WALAILAK JOURNAL
http://wjst.wu.ac.th Applied Mathematics

Gödel Type Solution of an *f(R)* Theory of Gravitation

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Received: 10 October 2013, Revised: 30 October 2015, Accepted: 1 November 2015

Abstract

In this paper, we have studied higher order theory of gravity which is based on conformal non-invariance of gravitational waves. These waves are an inevitable consequence of Einstein theory which is non-conformally invariant unlike electromagnetic waves. In this paper, we have solved these field equations by considering time independent cylindrically symmetric space-time and demonstrated Gödel space-time as a particular case. We also compared its result to Einstein's field equation with cosmological constant.

Keywords: *f(R)* theory of gravity, Gödel metric, Ricci Curvature Tensor, Einstein field equation, Hilbert-Lagrangian

Introduction

Einstein's theory of gravity has 2 important predictions - energy carrying gravitational waves and black holes. Black holes demand a very strong gravitational field, the gravitational potential should approach the square of the velocity of light for their formation while gravitational waves exists even in the weak field approximation [1]. Gravitational waves are non-conformal unlike electromagnetic waves and hence, Einstein's theory needs modification to put these gravitational waves as par with electromagnetic waves [2-4]. There are numerous ways to modify Einstein's theory of general relativity. An attractive possibility is to consider Einstein general relativity as a particular case of a more fundamental theory. This is the underlying philosophy of what are referred to as *f(R)* theories [5,6]. The usual general relativity is recovered in the limit *f(R) = R*. While in the weak field limit, the theory should give the usual Newtonian gravity. At a cosmological scale there is almost complete freedom in the choice of *f(R)*. This leaves open the way to a wide range of models [7].

Because of the non-conformal invariance of gravitational waves, we have chosen *f(R)* as a polynomial in *R* of a finite number of terms without associating it with any other field except gravitation [8,9]. Therefore, we look the Lagrangian in the form;

$$\mathcal{L} = R + \sum_{2}^{N} a_n R^n \tag{1}$$

The value of $n = 0$ and $n = 1$ results in Hilbert Lagrangian. An application of variational principle to this action yields the following field equations [10];

$$G_{ik} - \sum_{n=2}^{N} \frac{(na_n)}{6} (l^2 R)^{(n-1)} [R_{ik} - \frac{1}{2n} g_{ik} R - \frac{n-1}{R} (R_{ik} - g_{ik} \Box R) - \frac{(n-1)(n-2)}{R^2} (R_i R_k - g_{ik} R_\alpha R^\alpha)] = \kappa T_{ik} \tag{2}$$

Here, $\Box$ represents the 4 dimensional D'Alembertian operator, a_n's are arbitrary constants pertaining the values for preserving conformal non-invariance of gravitational waves, l is a constant making the equations dimensionless and stands for the energy momentum tensor, responsible for the production of a gravitational potential. It can be seen that the conservation law holds for these field equations as in the case of Einstein's general relativity [3].

$$T_{ik} = (-g)^{\frac{1}{2}} \left(\frac{\delta L_s}{\delta g^{ik}} \right) \tag{3}$$

The gravitational field surrounding spherically symmetric mass [11] and plane wave solution [12] have been studied for this *f(R)* theory of gravity. The beginning of the modern studies of singularities of space-time in general relativity has its roots in the mind of Gödel [13]. He has studied the solution of Einstein's field equation with cosmological constant or a perfect fluid with its pressure equal to energy density and discussed various interesting properties [14]. Existence of closed time like curves and geodesic brought a new thinking [15-17]. Gödel's work has been generalized to include other matter sources such as the vector field, scalar field, spinor field and tachyon field [18]. Scientists have explained many new features of our spaces which are revealed by modern astrophysical data like Pioneer anomaly [19,20] and teleparallel gravity [21] using Gödel space-time. There are many arguments that indicate that the Gödel universe is not an actual universe but still it cannot be easily ignored in general relativity.

The Gödel universe is heterogeneous with constant matter distribution and has rotational symmetry. It is geodesically complete and it has neither a singularity nor a horizon [14]. The Gödel solution of Einstein's field equation is given by space-time;

$$ds^2 = a^2 \left(dt^2 - dr^2 + \frac{e^{2r}}{2} d\phi^2 - dz^2 + 2e^r \, dt \, d\phi \right) \tag{4}$$

where a is a constant, and represents the scale factor. The quadratic form makes it evident that signature is -2 everywhere.

In what follows, we start with a field equation in vacuum with cylindrically symmetric Gödel type metric and its geometrical parameters followed by the solution of the field equation. Then we derive a few of its particular cases concerning linearity and sign of the root. Next, we discuss the solution and gave a comparison of this result to be obtained by solving Einstein's field equation. We conclude with some important results of the article.

Field equation and space-time

The field Eq. (1) for vacuum reduces to;

$$G_{ik} - \sum_{n=2}^{N} \frac{(na_n)}{6} (l^2 R)^{(n-1)} [R_{ik} - \frac{1}{2n} g_{ik} R - \frac{n-1}{R}(R_{ik} - g_{ik} \Box R) - \frac{(n-1)(n-2)}{R^2}(R_i R_k - g_{ik} R_\alpha R^\alpha)] = 0 \tag{5}$$

Eq. (5) is reducible to an Einstein field equation in a vacuum that can be obtained by putting $n = 0$ and $n = 1$ therefore n begins from 2 onwards.

Eq. (5) reduces to;

$$R_{ik} - \frac{1}{2} g_{ik} R + 2 a_2 R \left[R_{ik} - \frac{1}{4} g_{ik} R - \frac{2}{R}\left(R_{ik} - g_{ik} \Box R \right) \right] = 0 \tag{6}$$

for $n = 2$ and to

$$R_{ik}-\frac{1}{2}g_{ik}R+2a_2R\left[R_{ik}-\frac{1}{4}g_{ik}R-\frac{2}{R}\left(R_{ik}-g_{ik}\Box R\right)\right]+3a_3R^2[R_{ik}-\frac{1}{6}g_{ik}R-\frac{6}{R}(R_{ik}-g_{ik}\Box R)$$
$$-\frac{2}{R^2}(R_iR_k-g_{ik}R_\alpha R^\alpha)]=0 \tag{7}$$

for n = 3.

We are assuming the space-time is cylindrically symmetric and time independent. The space-time is represented by the metric;

$$ds^2=(dt+A\left(r\right)d\phi)^2-B^2\left(r\right)d\phi^2-dr^2-dz^2 \tag{8}$$

of which the scalar curvature is given by;

$$R=\frac{A'^2-4BB''}{2B^2} \tag{9}$$

where $A = A(r)$ and $B = B(r)$.
Again, the Ricci curvature tensor of (8) is;

$$R_{11}=\frac{A'^2-2BB''}{2B^2} \tag{10}$$

$$R_{22}=\frac{1}{2}\left(\left(1-\frac{A^2}{B^2}\right)A'^2-\frac{2AA'B'}{B}+2AA''-2BB''\right) \tag{11}$$

$$R_{42}=\frac{AA'^2-BA'B'+B^2A''}{2B^2} \tag{12}$$

$$R_{44}=\frac{A'^2}{2B^2} \tag{13}$$

Solution of the field equation

The Ricci curvature tensor of the metric (8) has been defined in the previous section. Substituting the values of the fundamental tensor g_{ik}, Ricci curvature tensor R_{ik} and scalar curvature R from Eqs. (9) - (13) in Eq. (6), we get;

$$-\frac{1}{8B^4}[3a_2A'^2(A'^2-64B'^2)+16a_2B(16A'B'A''+3A'^2B''+16B'2B'')-2B^2(A'^2+32a_2A'A''')$$
$$+8a_2(4A''^2+7B''^2+16B'B''')+12a_2B^3B^{iv}]=0 \tag{14}$$

$$-\frac{1}{8B^4}\left(B^2-A^2\right)\left(4BB''-A'^2\right)\left(2B^2-a_2A'^2+4a_2BB''\right)+\frac{1}{2}\left(\left(1+\frac{A^2}{B^2}\right)A'^2-\frac{2AA'B'}{B}+2AA''-2BB''\right)$$

$$+\left(1+\frac{a_2\left(4BB''-A'^2\right)}{B^2}\right)+\frac{1}{B^4}[4a_2\left(A^2-B^2\right)\{A'^2\left(BB''-3B'^2\right)+BA'\left(4B'A''-BA'''\right)$$

$$+B\left(4B'^2B''-B\left(A''^2+2B''^2+4B'B'''\right)+2B^2B^{iv}\right)\}]=0 \tag{15}$$

$$\frac{1}{8B^4}[a_2A'^2\left(A'^2+96B'^2\right)-8a_2B\left(16A'B'A''+5A'^2B''+16B'^2B''\right)+B^2\left(-2\mathrm{A}'^2+32a_2A'A'''\right)$$
$$+16a_2(2A''^2+5B''^2+8B'B''')+8B^3(B''-8a_2B^{iv})]=0 \tag{16}$$

$$\frac{1}{8B^4}[4B\left(BA''-A'B'\right)\left(B^2-a_2A'^2+4a_2BB''\right)+A\{-a_2A'^2\left(5A'^2+96B'^2\right)$$
$$+8a_2B\left(16A'B'A''+7A'^2B''+16B'^2B''\right)+2B^2(3A'^2-16a_2A'A'''-8a_2(2A''^2+5B''^2+8B'B'''))$$
$$-8B^3(B''-8a_2B^{iv})\}]=0 \tag{17}$$

$$\frac{1}{8B^4}[-a_2A'^2\left(5A'^2+96B'^2\right)+8a_2B\{16A'B'A''+7A'^2B''+2B^2\{3\mathrm{A}'^2-16a_2A'A'''$$
$$-8a_2\left(2A''^2+5B''^2+8B'B'''\right)\}-8B^3\left(B''-8a_2B^{iv}\right)\}]=0 \tag{18}$$

From Eqs. (17) and (18), we get;

$$4B(BA''-A'B')[B^2+a_2(4BB''-A'^2)]=0 \tag{19}$$

and adding Eqs. (16) and (18);

$$4A'^2[B^2+a_2(4BB''-A'^2)]=0 \tag{20}$$

Similarly, if we solve Eqs. (14) and (15), we get;

$$\left[B^2+a_2\left(4BB''-A'^2\right)\right]=0 \tag{21}$$

which is equivalent to;

$$1+2a_2R=0 \tag{22}$$

Here it should be noted that the scalar curvature R of space time (8) becomes constant as it depends only upon the constant in this case. Again, it is to be seen that is the coefficient in the Lagrangian (1), first term in the summation so it cannot be zero for higher order theory.

If we consider $n = 3$, the field Eq. (7) in view of the values for the space-time (8) yields

$$B^2 + a_2\left(4BB'' - A'^2\right) + \frac{3a_3}{4B^2}\left(4BB'' - A'^2\right)^2 = 0 \tag{23}$$

or

$$1 + 2a_2 R + 3a_3 R^2 = 0 \tag{24}$$

For $n=4$ we obtained;

$$1 + 2a_2 R + 3a_3 R^2 + 4a_4 R^3 = 0 \tag{25}$$

Eqs. (22), (24) and (25) indicate that each higher order term of field Eq. (5) contributes an additional term in the respective condition obtained by the solving field equation for Gödel type space-time. We can generalize the condition for field Eq. (5) using an application of mathematical induction for n;

$$1 + 2a_2 R + 3a_3 R^2 + 4a_4 R^3 + 5a_5 R^3 + \ldots + na_n R^{(n-1)} = 0 \tag{26}$$

Here it should be mentioned that Eq. (26) gives a relationship between A and B in space-time (8) which is related to Eq. (9). The relationship depends upon the arbitrary constants. These arbitrary constants are introduced in the Hilbert Lagrangian (1) in order to nullify the additional gravitational potential in the gravitational wave equation;

$$\mu'' + \mu\left(n^2 - \frac{a''}{a}\right) = 0 \tag{27}$$

under the background of Friedmann-Robertson-Walker metric written in conformal form as;

$$ds^2 = a^2(\eta)\left(d\eta^2 - d\sigma^2\right) \tag{28}$$

Space-time for a higher order theory

The solution of field Eq. (5) for the space-time (8) imposed one condition on the metric and hence, leaves a variety of space-time as its solution. In particular, we can choose a linear relationship between A and B; $A = cB$ where c is a constant. Under this condition, the solution of Eqs. (14) - (18) yields the space-time;

$$ds^2 = dt^2 - dr^2 - \left(1 - c^2\right)B^2 d\phi^2 - dz^2 + 2cB\,dt\,d\phi \tag{29}$$

The value of c lies between -1 and 1. Here, the nature of B depends upon the roots of Eq. (26) which has (n-1) roots. Assuming that ξ is a root of Eq. (26). Since these are arbitrary constants we can choose them in a fashion that becomes real in order to avoid physical absurdness in space-time.
If $\xi < 0$, we obtain;

$$B(r) = c_1 \left[Sin\left(\sqrt{\alpha\beta}\, r + c_2 \right) \right]^{1/\beta} \tag{30}$$

and if $\xi > 0$;

$$B(r) = \left[c_1 e^{\sqrt{\alpha\beta}\, r} + c_2 e^{-\sqrt{\alpha\beta}\, r} \right]^{1/\beta} \tag{31}$$

where $\alpha = \frac{|\xi|}{4}$ and $\beta = \frac{4-c^2}{4}$.

In particular, if we use $c_1 = \frac{1}{\sqrt{2}}, c = \sqrt{2}$ and $c_2 = 0$, the space-time becomes;

$$ds^2 = a^2 \left(dt^2 - dr^2 + \frac{e^{2r}}{2} d\phi^2 - dz^2 + 2e^r \, dt \, d\phi \right) \tag{32}$$

under the condition;

$$1 + 4a_2 + 12a_3 + \ldots + na_n 2^{(n-1)} = 0 \tag{33}$$

Eq. (32) is the metric used by K. Gödel in [14].

Discussion

The solution of field Eq. (5) is given by Eq. (26) on solving recursively for $n = 2, 3, 4, \ldots$. It can be easily seen that each term in Eq. (26) corresponds to each term in Eq. (5) respectively. Since there are 2 variables involved in Eq. (26) (*R* is a function of *A* and *B*, see Eq. (9)), so in order to solve this, we have chosen a simple relationship between *A(r)* and *B(r)* and space-time is obtained for various circumstances.

The Einstein field equation;

$$R_{ik} - \frac{1}{2} g_{ik} R = \kappa T_{ik} \tag{34}$$

In the case of a vacuum $T_{ik} = 0$ Eq. (34) provides $R_{ik} = 0$. So, the scalar curvature becomes null of the space-time given by Eq. (8) in a vacuum. Whereas the Einstein field equation with cosmological constant λ in vacuum, gives a solution described by Eq. (32) after solving for the space-time given by Eq. (8) [14]. So, the result obtained by solving a higher order theory of gravity provides a similar result as it comes out by solving the Einstein field equation with cosmological constant. A similar result has been obtained in [22] for a spherically symmetric metric. Thus the result indicates that a higher order theory of gravity provides a more realistic universe than the Einstein field equation.

$$R_{ik} - \frac{1}{2} g_{ik} R + \lambda g_{ik} = 0 \tag{35}$$

The universe we live in is homogenous and isotropic on a large scale and is adequately described by Friedmann solution of Einstein field equation. The metric of an isotropic universe is conformally Euclidean. So it is interesting to note that space-time (8) will be homogenous if and only if;

$$A'(r) = k_1 B(r) \tag{36}$$

and

$$B''(r) = k_2 B(r) \tag{37}$$

Here k_1 and k_2 are constants. If $k_2 \geq k_1^2$, there is no breakdown of causality and hence Eq. (8) can describe the physical universe.

Conclusions

We have obtained the conditions under which a time independent cylindrically symmetric Gödel type space-time will be a solution of an *f(R)* theory of gravity motivated by conformal non-invariance of Gravitational Waves and demonstrated that Gödel space-time is a particular case of its solution. Gödel space-time permits solution of Einstein's field equation with a non-vanishing cosmological term. So, the result supports the existence of the cosmological constant in the Einstein field equation and that cosmological constant is inherent in the higher order theory of gravity.

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