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Parameter Estimation for the Length Biased Beta-Pareto Distribution and Application

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Abstract

The length biased beta-Pareto distribution is a flexible model for nonnegative data with a power law probability tail. The objective of this article is to derive a parameter estimation for the length biased beta-Pareto distribution by maximum likelihood estimation and method of moments which investigates its inverse transformation. The results of parameter estimation from a Monte Carlo simulation shows that the method of moments provide sample mean of estimated parameters that are close to the true parameter values more than the maximum likelihood estimation. These methods are illustrated with an application from the exceedances of flood peaks of the Wheaton River near Carcross in the Yukon Territory data set.

Keywords: Parameter estimation, length biased beta-Pareto distribution, inverse transformation, maximum likelihood estimation, method of moments

Introduction

The Pareto distribution is a versatile probability model, which was first presented in 1909 by Vilfredo Pareto, the Italian-born Swiss economist, sociologist and philosopher. This distribution is used to describe data of personal income and wealth. The various forms of the Pareto distribution are very versatile, and a variety of uncertainties can be usefully modelled by them. For instance, they arise as tractable 'life time' models in actuarial sciences, economics, finance and life testing. Moreover, the Pareto distribution is used to describe the occurrence of extreme weather. Censoring is common in many lifetime studies due to time and money limitations and other data collection restrictions. In failure-censored tests, a known number of observations in an ordered sample may be missing at either end or at both ends [1-6].

Many transformations and generalization of the Pareto distribution have been proposed in order to get more flexible models [7-10]. Different methods may be used to introduce a shape parameter to the Pareto model. One such class of distributions generated from a beta random variable which extends the original beta family of distributions with the incorporation of 2 additional parameters that control the skewness and the tail weight. This class of generalized distributions has been receiving considerable attention over the last 10 years, in particular after the works of Eugene *et al.* [11]. Consequently, a beta-Pareto (BP) distribution was introduced by Akinsete *et al.* [12]. It can be applied to flood data sets and it provides a significantly better fit than the Pareto, Weibull and generalized Pareto distributions.

The problem of determining a proper model for information of interest is an important consideration for data analysts. One major benefit of a weighted distribution theory proposed by Rao [13] is that it provides a unifying approach for these problems. It has the ability to fit skewed data which is not properly fitted by existing distributions. Many authors include the concept of the weighted distribution. For example, Patil and Rao [14] examine some general models leading to the weighted distribution with weight functions not necessarily bounded by unity. They also studied a length-biased (size biased) sampling with applications to wildlife populations and human families. Characteristics of many length-biased distributions, preservation stability results and comparisons for weighted and length-biased distributions were presented [15,16]. More recently, the probability weighted moment inequalities and variability orderings for weighted and unweighted reliability measures and related functions were presented by Oluyede [17]. Also, the stochastic comparisons and moment inequalities were given. Afterwards, Seenoi *et al*. [18] presented a length-biased exponentiated inverted Weibull distribution and an application for uncensored data of 24 observed on distance between cracks in a pipe dataset showed this distribution fits the data and it is more flexible than a 2-parameter Weibull distribution.

More recently, Nanuwong and Bodhisuwan [19] presented a length-biased beta-Pareto (LBBP) distribution. An application to a real data set is one among the 20 sets of Norwegian fire claims showing that it provides a significantly better fit than length-biased Pareto and BP distributions. This distribution seems to be a very competitive model for lifetime data. If a random variable X has the LBBP distribution, then the probability density function (pdf) and cumulative distribution function (cdf) of X are given, respectively;

$$f(x)=\frac{\gamma}{\theta B(\alpha,\beta-1/\gamma)}\left[1-\left(\frac{x}{\theta}\right)^{-\gamma}\right]^{\alpha-1}\left(\frac{x}{\theta}\right)^{-\gamma\beta}, x\geq\theta, \beta>1/\gamma; \alpha,\beta,\theta,\gamma>0, \tag{1}$$

where $B(a,b)=\dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ and;

$$F(x)=1-\frac{B(z;\beta-1/\gamma,\alpha)}{B(\alpha,\beta-1/\gamma)}, z=\left(\frac{x}{\theta}\right)^{-\gamma}. \tag{2}$$

Then the r^{th} moment of X is given by;

$$E(X^r)=\theta^r\frac{B(\alpha,\beta-(r+1)/\gamma)}{B(\alpha,\beta-1/\gamma)}=\theta^r\frac{\Gamma(\beta-(r+1)/\gamma)\Gamma(\alpha+\beta-1/\gamma)}{\Gamma(\alpha+\beta-(r+1)/\gamma)\Gamma(\beta-1/\gamma)}. \tag{3}$$

From Eq. (3), it is simple to deduce the mean and variance of X which are given, respectively, by;

$$E(X)=\theta\frac{\Gamma(\beta-2/\gamma)\Gamma(\alpha+\beta-1/\gamma)}{\Gamma(\alpha+\beta-2/\gamma)\Gamma(\beta-1/\gamma)}, \tag{4}$$

and

$$Var(X)=\theta^2\left\{\frac{\Gamma(\beta-3/\gamma)\Gamma(\alpha+\beta-1/\gamma)}{\Gamma(\alpha+\beta-3/\gamma)\Gamma(\beta-1/\gamma)}-\left[\frac{\Gamma(\beta-2/\gamma)\Gamma(\alpha+\beta-1/\gamma)}{\Gamma(\alpha+\beta-2/\gamma)\Gamma(\beta-1/\gamma)}\right]^2\right\}. \tag{5}$$

Moreover, the skewness and kurtosis of X are as follows, respectively;

$$Skewness(X)=T^{-3}\left\{\omega(\alpha,\beta,\gamma;4)+2\omega^3(\alpha,\beta,\gamma;2)-3\omega(\alpha,\beta,\gamma;3)\omega(\alpha,\beta,\gamma;2)\right\}, \tag{6}$$

and

$$Kurtosis(X) = T^{-4}\left\{\omega(\alpha,\beta,\gamma;5) - 3\omega^4(\alpha,\beta,\gamma;2) - 4\omega(\alpha,\beta,\gamma;4)\omega(\alpha,\beta,\gamma;2) + 6\omega(\alpha,\beta,\gamma;3)\omega^2(\alpha,\beta,\gamma;2)\right\}, \tag{7}$$

where $\omega(\alpha,\beta,\gamma;i) = \dfrac{\Gamma(\beta - i/\gamma)\Gamma(\alpha+\beta-1/\gamma)}{\Gamma(\alpha+\beta-i/\gamma)\Gamma(\beta-1/\gamma)}, \beta > i/\gamma, i \in I^+$ and $T = \sqrt{\omega(\alpha,\beta,\gamma;3) - \omega^2(\alpha,\beta,\gamma;2)}$.

The estimation for the parameters of a statistical distribution is one of the fundamental issues in statistics. If the phenomenon is to be modeled using a parametric model, it is necessary to assign values to the parameters. Choosing an appropriate estimator is an important task. Consequently, the first order of business should be the consideration of several optimal criterions for applying a data set and a given statistical model. The maximum likelihood estimation (MLE) is a method of estimating the parameters of statistical model when applied to a data set and given statistical model [20-24]. Aban *et al*. [25] discussed parameter estimation for the truncated Pareto distribution by developing the MLE for all parameters of a truncated Pareto distribution and proving the existence and uniqueness of the MLE under certain easy-to-check conditions. These are shown to hold with the probability approaching 1 as the sample size increases. Asymptotic normality is established for the estimator of the tail parameter, α . It is typical that for extreme upper values the truncated Pareto model gives the lowest estimate, and the Pareto model with truncated Pareto parameters gives the highest estimate.

The objectives of this article are as follows: to derive the parameter estimation of the LBBP distribution by using the MLE and method of moments (MM), to compare the efficiency of both parameter estimation methods for simulation study by using mean square error criteria, and to apply the LBBP distribution to real data sets by comparing the efficiency of the parameter estimation methods.

Materials and methods

In this article, the methods of study are as follows: we derive estimated parameters of the LBBP distribution by the MLE and MM to generate random variates of the LBBP for simulation study. Next, the simulation study is applied in order to compare, for some situations, numerical results under estimating parameters of the LBBP distribution with the MLE and MM by using the mean square error (MSE) criteria. In addition, we use a real data set by comparing the efficiencies of fitting distribution among the MLE and MM by using the Kolmogorov-Smirnov (K-S) test [26], Akaike information criterion (AIC) [27], and Bayesian information criterion (BIC) [28] to evaluate goodness of fit.

In this section, we present a simulation study for the estimation of parameters. Our objective here is to compare the true values of the LBBP$(\alpha,\beta,\theta,\gamma)$ in Eq. (1) and their estimates from the MLE and MM. In this study, the summary of the specified parameter set is eight cases as shown in **Table 1**. The mean, variance, skewness and kurtosis for a LBBP random variable are calculated by Eqs. (4) - (7), respectively. All cases in the Monte Carlo simulation study were then generated from the LBBP distribution with sample sizes of n = 15, 50, 200 and 500, respectively. A simulation study of the all possible cases mentioned above was performed via the R program version 3.1.2 [29].

Table 1 The summary of the specified parameter set in the simulation study of a LBBP random variable.

Cases	θ	α	β	γ	Mean	Variance	Skewness	Kurtosis
1	2.0	0.75	2.0	5.0	2.1973	0.0647	3.2558	23.2135
2	2.0	0.75	2.0	10.0	2.0879	0.0113	2.6575	14.6414
3	2.0	0.75	7.5	5.0	2.0428	0.0026	2.4881	12.7552
4	2.0	0.75	7.5	10.0	2.0209	0.0006	2.3931	11.7955
5	2.0	5.0	2.0	5.0	2.7642	0.2165	2.0084	11.6366
6	2.0	5.0	2.0	10.0	2.3303	0.0315	1.4673	7.0239
7	2.0	5.0	7.5	5.0	2.2356	0.0130	1.1224	5.1367
8	2.0	5.0	7.5	10.0	2.1126	0.0028	1.0196	4.6838

Generation of a LBBP random variate

We use an inverse transformation technique to generate random data from the LBBP distribution by setting $x = F^{-1}(U)$, where U is distributed as uniform on $(0,1)$, denoted as $U(0,1)$. The cdf of the LBBP distribution in Eq.(2) can be used to generate random data $x_i; i = 1,\ldots,n$, one can use the following steps;

1) Generate $U_i; i = 1,\ldots,n$ from $U(0,1)$.

2) Set $U_i = F(x_i)$, then $U_i = 1 - \dfrac{B(z_i;\beta - 1/\gamma,\alpha)}{B(\alpha,\beta - 1/\gamma)}$ and $(1-U_i)B(\alpha,\beta-1/\gamma) = B(z_i;\beta-1/\gamma,\alpha)$, where $z_i = \left(\dfrac{x_i}{\theta}\right)^{-\gamma}$.

3) Assigned $v_i = (1-U_i)B(\alpha,\beta-1/\gamma)$.

4) Set $z_i = Ibeta.Inv(v_i,\beta-1/\gamma,\alpha)$, when $Ibeta.Inv(v_i,\beta-1/\gamma,\alpha)$ is an inverted incomplete beta function in the R program.

5) Set $z_i = \left(\dfrac{x_i}{\theta}\right)^{-\gamma}$.

6) Then, $x_i = \theta \times z_i^{-1/\gamma}$.

Maximum likelihood estimation

The estimation of parameters for the LBBP distribution via the MLE will be discussed. Let $X_1,\ldots,X_n$ be a random sample from $X \sim \text{LBBP}(\alpha,\beta,\theta,\gamma)$ in Eq.(1) and let $\Theta = (\alpha,\beta,\theta,\gamma)^T$ be a vector of the model parameters. The likelihood function for Θ can be expressed as;

$$L(\Theta) = \prod_{j=1}^{n} \left\{ \frac{\gamma}{\theta B(\alpha,\beta-1/\gamma)} \left[1 - \left(\frac{x_j}{\theta}\right)^{-\gamma}\right]^{\alpha-1} \left(\frac{x_j}{\theta}\right)^{-\gamma\beta} \right\},$$

with the corresponding log-likelihood function;

$$\log L(\Theta) = n\log\gamma - n\log\theta - n\log\Gamma(\alpha) - n\log\Gamma(\beta - 1/\gamma) + n\log\Gamma(\alpha+\beta-1/\gamma) - \gamma\beta\sum_{j=1}^{n}\log\left(\frac{x_j}{\theta}\right)$$
$$+(\alpha-1)\sum_{j=1}^{n}\log\left[1-\left(\frac{x_j}{\theta}\right)^{-\gamma}\right]. \tag{8}$$

Note that since $x \geq \theta$, $\hat{\theta}$ is the first-order statistic. The first step for finding the optimal values of the parameters is obtained by differentiating in Eq. (8) with respect to α, β and γ. Thus, it gives rise to the following equations;

$$\frac{\partial}{\partial\alpha}\log L(\Theta) = n\left[\Psi(\alpha+\beta-1/\gamma) - \Psi(\alpha)\right] + \sum_{j=1}^{n}\log\left[1-\left(\frac{x_j}{\theta}\right)^{-\gamma}\right], \tag{9}$$

$$\frac{\partial}{\partial\beta}\log L(\Theta) = n\left[\Psi(\alpha+\beta-1/\gamma) - \Psi(\beta-1/\gamma)\right] - \gamma\sum_{j=1}^{n}\log\left(\frac{x_j}{\theta}\right), \tag{10}$$

and

$$\frac{\partial}{\partial\gamma}\log L(\Theta) = \frac{n}{\gamma} + \frac{n}{\gamma^2}\left[\Psi(\alpha+\beta-1/\gamma) - \Psi(\beta-1/\gamma)\right] - \sum_{j=1}^{n}\left\{\beta + (\alpha-1)\left[1-\left(\frac{x_j}{\theta}\right)^{\gamma}\right]^{-1}\right\}\log\left(\frac{x_j}{\theta}\right), \tag{11}$$

where $\Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ is a digamma function. The MLE for the parameters α, β and γ are denoted by $\hat{\alpha}_{MLE}, \hat{\beta}_{MLE}$ and $\hat{\gamma}_{MLE}$ respectively, and are obtained by solving Eqs. (9) - (11) iteratively to zero. The differential equations of the MLE solutions are not in explicit forms, and can be obtained by solving the resulting equations simultaneously using a numerical procedure. We use optim function in statistical package of R program [29].

Method of moments

By definition of the MM estimation, let $X_1, \ldots, X_n$ be a sample from a population with the LBBP distribution. The MM estimators are found by equating the first r sample moments to the corresponding r population moments, and solving the system of equations simultaneously. Given that $\hat{\theta}$ is the first-order statistic, by Eq.(3) the r^{th} sample moments (m_r) and the population moments μ_r', the parameter α, β and γ are defined as, respectively;

$$\mu_1'(\alpha,\beta,\gamma) = m_1 = E(X) = \theta\frac{\Gamma(\beta-2/\gamma)\Gamma(\alpha+\beta-1/\gamma)}{\Gamma(\alpha+\beta-2/\gamma)\Gamma(\beta-1/\gamma)}, \tag{12}$$

$$\mu_2'(\alpha,\beta,\gamma) = m_2 = E(X^2) = \theta^2\frac{\Gamma(\beta-3/\gamma)\Gamma(\alpha+\beta-1/\gamma)}{\Gamma(\alpha+\beta-3/\gamma)\Gamma(\beta-1/\gamma)}, \tag{13}$$

and

$$\mu_3'(\alpha,\beta,\gamma) = m_3 = E(X^3) = \theta^3\frac{\Gamma(\beta-4/\gamma)\Gamma(\alpha+\beta-1/\gamma)}{\Gamma(\alpha+\beta-4/\gamma)\Gamma(\beta-1/\gamma)}. \tag{14}$$

The MM estimators $\hat{\alpha}_{MM}, \hat{\beta}_{MM}$ and $\hat{\gamma}_{MM}$ are obtained by solving the following system of equations for parameters α, β and γ which are respectively given by Eqs. (12) - (14). Since they are not in simple form, a numerical method can be employed to obtain the expectations of the parameter estimators of the MM. We use the package gmm of the R program [29].

Criterion for a comparison of parameter estimation methods

Here, we compare the performance of the MLE and MM for the parameters of the LBBP distribution. The comparison for parameters, $\alpha, \beta, \theta, \gamma$, are based on the sample mean, variance and MSE of the estimates are calculated based on 1,000 Monte Carlo simulations, with the criterion defined by, respectively;

$$\text{Mean}\left(\hat{\theta}_i\right) = \bar{\hat{\theta}}_i = \frac{1}{m}\sum_{j=1}^{m}\hat{\theta}_{ij}, \tag{15}$$

$$\text{Variance}\left(\hat{\theta}_i\right) = \frac{1}{m-1}\sum_{j=1}^{m}\left(\hat{\theta}_{ij} - \bar{\hat{\theta}}_i\right)^2, \tag{16}$$

$$\text{MSE}\left(\hat{\theta}_i\right) = \frac{1}{m}\sum_{j=1}^{m}\left(\hat{\theta}_{ij} - \theta_i\right)^2, \tag{17}$$

where $\hat{\theta}_{ij}$ is the estimated value for the i^{th} parameter and the j^{th} replication,
θ_i is the true value of the i^{th} parameter,
m is the number of replications, $j = 1, 2, \ldots, 1000$.

Results and discussion

Monte Carlo simulation

Comparison of the parameter estimation for the LBBP distribution uses the MLE and MM in a Monte Carlo simulation. The sample mean, variance and MSE of the estimated parameters are calculated by Eqs. (15) - (17), respectively. The MSE is illustrated to compare the performances of the parameter estimation methods. The results are illustrated in **Tables 2 - 9**. We found that in most of the considered cases, the parameter estimates based on the MLE are closer to the true parameter values as the sample size increases. However, the MM provides the sample mean of parameter estimation that is close to the true parameter values for almost all sample sizes. Moreover, in cases 1 and 2 and with $n = 15, 50, 200$, we found that $\hat{\alpha}_{MLE}$ is smaller than the true parameter values, $\hat{\alpha}_{MM}$ is higher than the true parameter values, however $\hat{\alpha}_{MLE}$ and $\hat{\alpha}_{MM}$ are closer to the true parameter values when $n = 500$. In case 3, $\hat{\alpha}_{MM}$ is higher than the true parameter values for all sample sizes, on the other hand, $\hat{\alpha}_{MLE}$ is smaller than the true parameter values when $n = 15, 50$, nevertheless $\hat{\alpha}_{MLE}$ is higher than the true parameter values for $n = 200, 500$. In case 4, $\hat{\alpha}_{MM}$ is higher than the true parameter values for all sample sizes, whereas, $\hat{\alpha}_{MLE}$ is smaller than the true parameter values for $n = 15$, nevertheless, $\hat{\alpha}_{MLE}$ is higher than the true parameter values for $n = 50, 200, 500$. In cases 5-8, $\hat{\alpha}_{MM}$ and $\hat{\alpha}_{MLE}$ are smaller than the true parameter values for all sample sizes.

The sample mean of $\hat{\gamma}_{MM}$ is higher than the true parameter values in all sample sizes and all the specified parameter values in the simulation study. In contrast, $\hat{\gamma}_{MLE}$ is higher than the true parameter values for all sample sizes in cases 1 - 4, however, in cases 5 - 8, $\hat{\gamma}_{MLE}$ is higher than the true parameter values for $n = 15, 50$, nevertheless, $\hat{\alpha}_{MLE}$ is smaller than the true parameter values for $n = 200, 500$. We found that in all the specified parameter values $\hat{\beta}_{MM}$ provides the parameter estimation closest to the true parameter values for all sample sizes. The results for almost cases of $\hat{\beta}_{MLE}$ shows it is smaller than the true parameter values for $n=15$, nevertheless, $\hat{\beta}_{MLE}$ is higher than the true parameter values for $n = 50, 200, 500$, yet in case 3, $\hat{\beta}_{MLE}$ is smaller than the true parameter values for all sample sizes.

By using the MSE, the estimation of $\hat{\alpha}$ and $\hat{\beta}$ are shown in **Figures 1** - **2**. The results show that the MSE of $\hat{\gamma}_{MLE}$ is the highest in all sample sizes and all the specified parameter set in the simulation study, while the MSE of $\hat{\gamma}_{MM}$ is smaller than the MSE of $\hat{\gamma}_{MLE}$. The results show that, the MSE of $\hat{\alpha}_{MM}$ and $\hat{\alpha}_{MLE}$ are smaller for larger sample sizes for all cases in the study. In contrast, in cases 1 - 4, the MSE of $\hat{\alpha}_{MLE}$ is smaller than the MSE of $\hat{\alpha}_{MM}$, nevertheless, in the other cases, the MSE of $\hat{\alpha}_{MM}$ is smaller than the MSE of $\hat{\alpha}_{MLE}$. The MSE of $\hat{\beta}_{MM}$ is smaller for larger sample size in all cases. In case 3, the MSE of $\hat{\beta}_{MLE}$ is smaller for larger sample sizes, yet in cases 4 - 8, the MSE of $\hat{\beta}_{MLE}$ is higher for larger sample sizes. Moreover, the MSE of $\hat{\beta}_{MM}$ is smaller than the MSE of $\hat{\beta}_{MLE}$ for all cases in the study. Additionally, the results for almost every case of the variance for all parameter estimates are smaller as the sample size increases for both parameter estimations.

Table 2 Parameter estimates of a LBBP random variable with α = 0.75, β = 2, γ = 5 and θ = 2.

n	$\hat{\theta}$	Parameter	MLE			MM		
			Estimate	Variance	MSE	Estimate	Variance	MSE
15	2.0069	$\hat{\alpha}$	0.3811	0.0118	0.1478	1.1474	0.2501	0.4078
		$\hat{\beta}$	1.1979	11.3326	11.9646	2.8886	1.8244	2.6122
		$\hat{\gamma}$	286.4707	98,478.1479	177,605.4021	5.3209	1.6038	1.7051
50	2.0013	$\hat{\alpha}$	0.6098	0.0184	0.0381	0.9074	0.0664	0.0912
		$\hat{\beta}$	2.2601	14.6648	14.7178	2.2884	0.4249	0.5077
		$\hat{\gamma}$	132.8032	58,704.6341	74,979.5809	5.1517	0.6460	0.6684
200	2.0002	$\hat{\alpha}$	0.7231	0.0058	0.0065	0.8083	0.0231	0.0264
		$\hat{\beta}$	2.7920	9.6677	10.2853	2.0712	0.1218	0.1268
		$\hat{\gamma}$	30.4044	3,487.1094	4,129.0034	5.1266	0.2917	0.3075
500	2.0001	$\hat{\alpha}$	0.7467	0.0021	0.0021	0.7800	0.0123	0.0132
		$\hat{\beta}$	2.6999	6.2021	6.6857	2.0085	0.0416	0.0417
		$\hat{\gamma}$	14.4605	435.1959	524.2620	5.1274	0.0890	0.1051

Table 3 Parameter estimates of a LBBP random variable $\alpha = 0.75$, $\beta = 2$, $\gamma = 10$ and $\theta = 2$.

n	$\hat{\theta}$	Parameter	MLE			MM		
			Estimate	Variance	MSE	Estimate	Variance	MSE
15	2.0030	$\hat{\alpha}$	0.4449	0.0143	0.1074	0.9777	0.1769	0.2285
		$\hat{\beta}$	1.6146	15.5075	15.6405	2.6420	1.5451	1.9557
		$\hat{\gamma}$	409.0856	265,460.9510	424,464.8118	10.3072	2.5756	2.6674
50	2.0007	$\hat{\alpha}$	0.6614	0.0167	0.0245	0.8466	0.0569	0.0662
		$\hat{\beta}$	2.9738	18.4955	19.4252	2.2067	0.3423	0.3847
		$\hat{\gamma}$	153.0803	68,804.8481	89,208.0089	10.2478	0.9365	0.9970
200	2.0001	$\hat{\alpha}$	0.7413	0.0049	0.0049	0.7866	0.0174	0.0188
		$\hat{\beta}$	3.1957	11.1742	12.5928	2.0576	0.0768	0.0801
		$\hat{\gamma}$	41.4075	3,928.6091	4,911.1088	10.1495	0.4801	0.5019
500	2.0000	$\hat{\alpha}$	0.7572	0.0017	0.0018	0.7652	0.0076	0.0079
		$\hat{\beta}$	3.2631	7.8223	9.4098	2.0268	0.0304	0.0311
		$\hat{\gamma}$	22.5306	887.5644	1,043.6923	10.0510	0.4082	0.4104

Table 4 Parameter estimates of a LBBP random variable with $\alpha = 0.75$, $\beta = 7.5$, $\gamma = 5$ and $\theta = 2$.

n	$\hat{\theta}$	Parameter	MLE			MM		
			Estimate	Variance	MSE	Estimate	Variance	MSE
15	2.0015	$\hat{\alpha}$	0.5382	0.0207	0.0656	0.8524	0.1289	0.1393
		$\hat{\beta}$	3.0583	67.9480	87.6091	8.0404	2.2704	2.5601
		$\hat{\gamma}$	479.8515	449,183.7650	674,218.5599	5.6889	3.7584	4.2292
50	2.0003	$\hat{\alpha}$	0.7004	0.0159	0.0183	0.8343	0.0458	0.0528
		$\hat{\beta}$	3.2267	20.6001	38.8407	7.8744	0.9644	1.1036
		$\hat{\gamma}$	175.1441	72,958.6693	101,834.7332	5.3532	1.6149	1.7380
200	2.0001	$\hat{\alpha}$	0.7599	0.0037	0.0038	0.7825	0.0112	0.0123
		$\hat{\beta}$	3.8390	12.5783	25.9689	7.7379	0.3259	0.3822
		$\hat{\gamma}$	53.8274	7,008.0234	9,385.1345	5.0654	0.4501	0.4540
500	2.0000	$\hat{\alpha}$	0.7709	0.0015	0.0019	0.7653	0.0052	0.0054
		$\hat{\beta}$	4.2203	9.1294	19.8765	7.6663	0.1602	0.1877
		$\hat{\gamma}$	29.3197	1,894.3179	2,483.8725	4.9847	0.1961	0.1962

Table 5 Parameter estimates of a LBBP random variable with α = 0.75, β = 7.5, γ = 10 and θ = 2.

n	$\hat{\theta}$	Parameter	MLE			MM		
			Estimate	Variance	MSE	Estimate	Variance	MSE
15	2.0008	$\hat{\alpha}$	0.6668	0.0287	0.0356	0.8244	0.0853	0.0908
		$\hat{\beta}$	5.9745	95.8896	98.1208	8.1271	3.4539	3.8437
		$\hat{\gamma}$	322.9893	314,151.1002	411,799.2558	10.5836	2.1192	2.4577
50	2.0002	$\hat{\alpha}$	0.7772	0.0141	0.0148	0.8098	0.0268	0.0304
		$\hat{\beta}$	10.8523	119.1055	130.2241	7.7654	1.3382	1.4073
		$\hat{\gamma}$	69.1209	37,402.8801	40,860.7531	10.4898	0.8038	1.0429
200	2.0000	$\hat{\alpha}$	0.7980	0.0037	0.0060	0.7892	0.0079	0.0094
		$\hat{\beta}$	15.1283	149.8692	207.9100	7.6021	0.3475	0.3576
		$\hat{\gamma}$	16.1107	2,002.1112	2,037.4502	10.3335	0.4453	0.5561
500	2.0000	$\hat{\alpha}$	0.7962	0.0016	0.0037	0.7812	0.0034	0.0044
		$\hat{\beta}$	17.7367	174.3853	279.0001	7.5663	0.1550	0.1593
		$\hat{\gamma}$	11.0572	500.6823	501.2992	10.2785	0.2091	0.2864

Table 6 Parameter estimates of a LBBP random variable with α = 5, β = 2, γ = 5 and θ = 2.

n	$\hat{\theta}$	Parameter	MLE			MM		
			Estimate	Variance	MSE	Estimate	Variance	MSE
15	2.2457	$\hat{\alpha}$	0.2309	0.0011	22.7454	3.2970	1.5052	4.4039
		$\hat{\beta}$	0.0337	0.1447	4.0110	2.3517	0.6889	0.8120
		$\hat{\gamma}$	341.3545	12,863.4037	125,984.8943	5.6679	1.5354	1.9800
50	2.1672	$\hat{\alpha}$	1.4450	0.0977	12.7355	3.4584	1.0555	3.4310
		$\hat{\beta}$	15.2933	44.3595	221.0276	1.9581	0.3681	0.3695
		$\hat{\gamma}$	21.7544	14,332.0223	14,598.3990	5.7058	1.6910	2.1875
200	2.1170	$\hat{\alpha}$	2.3184	0.1166	7.3076	3.7226	0.6286	2.2596
		$\hat{\beta}$	19.3467	26.5876	327.4690	1.7420	0.1739	0.2403
		$\hat{\gamma}$	0.5482	0.0714	19.8895	5.9092	1.2798	2.1052
500	2.0952	$\hat{\alpha}$	2.7384	0.1081	5.2227	3.9544	0.4339	1.5267
		$\hat{\beta}$	17.1651	21.9416	251.9007	1.7026	0.1294	0.2177
		$\hat{\gamma}$	0.6851	0.0810	18.6996	5.9348	1.1155	1.9882

Table 7 Parameter estimates of a LBBP random variable with $\alpha = 5$, $\beta = 2$, $\gamma = 10$ and $\theta = 2$.

n	$\hat{\theta}$	Parameter	MLE			MM		
			Estimate	Variance	MSE	Estimate	Variance	MSE
15	2.1117	$\hat{\alpha}$	0.2951	0.0214	22.1578	2.9937	1.3306	5.3544
		$\hat{\beta}$	0.3161	2.9158	5.7484	2.3274	0.7263	0.8328
		$\hat{\gamma}$	603.3858	51,210.8136	403,266.2756	10.8861	2.3833	3.1660
50	2.0792	$\hat{\alpha}$	1.5365	0.0824	12.0781	3.4005	1.1744	3.7315
		$\hat{\beta}$	16.0119	36.3180	232.6141	2.0329	0.3666	0.3673
		$\hat{\gamma}$	11.1261	7,938.4128	7,931.7424	10.9734	2.3855	3.3307
200	2.0558	$\hat{\alpha}$	2.3693	0.1197	7.0400	3.8523	0.7823	2.0988
		$\hat{\beta}$	18.4100	22.1988	291.4658	1.9413	0.1778	0.1811
		$\hat{\gamma}$	1.1379	0.2154	78.7526	10.9073	2.0983	2.9193
500	2.0451	$\hat{\alpha}$	2.7915	0.1072	4.9847	4.2006	0.5157	1.1542
		$\hat{\beta}$	16.2595	17.5824	220.8987	1.9547	0.1506	0.1525
		$\hat{\gamma}$	1.4514	0.4169	73.4957	10.9254	1.7088	2.5636

Table 8 Parameter estimates of a LBBP random variable with $\alpha = 5$, $\beta = 7.5$, $\gamma = 5$ and $\theta = 2$.

n	$\hat{\theta}$	Parameter	MLE			MM		
			Estimate	Variance	MSE	Estimate	Variance	MSE
15	2.0861	$\hat{\alpha}$	0.3901	0.0696	21.3207	3.1736	0.9898	4.3246
		$\hat{\beta}$	1.1864	11.0700	50.9199	7.9345	1.7322	1.9192
		$\hat{\gamma}$	700.6773	142,892.5587	626,716.6219	5.4420	1.8675	2.0610
50	2.0614	$\hat{\alpha}$	1.6296	0.0933	11.4526	3.6803	0.8960	2.6368
		$\hat{\beta}$	17.3709	34.1797	131.5798	7.8146	1.6066	1.7040
		$\hat{\gamma}$	7.4775	4,513.9717	4,515.5955	5.2311	1.1211	1.1734
200	2.0436	$\hat{\alpha}$	2.5294	0.1500	6.2538	4.0137	0.6375	1.6096
		$\hat{\beta}$	20.3258	25.2707	189.7476	7.8887	0.9496	1.0997
		$\hat{\gamma}$	1.4724	0.1746	12.6183	5.0032	0.7752	0.7745
500	2.0350	$\hat{\alpha}$	3.0294	0.1443	4.0275	4.1957	0.5051	1.1515
		$\hat{\beta}$	19.5709	17.8093	163.4983	7.9271	0.9002	1.0817
		$\hat{\gamma}$	1.7097	0.2954	11.1213	4.9351	0.6157	0.6193

Table 9 Parameter estimates of a LBBP random variable with $\alpha = 5$, $\beta = 7.5$, $\gamma = 10$ and $\theta = 2$.

n	$\hat{\theta}$	Parameter	MLE			MM		
			Estimate	Variance	MSE	Estimate	Variance	MSE
15	2.0423	$\hat{\alpha}$	0.6504	0.1359	19.0546	3.4015	0.6905	3.2448
		$\hat{\beta}$	4.0648	33.7383	45.5048	8.6939	1.3015	2.7256
		$\hat{\gamma}$	803.8482	562,091.4458	1,191,724.3467	10.5069	1.8166	2.0717
50	2.0299	$\hat{\alpha}$	1.6995	0.1008	10.9938	3.9771	0.5017	1.5475
		$\hat{\beta}$	19.3536	38.8991	179.3688	8.4736	0.9170	1.8640
		$\hat{\gamma}$	5.7444	2,797.0990	2,812.4118	10.2755	1.6148	1.6891
200	2.0212	$\hat{\alpha}$	2.6146	0.1579	5.8476	4.3153	0.3045	0.7731
		$\hat{\beta}$	22.9882	27.4286	267.2841	8.2780	0.4587	1.0635
		$\hat{\gamma}$	2.6560	0.5230	54.4572	10.1266	0.9553	0.9704
500	2.0173	$\hat{\alpha}$	3.0413	0.1487	3.9852	4.4146	0.3098	0.6522
		$\hat{\beta}$	22.0947	21.1511	234.1365	8.1012	0.5691	0.9299
		$\hat{\gamma}$	3.0298	0.5165	49.1002	10.0716	0.9429	0.9471

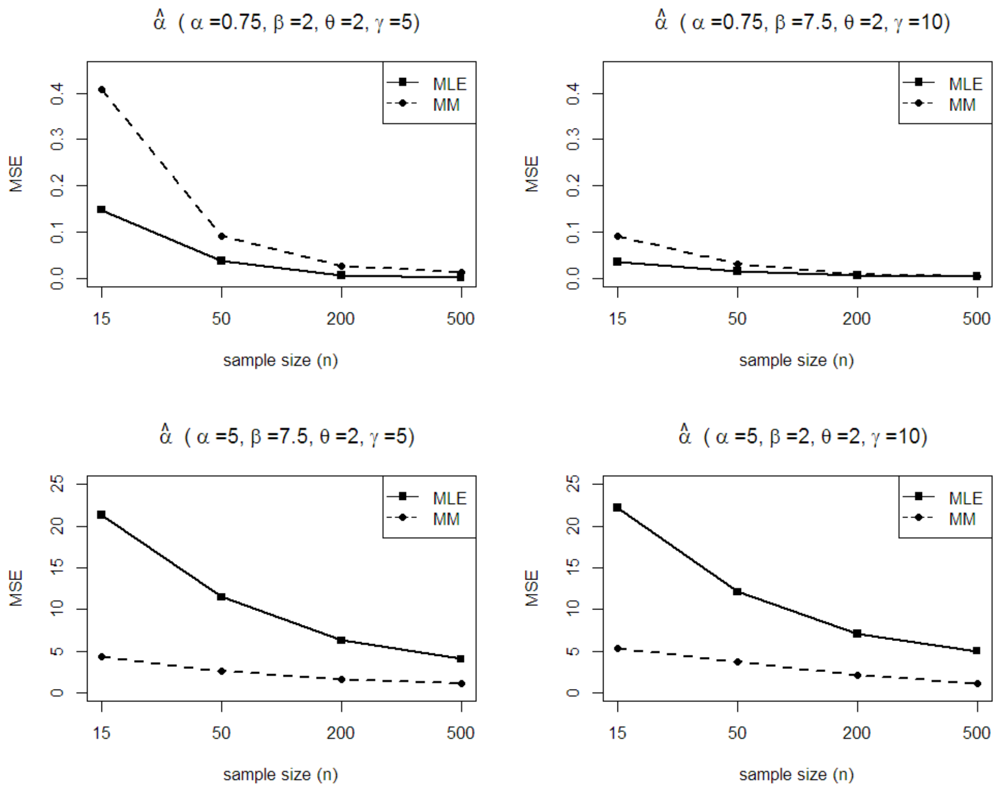

Figure 1 Comparison for the MSE of $\hat{\alpha}$ among the MLE and MM of $X \sim \text{LBBP}(\alpha,\beta,\theta,\gamma)$.

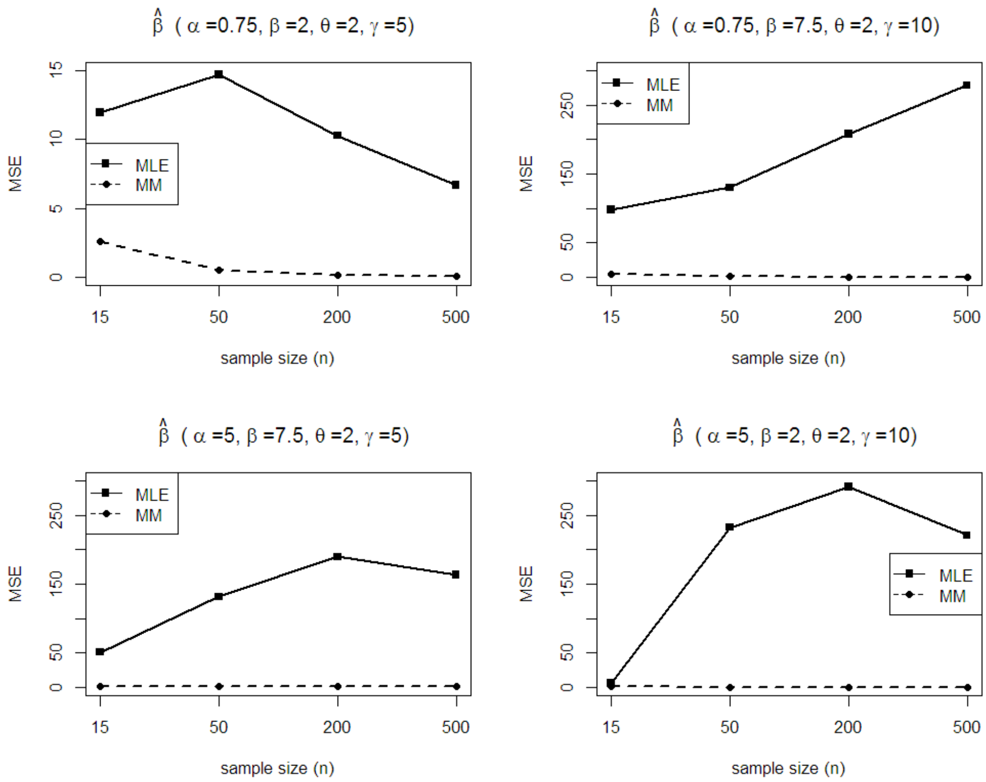

Figure 2 Comparison for the MSE of $\hat{\beta}$ among the MLE and MM of $X \sim \text{LBBP}(\alpha,\beta,\theta,\gamma)$.

Table 10 Parameter estimates for the exceedances of flood peaks of the Wheaton River near Carcross in Yukon Territory data.

Estimation	**MLE**			**MM**		
Distribution	**Pareto**	**BP**	**LBBP**	**Pareto**	**BP**	**LBBP**
Parameter estimates	$\hat{\theta}$=0.1 $\hat{\gamma}$=0.2438	$\hat{\theta}$=0.1 $\hat{\gamma}$=0.0177 $\hat{\alpha}$=3.1474 $\hat{\beta}$=41.9741	$\hat{\theta}$=0.1 $\hat{\gamma}$=0.0423 $\hat{\alpha}$=3.1471 $\hat{\beta}$=40.6408	$\hat{\theta}$=0.1 $\hat{\gamma}$=1.0082	$\hat{\theta}$=0.1 $\hat{\gamma}$=1.8825 $\hat{\alpha}$=169.8540 $\hat{\beta}$=1.5940	$\hat{\theta}$=0.1 $\hat{\gamma}$=0.4232 $\hat{\alpha}$=10.1570 $\hat{\beta}$=4.7703
-log likelihood	303.0642	283.7339	283.7661	426.5559	752.2756	281.4186
AIC	610.1283	575.4678	575.5322	857.1118	1,512.5510	570.8372
BIC	616.0401	584.5745	584.6389	861.6652	1,521.6580	579.9438
K-S statistics	0.3324	0.1734	0.1732	0.8047	0.5967	0.1268
p-value	< 0.0001	0.0263	0.0265	< 0.0001	< 0.0001	0.1970

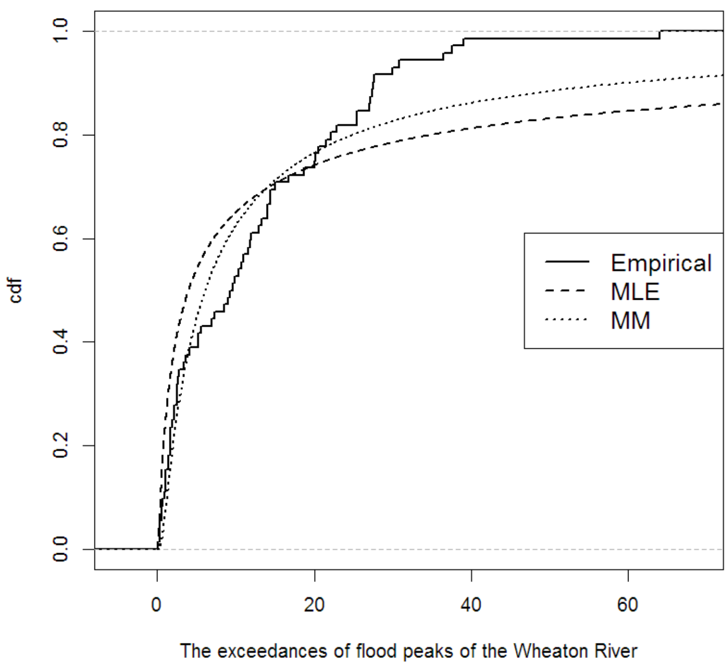

Figure 3 The cdf of the exceedances of flood peaks of the Wheaton River for the LBBP distribution.

Application

In this section the parameter estimation for the LBBP distribution is fitted to a real data set, we compared the MLE with the MM in the parameter estimation for the data that are the exceedances of flood peaks (in m^3/s) of the Wheaton River near Carcross in the Yukon Territory, Canada. The data consist of 72 exceedances for the years 1958 - 1984 [12]. The estimators obtained by both approaches, AIC, BIC and K-S statistics are shown in **Table 10**. In this case, the values of the K-S statistics for the LBBP distribution based on the MM is smaller than the values of the K-S statistics based on the MLE. Moreover, in **Figure 3**, the MM outperforms the MLE for the exceedances of flood peaks of the Wheaton River near Carcross dataset.

Conclusions

We found that, the results of parameter estimation methods from 1,000 Monte Carlo simulations showed that the MM provides a sample mean of the parameter estimate that is closer to the true parameter values than that of the MLE, in almost every case. Moreover, the results of the parameter estimation show the MSE of the sample mean for the MM is smaller than the MSE for the MLE for almost every sample size. Additionally, the parameter estimation for the LBBP distribution is fitted to the exceedances of flood peaks of the Wheaton River near Carcross in the Yukon Territory data set and shows that the MM is the best parameter estimation for this data. The results of this study show that the parameter estimation methods for the new lifetime distribution is highly significant because selecting an improper method for parameter estimation leads to incorrect results. We hope the MM will attract extensive applications in lifetime distribution. Future research would do well to consider using Bayesian approaches in parameter estimation.

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References

[1] AJ Fernández. Smallest Pareto confidence regions and applications. *Comput. Statist. Data Anal.* 2013; **62**, 11-25.
[2] A Soliman. Estimations for Pareto model using general progressive censored data and asymmetric loss. *Commun. Statist-Theory Meth*. 2008; **37**, 1353-70.
[3] S Wu. Interval estimation for a Pareto distribution based on a doubly type II censored sample. *Comput. Statist. Data Anal*. 2008; **52**, 3779-88.
[4] C Hong, J Wu and C Cheng. Computational procedure of performance assessment of lifetime index of businesses for the Pareto lifetime model with the right type II censored sample. *Appl. Math. Comput.* 2007; **184**, 336-50.
[5] J Wu, W Lee and S Chen. Computational comparison for weighted moments estimators and BLUE of the scale parameter of a Pareto distribution with known shape parameter under type II multiply censored sample. *Appl. Math. Comput*. 2006; **181**, 1462-70.
[6] N Abdel-All, M Mahmoud and H Abd-Ellah. Geometrical properties of Pareto distribution. *Appl. Math. Comput*. 2003; **145**, 321-39.
[7] MEJ Newman. Power laws, Pareto distributions and Zipf's law. *Contemporary Phys*. 2005; **46**, 323-51.
[8] MM Ali and S Nadarajah. A truncated Pareto distribution. *Comput. Commun*. 2006; **30**, 1-4.
[9] A Manas. The paretian ratio distribution-an application to the volatility of GDP. *Econ. Lett*. 2011; **111**, 180-3.
[10] M Nassar and N Nada. A new generalization of the Pareto-geometric distribution. *J. Egy. Math. Soc*. 2013; **21**, 148-55.
[11] N Eugene, C Lee and F Famoye. The beta-normal distribution and its applications. *Commun. Statist-Theory Meth*. 2002; **31**, 497-512.
[12] A Akinsete, F Famoye and C Lee. The beta-Pareto distribution. *Statistics* 2008; **42**, 547-63.
[13] CR Rao. On discrete distributions arising out of methods of ascertainment. *Sankhyā: Indian J. Statist.* 1965; **27**, 311-24.
[14] GP Patil and CR Rao. Weighted distributions and size-biased sampling with applications to wildlife populations and human families. *Biometrics* 1978; **34**, 179-89.
[15] R Khattree. Characterization of inverse-Gaussian and gamma distributions through their length-biased distributions. *IEEE Trans. Reliab*. 1989; **38**, 610-1.
[16] BO Oluyede and EO George. On stochastic inequalities and comparisons of reliability measures for weighted distributions. *Math. Prob. Eng*. 2002; **8**, 1-13.
[17] BO Oluyede. A note on probability weighted moment inequalities for reliability measures. *J. Inequal. Pure Appl. Math*. 2006; **7**, 1-11.
[18] P Seenoi, W Bodhisuwan and T Supapakorn. The length-biased exponentiated inverted Weibull distribution. *Int. J. Pure Appl. Math.* 2014; **92**, 191-206.
[19] N Nanuwong and W Bodhisuwan. Length biased beta-Pareto distribution and its structural properties with application. *J. Math. Stat.* 2014; **10**, 49-57.
[20] J Nelder and R Wedderburn. Generalized linear models. *J. R. Stat. Soc*. 1972; **135**, 370-84.
[21] S Eliason. *Maximum Likelihood Estimation: Logic and Practice*. SAGE Publications, California, 1993.
[22] J Aldrich. R.A. Fisher and the making of maximum likelihood 1912-1922. *Statist. Sci.* 1997; **12**, 162-76.
[23] G King. *Unifying Political Methodology: The Likelihood Theory of Statistical Inference*. Cambridge University Press, New York. 1998, p. 1-35.

[24] W Panichkitkosolkul. A modified weighted symmetric estimator for a Gaussian first-order autoregressive model with additive outliers. *Walailak J. Sci. & Tech*. 2012; **9**, 255-62.
[25] IB Aban, MM Meerschaert and AK Panorska. Parameter estimation for the truncated Pareto distribution. *J. Am. Statist. Assoc*. 2006; **101**, 270-7.
[26] Z Birnbaum and F Tingey. One-sided confidence contours for probability distribution functions. *Annal. Math. Stat*. 1951; **22**, 592-6.
[27] H Akaike. A new look at the statistical model identification. *IEEE-TAC*. 1974; **19**, 716-23.
[28] E Wit, E Heuvel and J Romeijn. 'All models are wrong...': An introduction to model uncertainty. *Stat. Neerl*. 2012; **66**, 217-36.
[29] The R Core Team. *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing. Vienna, 2013, Available at: http://www.R-project.org, accessed February 2015.