

## **A Note on Exact Solution for Thermal Radiative Flow over a Stretching/Shrinking Sheet with Convective Boundary Condition**

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### **Abstract**

An analytical study of thermal radiation in the boundary layer flow through porous medium of an electrically conducting incompressible fluid over a stretching/shrinking sheet in the presence of convective boundary condition is presented. The flow is permeated by an externally applied magnetic field normal to the plane of flow. The equations governing the flow and heat transfer are reduced into a set of nonlinear ordinary differential equations and exact solutions are obtained. The effects of various parameters entering into the problem on the velocity and temperature distribution are discussed and depicted graphically. This study reveals that convective boundary condition results in temperature slip at the sheet and this temperature slip is significantly influenced by the Biot number.

**Keywords:** Convective boundary condition, thermal radiation, exact solution

### **Introduction**

The study of heat transport in the boundary layer is a key to the effectiveness of many thermal technologies. In particular, heat transport in laminar boundary layer flow over a stretching surface is of great importance in many industrial applications, such as extrusion of plastic sheets, glass blowing, paper production, drawing plastic films, cooling of metallic plates in cooling baths, and so forth. The investigation of drag and heat transport in such situations belongs to a different class of problems, distinguishing itself from the study of flows over a static surface. The boundary layer flow over a moving surface in a viscous fluid at rest was studied by Sakiadis [1], whose work was subsequently extended by many authors. After this, Crane [2] provided an analytic solution for steady flow over a two-dimensional linear stretching surface. Banks [3] studied the similarity solutions of the boundary layer equation for a stretching wall. Gupta and Gupta [4] investigated the heat and mass transfer on a stretching sheet with suction and blowing. Liao and Pop [5] studied the explicit analytic solution of the boundary layer equation. Recently, Hussain and Ahmad [6] explored the numerical solution of magnetohydrodynamic (MHD) flow over a stretching surface.

The radiative heat transfer with convective boundary condition has a lot of applications in many engineering areas, especially in space technology and processes occurring at high temperatures. Thermal design for devices operating in outer space, the design of engines, and combustion chambers to operate at increased temperature to raise thermal efficiency, developments for the utilization of solar energy, various manufacturing processes including growth of translucent crystals, semiconductor processing and forming, and tempering glass are some applications of radiative heat transfer. In light of these applications, recently, Hussain *et al.* [7] studied the effects of radiation on free convection flow of fluid with variable viscosity from a porous vertical plate by applying the Rosseland approximation. Raptis *et al.* [8] analyzed the MHD asymmetric flow of an electrically conducting fluid past a semi-infinite stationary plate in the presence of

radiation. The effects of thermal radiation and heat transfer over an unsteady stretching surface in a porous medium in the presence of a heat source or sink are studied by Elbahsbeshy and Emam [9]. On the other hand, the heat transfer problem for boundary layer flow concerning convective boundary condition was investigated by Aziz [10]. Makinde and Aziz [11] studied the MHD mixed convection from a vertical plate embedded in a porous medium. Yao *et al.* [12] investigated heat transfer of a generalized stretching/shrinking wall problem with convective boundary condition. Some other important work related to the radiation phenomenon is presented by Hsiao [13, 14]. The aim of the present paper is to report closed form exact analytic solutions for thermal radiative slip flow over a stretching/shrinking sheet in the presence of convective boundary condition.

### Governing problem and exact solution

Let us consider the steady two-dimensional flow of an incompressible and electrically conducting fluid through porous medium on a continuously stretching/shrinking porous sheet. We take the  $x$ -axis along the sheet and the  $y$ -axis being normal to it. A magnetic induction of strength  $B_0$  is applied perpendicular to the stretching/shrinking sheet. It is assumed that the velocity of the stretching sheet is  $u_w(x) = bx$ , where  $b$  is a positive constant. It is further assumed that constant mass transfer velocity is  $v_w$  with  $v_w < 0$  for mass suction and  $v_w > 0$  for mass injection, respectively. The sheet surface temperature is maintained by convective heat transfer at a certain value  $T_w$  which is to be determined later with ambient temperature  $T_\infty$ . Further, the radiative heat flux in the  $x$ -direction is considered negligible in comparison to that in the  $y$ -direction. Under these assumptions, and neglecting the induced magnetic field by assuming the magnetic Reynolds number to be very small, the governing boundary layer equations are;

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} - \left( \frac{\sigma B_0^2}{\rho} + \frac{\nu \phi}{k_1} \right) u, \quad (2)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{k}{\rho c_p} \frac{\partial^2 T}{\partial y^2} - \frac{1}{\rho c_p} \frac{\partial q_r}{\partial y}, \quad (3)$$

with  $q_r = -\frac{16\sigma^*}{3k^*} T_\infty^3 \frac{\partial T}{\partial y}$ , and the corresponding slip boundary conditions are;

$$u = bx + L \frac{\partial u}{\partial y}, \quad v = v_w, \quad -k \frac{\partial T}{\partial y} = h(T_f - T_w) \quad \text{at } y = 0, \quad (4)$$

$$u \rightarrow 0, \quad T \rightarrow T_\infty \quad \text{as } y \rightarrow \infty. \quad (5)$$

In the above equations,  $u$  and  $v$  are the velocity components along the  $x$ - and  $y$ -directions respectively,  $\nu$  the kinematic viscosity,  $\sigma$  the electrical conductivity,  $\phi$  the porosity of the medium,  $k_1$  the permeability of porous medium,  $\rho$  the fluid density,  $L$  the slip parameter having dimension of length,  $T$  the temperature,  $c_p$  the specific heat at constant pressure,  $k$  the thermal conductivity,  $q_r$  the

radiative heat flux,  $\sigma^*$  the Stefan-Boltzmann constant,  $k^*$  the mean absorption coefficient,  $h$  the convective heat transfer coefficient, and  $T_f$  the convective fluid temperature below the moving sheet.

We introduce the following similarity variables

$$\eta = y\sqrt{\frac{c}{\nu}}, \quad u = cx f'(\eta), \quad v = -\sqrt{c\nu} f(\eta), \quad \theta(\eta) = \frac{T - T_\infty}{T_f - T_\infty}, \quad (6)$$

where prime denotes differentiation with respect to  $\eta$ , and  $c$  is a positive constant. In view of the above similarity transformations, the continuity Eq. (1) is identically satisfied, and Eqs. (2) - (5) now become;

$$f''' + ff'' - f'^2 - \left(M + \frac{1}{K}\right) f' = 0, \quad (7)$$

$$\theta'' + \frac{3Pr}{3 + 4R} f \theta' = 0, \quad (8)$$

$$f(\eta) = s, \quad f'(\eta) = \alpha + \alpha_1 f''(\eta), \quad \theta'(\eta) = -\gamma[1 - \theta(\eta)] \quad \text{at } \eta = 0, \quad (9)$$

$$f'(\eta) = 0, \quad \theta(\eta) = 0 \quad \text{as } \eta \rightarrow \infty. \quad (10)$$

where  $M = \frac{\sigma B_0^2}{\rho c}$  is the magnetic parameter,  $R = \frac{4\sigma^* T_\infty^3}{kk^*}$  the radiation parameter,  $\frac{1}{K} = \frac{\nu\phi}{k_1}$  the permeability parameter,  $Pr = \frac{\rho\nu c_p}{k}$  the Prandtl number, and  $\gamma = \frac{h}{k} \sqrt{\frac{\nu}{c}}$  the Biot number. Also,  $s = -\frac{v_w}{\sqrt{c\nu}}$  is the mass transfer parameter, with  $s > 0$  for mass suction and  $s < 0$  for mass injection, respectively. Further,  $\alpha = \frac{b}{c}$  is the stretching/shrinking parameter, with  $\alpha > 0$  for wall stretching and  $\alpha < 0$  for wall shrinking, respectively, and  $\alpha_1$  the velocity slip parameter, with  $\alpha_1 = L\sqrt{\frac{c}{\nu}}$ .

The physical quantities of interest include the skin friction coefficient,  $C_f$ , and the Nusselt number,  $N_u$ , defined as;

$$C_f = \frac{2\sqrt{c\nu}}{\alpha x} f''(0), \quad N_u = -\sqrt{\frac{cx^2}{\nu}} \theta'(0). \quad (11)$$

It is observed that Eq. (7), together with boundary conditions (9) and (10), admits an exact analytical solution of the form;

$$f(\eta) = s + \frac{\alpha}{\beta(1 + \alpha_1\beta)} (1 - e^{-\beta\eta}), \quad (12)$$

where  $\beta$  is the root of following third order algebraic equation;

$$K\alpha_1\beta^3 + K(1 - s\alpha_1)\beta^2 - (KM\alpha_1 + \alpha_1 + Ks)\beta - (K\alpha + KM + 1) = 0. \quad (13)$$

The roots of Eq. (13) are given by;

$$\beta_i = -\frac{1}{3a_1} \left( a_2 + t_i C + \frac{\Delta_0}{t_i C} \right), \quad i = 1, 2, 3,$$

where  $t_1 = 1$ ,  $t_2 = \frac{-1+i\sqrt{3}}{2}$ ,  $t_3 = \frac{-1-i\sqrt{3}}{2}$ ,  $C = \sqrt[3]{\frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}}$  with  $\Delta_0 = a_2^2 - 3a_1a_3$ ,  $\Delta_1 = 2a_2^3 - 9a_1a_2a_3 + 27a_1^2a_4$ ,  $a_1 = K\alpha_1$ ,  $a_2 = K(1 - s\alpha_1)$ ,  $a_3 = -(KM\alpha_1 + \alpha_1 + Ks)$ ,  $a_4 = -(K\alpha + KM + 1)$ . Based on the flow configuration, only positive real root  $\beta$  corresponds to a physically feasible solution.

Using solution (12) in Eq. (6), the velocity components are obtained in the form;

$$u = \frac{bx}{1 + \alpha_1\beta} e^{-\beta\eta} \quad \text{and} \quad v = -\sqrt{cv} \left[ s + \frac{\alpha}{\beta(1 + \alpha_1\beta)} (1 - e^{-\beta\eta}) \right]. \quad (14)$$

In order to find solution of Eq. (8), we introduce a new variable;  $\varepsilon = \frac{3Pr}{(3+4R)\beta^2} e^{-\beta\eta}$ . Consequently, Eq. (8) reduces to;

$$\varepsilon \frac{d^2\theta}{d\varepsilon^2} + \left( 1 - \frac{3Pr(\alpha + s\beta(1 + \alpha_1\beta))}{(3 + 4R)(1 + \alpha_1\beta)\beta^2} + \frac{\alpha}{1 + \alpha_1\beta} \varepsilon \right) \frac{d\theta}{d\varepsilon} = 0, \quad (15)$$

with the corresponding boundary conditions;

$$\theta' \left( \frac{3Pr}{(3 + 4R)\beta^2} \right) = -\gamma \left[ 1 - \theta \left( \frac{3Pr}{(3 + 4R)\beta^2} \right) \right], \quad \theta(0) = 0. \quad (16)$$

The solution of Eq. (15) under boundary condition (16) is given by;

$$\theta(\varepsilon) = c_1 + c_2 \Gamma \left( \frac{3Pr(\alpha + s\beta(1 + \alpha_1\beta))}{(3 + 4R)(1 + \alpha_1\beta)\beta^2}, \frac{\alpha}{1 + \alpha_1\beta} \varepsilon \right), \quad (17)$$

where  $\Gamma(a, x)$  is the incomplete Gamma function. Therefore;

$$\begin{aligned}
 \theta(\eta) = & \left[ \Gamma\left(\frac{3 \operatorname{Pr}(\alpha + s\beta(1 + \alpha_1\beta))}{(3 + 4R)(1 + \alpha_1\beta)\beta^2}, 0\right) \right. \\
 & \left. - \Gamma\left(\frac{3 \operatorname{Pr}(\alpha + s\beta(1 + \alpha_1\beta))}{(3 + 4R)(1 + \alpha_1\beta)\beta^2}, \frac{3\alpha \operatorname{Pr}}{(3 + 4R)(1 + \alpha_1\beta)\beta^2} e^{-\beta\eta}\right) \right] \\
 & \left[ \Gamma\left(\frac{3 \operatorname{Pr}(\alpha + s\beta(1 + \alpha_1\beta))}{(3 + 4R)(1 + \alpha_1\beta)\beta^2}, 0\right) - \Gamma\left(\frac{3 \operatorname{Pr}(\alpha + s\beta(1 + \alpha_1\beta))}{(3 + 4R)(1 + \alpha_1\beta)\beta^2}, \frac{3\alpha \operatorname{Pr}}{(3 + 4R)(1 + \alpha_1\beta)\beta^2}\right) \right] \\
 & + \frac{\beta}{\gamma} e^{-\frac{3\alpha \operatorname{Pr}}{(3 + 4R)(1 + \alpha_1\beta)}} \left[ \frac{3\alpha \operatorname{Pr}}{(3 + 4R)(1 + \alpha_1\beta)\beta^2} \right]
 \end{aligned} \tag{18}$$

Now, it is important to remark that, for  $\frac{1}{k} = 0$ ,  $\alpha = 1$ , Eq. (12) can be reduced to the similar solution as obtained by Fang *et al.* [15], and for  $\alpha = -1$ ,  $\frac{1}{k} = \alpha_1 = 0$ , Eq. (12) reduces to the solution by Fang and Zhang [16]. Furthermore, for  $M = 0$ ,  $\frac{1}{k} = 0$ , Eq. (12) reduces to the solution by Wang [17], and for  $M = 0$ ,  $\frac{1}{k} = 0$ ,  $s = 0$ , Eq. (12) reduces to those by Andersson [18]. For  $M = \frac{1}{k} = \alpha_1 = R = 0$ , Eqs. (12) and (18) can be reduced to the similar solution obtained by Yao *et al.* [12].

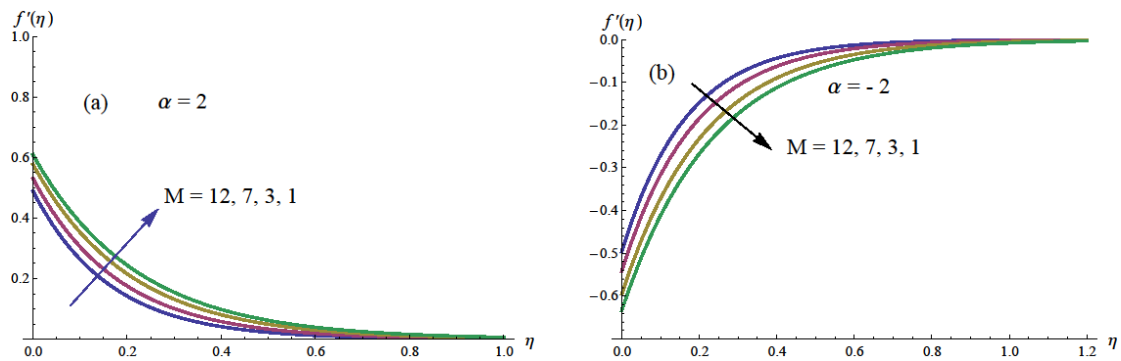
### Results and discussion

This section presents graphical representations of the velocity and temperature fields considered in the present study. Some of the qualitatively interesting results are presented through **Figures 1 - 8**. We now proceed with the discussion and results.

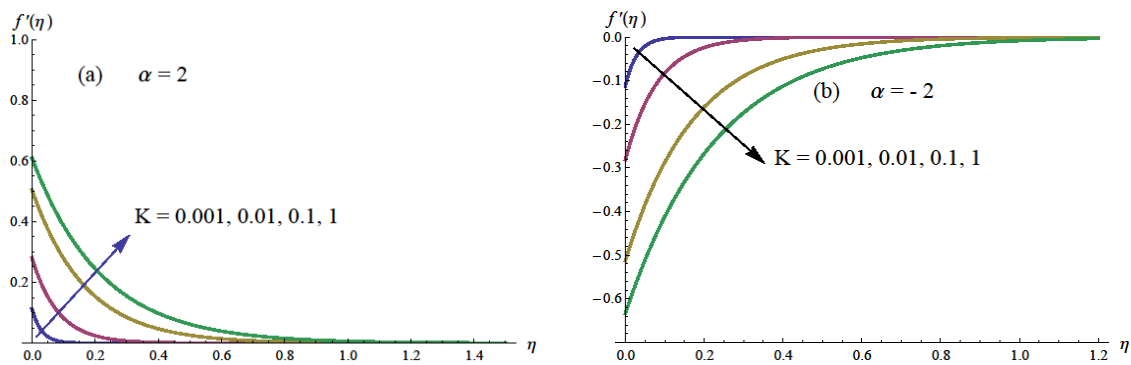
**Figures 1a** and **1b** are plotted to see the effects of the magnetic parameter  $M$  on the longitudinal velocity for both stretching and shrinking sheets. It is observed that a steady decrease in longitudinal velocity accompanies a rise in  $M$ , with all profiles tending asymptotically to the horizontal axis. Further, the longitudinal velocity is observed to be a maximum at the wall. The effects of the permeability parameter  $K$  on the longitudinal velocity for the cases of stretching and shrinking sheets are depicted in **Figures 2a** and **2b**. Qualitatively, we observe a quite opposite effect of the permeability parameter to that of the magnetic parameter. The effect of the mass transfer parameter  $s$  on the longitudinal velocity is presented in **Figure 3**. It can be seen that the longitudinal velocity increases with the decrease of mass transfer parameter. The effects of the slip parameter  $\alpha_1$  on the longitudinal velocity are shown in **Figures 4a** and **4b** for the case of stretching and shrinking sheets. It is noticed that longitudinal velocity decreases by increasing the values of slip parameter for both stretching and shrinking sheets.

**Figures 5a** and **5b** depict the temperature profile  $\theta(\eta)$  for different values of Biot number  $\gamma$ . From these figures, it can be noticed that the wall temperature increases for both wall stretching and shrinking as the value of Biot number  $\gamma$  increases. However, for the same Biot number, the wall shrinking case results in a slightly higher wall temperature than the wall stretching case. Further, it is interesting to notice that, as the Biot number tends to infinity, the convective boundary condition will reduce to the prescribed wall temperature. In **Figures 6a** and **6b**, the effects of radiation parameter  $R$  on the temperature profile for both wall stretching and shrinking cases are observed. These figures show marked increase in the temperature distribution, with an increase in  $R$  for both wall stretching and shrinking cases. The reason for this trend can be explained by the fact that higher values of the radiation parameter  $R$  leads to a decrease in the Rosseland mean absorption coefficient  $k^*$  for a given  $k$  and  $T_\infty$ . Thus, we can deduce from Eq. (3) that the

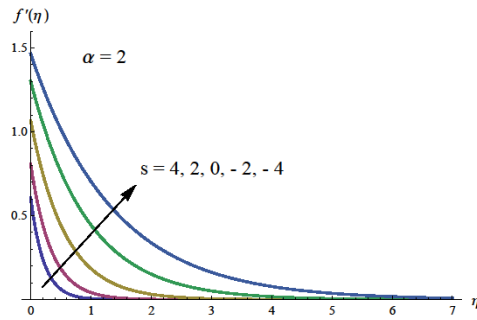
divergence of the radiative heat flux  $\frac{\partial q_r}{\partial y}$  rises as  $k^*$  decreases; consequently, the fluid temperature increases. **Figures 7a** and **7b** are plotted for different values of the Prandtl number  $Pr$  for both stretching and shrinking sheets. These graphs reveal that the increase in the Prandtl number  $Pr$  yields a decrease in temperature distribution. This is due to the fact that an increase in Prandtl number means a slow rate of thermal diffusion. Further, it is observed that the thermal boundary layer is thicker for mass injection as compared to mass suction. In **Figure 8**, the influence of mass transfer parameter  $s$  on the temperature distribution is investigated. In the case of mass suction, there is a rapid decrease in temperature and thermal boundary layer thickness. On the other hand, for the case of mass injection, a rapid increase in temperature distribution and thermal boundary layer thickness may be observed. A comparison of  $f'(\eta)$  with the previous literature in the limiting case is presented through **Table 1**. Quantification of these results shows that the agreement is very good.



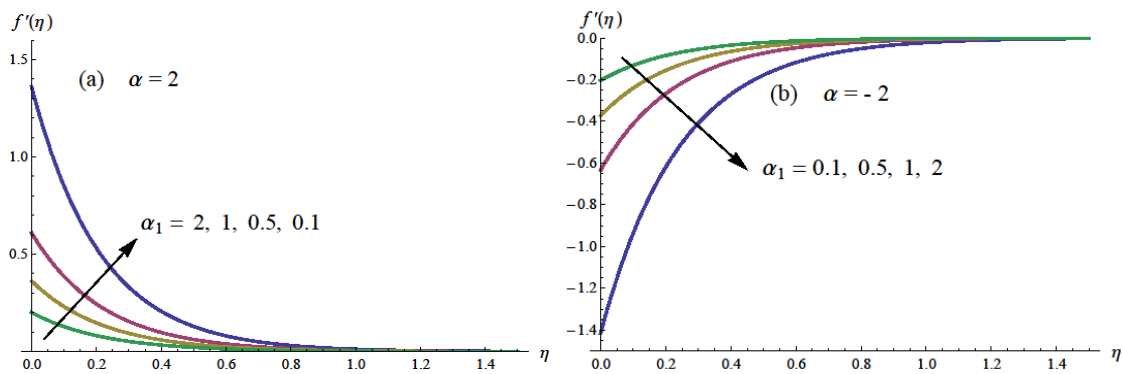
**Figure 1** Effect of magnetic parameter  $M$  on  $f'(\eta)$  for (a) stretching and (b) shrinking cases, when  $K = 1$ ,  $\alpha_1 = 0.5$ , and  $s = 4$  are fixed.



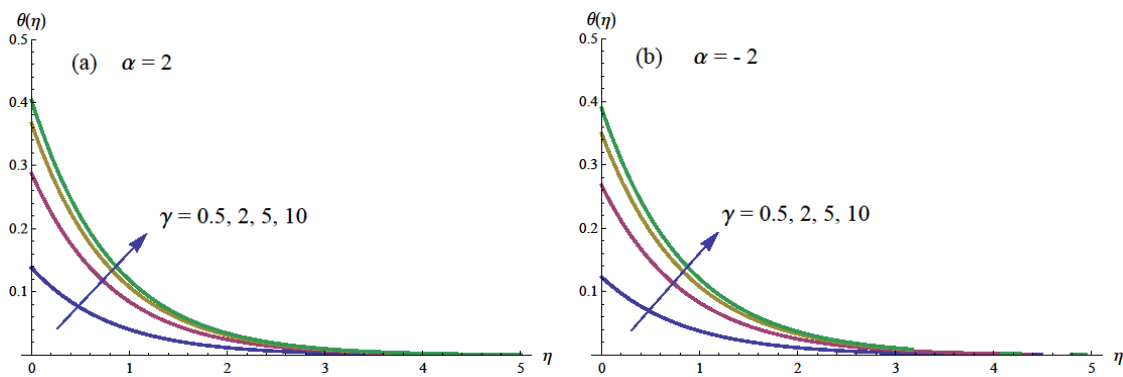
**Figure 2** Effect of permeability parameter  $K$  on  $f'(\eta)$  for (a) stretching and (b) shrinking cases, when  $M = 1$ ,  $\alpha_1 = 0.5$ , and  $s = 4$  are fixed.



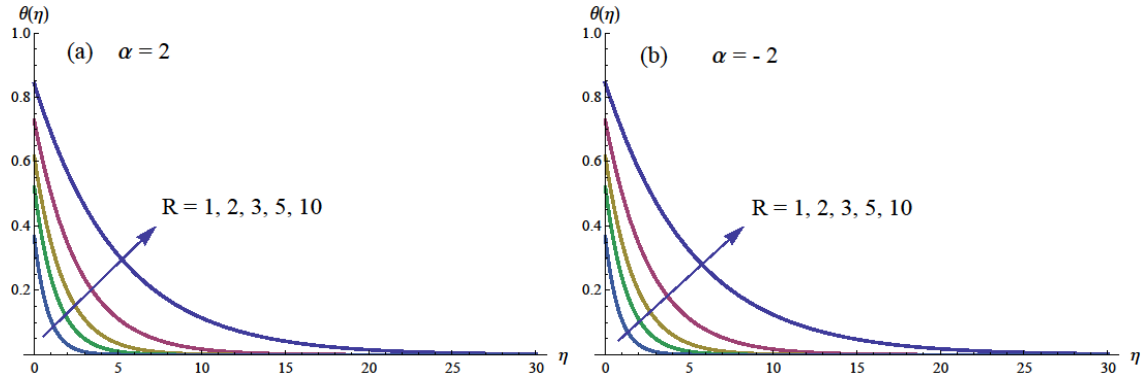
**Figure 3** Effect of mass transfer parameter  $s$  on  $f'(\eta)$ , when  $M = K = 1$  and  $\alpha_1 = 0.5$  are fixed.



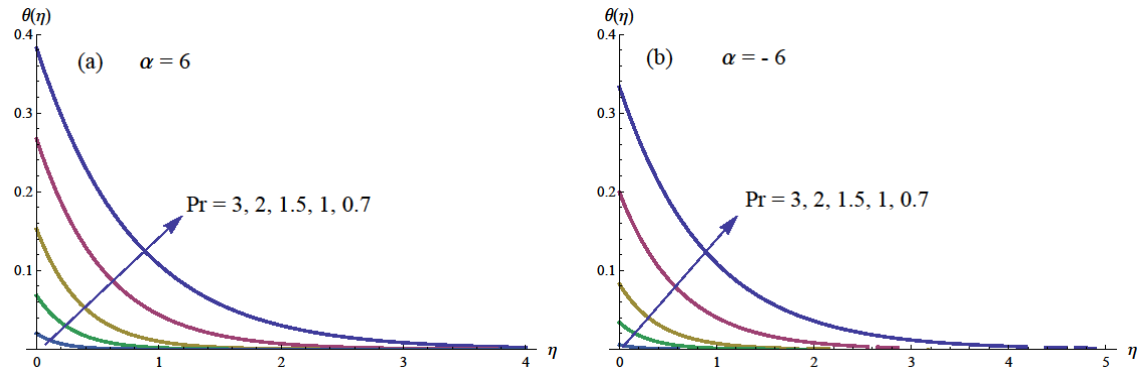
**Figure 4** Effect of slip parameter  $\alpha_1$  on  $f'(\eta)$  for (a) stretching and (b) shrinking cases, when  $M = K = 1$  and  $s = 4$  are fixed.



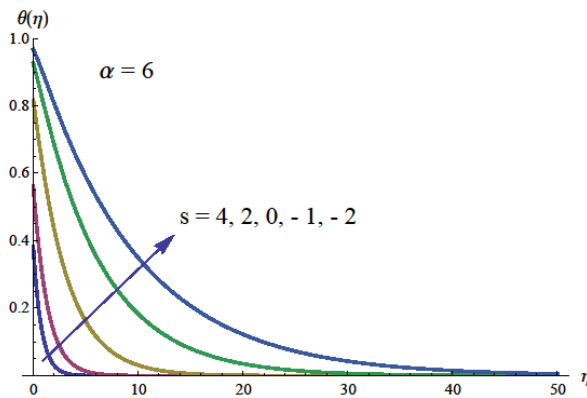
**Figure 5** Effect of Biot number  $\gamma$  on  $\theta(\eta)$  for (a) shrinking and (b) stretching cases, when  $M = K = R = 1$ ,  $s = 4$ , and  $Pr = 0.7$  are fixed.



**Figure 6** Effect of radiation parameter  $R$  on  $\theta(\eta)$  for (a) shrinking and (b) stretching cases, when  $M = K = 1$ ,  $s = 4$ ,  $Pr = 0.7$ , and  $\gamma = 5$  are fixed.



**Figure 7** Effect of Prandtl number  $Pr$  on  $\theta(\eta)$  for (a) stretching and (b) shrinking cases, when  $M = K = R = 1$ ,  $s = 6$ , and  $\gamma = 5$  are fixed.



**Figure 8** Effect of mass transfer parameter  $s$  on  $\theta(\eta)$  for stretching case.



**Table 1** A comparison of  $f'(\eta)$  for various values of  $M$  when  $s = \alpha = 1$  and  $\frac{1}{K} = \alpha_1 = 0$  are fixed.

	$\eta$	Present result $f'(\eta)$	Hussain & Ahmad [6] $f'(\eta)$
$M = 0$	0.0	1.000000	1.000000
	1	0.36787	0.366339
	2	0.13533	0.132009
	3	0.04978	0.044872
	4	0.01831	0.012267
	5	0.00673	0.00000
$M = 2$	0.0	1.000000	1.000000
	1	0.176921	0.176925
	2	0.031301	0.031308
	3	0.005537	0.005523
	4	0.000979	0.000926
	5	0.000173	0.00000

### Conclusions

In this paper, an analysis has been carried out to study the MHD flow and heat transfer characteristics with convective boundary condition at the wall. The flow of a viscous fluid from a permeable stretching/shrinking sheet through porous medium in the presence of thermal radiation was considered. The basic boundary layer equations have been converted into a set of non-linear ordinary differential equations by similarity transformations and solved analytically. In this study, it was concluded that:

- 1) The velocity profile decreases with the increase of magnetic, mass transfer parameter, and slip parameter, while it increases with the increase of permeability parameter, for both wall stretching and shrinking cases.
- 2) The wall temperature increases with the increase of Biot number and radiation parameter for both wall stretching and shrinking cases.
- 3) A remarkable decrease is noticed in the temperature distribution with the increase in Prandtl number for both wall stretching and shrinking cases.
- 4) It was interesting to notice that the convective boundary condition results in temperature slip relative to the convective fluid temperature at the boundary.

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### References

- [1] BC Sakiadis. Boundary-layer behavior on continuous surface: I. Boundary layer equations for two-dimensional and axisymmetric flow. *AIChE J.* 1961; **7**, 26-8.
- [2] LJ Crane. Flow past a stretching plate. *Z. Angew. Math. Phys.* 1970; **21**, 645-7.
- [3] WHH Banks. Similarity solutions of the boundary layer equations for a stretching wall. *J. Mecan. Theoret. Appl.* 1983; **2**, 375-92.
- [4] AS Gupta and PS Gupta. Heat and mass transfer on a stretching sheet with suction or blowing. *Cand. J. Chem. Eng.* 1977; **55**, 744-6.
- [5] SJ Liao and I Pop. Explicit analytic solution for similarity boundary layer equation. *Int. J. Heat Mass Transf.* 2004; **47**, 75-85.
- [6] S Hussain and F Ahmad. Numerical solution of MHD flow over a stretching sheet. *J. Basic Appl. Sci. Res.* 2014; **4**, 98-102.
- [7] MA Hussain, MA Alim and DAS Raees. The effect of radiation on free convection from a porous vertical plate. *Int. J. Heat Mass Transf.* 1999; **42**, 181-91.
- [8] A Raptis, C Perdikis and HS Thakhar. Effect of thermal radiation on MHD flow. *Appl. Math. Comput.* 2004; **153**, 645-9.
- [9] EMA Elbashbeshy and TG Ema. Effects of thermal radiation and heat transfer over an unsteady stretching surface embedded in a porous medium in the presence of heat source or sink. *Therm. Sci.* 2011; **15**, 477-85.
- [10] A Aziz. A similarity solution for laminar thermal boundary layer over a flat plate with convective boundary condition. *Commun. Non-linear Sci. Num. Simulat.* 2009; **14**, 1046-8.
- [11] OD Makinde and A Aziz. MHD mixed convection from a vertical plate embedded in a porous medium with convective boundary condition. *Int. J. Therm. Sci.* 2010; **49**, 1813-20.
- [12] S Yao, T Fang and Y Zhong. Heat transfer of generalized stretching/shrinking wall problem with convective boundary condition. *Commun. Non-linear Sci. Num. Simulat.* 2011; **16**, 752-60.
- [13] KL Hsia. Energy conversion conjugate conduction-convection and radiation over non-linearly extrusion stretching sheet with physical multimedia effects. *Energy* 2013; **59**, 494-502.
- [14] KL Hsiao. Multimedia feature for unsteady fluid flow over a non-uniform heat source stretching sheet with magnetic radiation physical effects. *Appl. Math. Inf. Sci.* 2012; **6**, 59S-65S.
- [15] T Fang, J Zhang and S Yao. Slip MHD viscous flow over a stretching sheet-An exact solution. *Commun. Non-linear Sci. Num. Simulat.* 2009; **14**, 3731-7.
- [16] T Fang and J Zhang. Closed-form exact solution of MHD viscous flow over a shrinking sheet. *Commun. Non-linear Sci. Num. Simulat.* 2009; **14**, 2853-7.
- [17] CY Wang. Flow due to a stretching boundary with partial slip-an exact solution of the Navier-Stokes equations. *Chem. Eng. Sci.* 2002; **57**, 3745-7.
- [18] HI Andersson. Slip flow past a stretching surface. *Acta Mechanica* 2002; **158**, 121-5.