_2$ or $(b_1, b_2) \in P \times R_2$. By the definition of a left prime ideal, we have $P \times R_2$ as a left prime ideal of R. Conversely, this follows from Theorem 2.23.

Corollary 2.25 Let $R = R_1 \times R_2$, where each R_i is an Γ-LA-ring with a left identity. If P is a weakly left prime (left prime) ideal of R_2, then $R_1 \times P$ is a weakly left prime (left prime) ideal of R.

Proof. This follows from Theorem 3.23.

Corollary 2.26 Let $R = \prod_{i=1}^{n} R_i$, where each R_i is an Γ-LA-ring with a left identity. If P is a weakly left prime (left prime) ideal of R_j, then $R_1 \times R_2 \times \ldots \times R_{j-1} \times P \times R_{j+2} \times \ldots \times R_n$ is a weakly left prime (left prime) ideal of R.

Proof. This follows from Theorem 2.23 and Corollary 2.25.

Theorem 2.27 Let $R = R_1 \times R_2$, where each R_i, is an Γ-LA-ring with a left identity. If P is a weakly left prime ideal of R, then $P = \{(0,0)\}$ or P is a left prime ideal of R.

Proof. Let $P = P_1 \times P_2$ be a weakly left prime ideal of R. We can assume that $P \neq \{(0,0)\}$. So, there is an element (a_1, a_2) of P, with $(a_1, a_2) \neq (0,0)$. Then, $(0,0) \neq (a_1, a_2) = (a_1, e)\gamma(e, a_2) \in P$ gives $(a_1, e) \in P$ or $(e, a_2) \in P$. If $(a_1, e) \in P$, then $P = P_1 \times R_2$. We will show that P_1 is a left prime, hence, P is weakly left prime by Theorem 2.23. Let $x\gamma y \in P_1$, where $x, y \in R_1$ and $\gamma \in \Gamma$. Then,

$$0 \neq (x\gamma y, e) = (x,e)\gamma(y,e) \in P_1 \times R_2 = P$$

So, $(x,e) \in P = P_1 \times R_2$ or $(y,e) \in P = P_1 \times R_2$ and, hence, $x \in P_1$ or $y \in P_1$. If $(e,a_2) \in P$, then $P = R_1 \times P_2$. By a similar argument, $R_1 \times P_2$ is a weakly left prime ideal of R.

Theorem 2.28 Let $A \subseteq P$ be a proper ideal of an Γ-LA-ring with left identity R. Then, the following holds:

1. If P is a weakly left prime left ideal of R, then P/A is a weakly left prime ideal of R/A.
2. If A and P/A are weakly left prime ideals of R/A, then P is a weakly left prime ideal of R.

Proof. 1. Let $x, y \in R$ and $\gamma \in \Gamma$, such that $A \neq (x+A)\gamma(y+A) = x\gamma y + A \in P/A$. Then, $x\gamma y \in P$. If $x\gamma y = 0 \in A$, then $(x+A)\gamma(y+A) = x\gamma y + A = 0 + A = A$. This is a contradiction. So, if P is a weakly left prime ideal of R, then $x \in P$ or $y \in P$, hence $x + A \in P/A$ or $y + A \in P/A$, as required.

2. Let $x, y \in R$ and $\gamma \in \Gamma$, such that $0 \neq x\gamma y \in P$. Then,

$$(x+A)\gamma(y+A) = x\gamma y + A \in P/A.$$

For $x\gamma y \in A$, if A is a weakly left prime, then $x \in A \subseteq P$ or $y \in A \subseteq P$. So, we may assume that $x\gamma y \notin A$. Then, $x + A \in P/A$ or $y + A \in P/A$. It follows that $x \in P$ or $y \in P$, as needed.

Theorem 2.29 Let P and Q be two weakly left prime ideals of an Γ-LA-ring with left identity R that are not left prime. Then, $P + Q$ is a weakly left prime ideal of R.

Proof. Since $(P+Q)/Q \approx Q/(P \cap Q)$, we get that $(P+Q)/Q$ is a weakly left prime by Theorem 2.28 (1). Now, the assertion follows from Theorem 2.28 (2).

Conclusions

Many new classes of Γ-LA-rings have been discovered recently. All of these have attracted researchers in the field to investigate them in detail. This article investigates the prime ideal, the left prime ideal, and weakly left prime ideals in Γ-LA-rings. We show that a left ideal P is a left prime ideal of R if and only if $a\gamma b \in P$ implies that $a \in P$ or $b \in P$ where $\gamma \in \Gamma$ and $a,b \in R$ Finally, we show that if P is a weakly left prime ideal of $R_1 \times R_2$ then $P=\{(0,0)\}$ or P is a left prime ideal of $R_1 \times R_2$.

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