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Fuzzy and *l*-Fuzzy Subset in a Locally Convex Topology

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Abstract

In this paper, the concepts of sectional fuzzy continuous mappings, and *l*-fuzzy compact sets, are introduced in locally convex topology generated by fuzzy *n*-norms. Schauder-type and other fixed point theorems are established in locally convex topology generated by fuzzy *n*-norms.

Keywords: Locally convex topology, Chauder fixed point theorem

Introduction and preliminaries

In Gähler [1] introduced n-norms on a linear space. A detailed theory of *n*-normed linear space can be found in [2-8]. Gunawan and Mashadi [2] gave a simple way to derive an $(n-1)$-norm from the n-norm in such a way that the convergence and completeness in the n-norm is related to those in the derived $(n-1)$-norm. Narayanan and Vijayabalaji extended *n*-normed linear space to fuzzy *n*-normed linear space. The main objective of this paper is to introduce concepts of sectional fuzzy continuous mappings and *l*-fuzzy compact sets, and in the same time, to perform the Schauder-type [9] and other fixed point theorems in locally convex topology generated by fuzzy *n*-norms. In section 1, we quote some basic definitions, and in section 2, we introduce concepts of sectional fuzzy continuous mappings and *l*-fuzzy compact sets, as well as presenting our new results. Let *n* be a positive integer, and let *X* be a real vector space of dimension of at least *n*. We recall the definitions of an *n*-seminorm and a fuzzy *n*-norm from [10,11].

Definition 1 *A function* $(x_1, x_2, \dots, x_n) \mapsto \| x_1, \dots, x_n \|$ *from* X^n *to* $[0,\infty)$ *is called an n-seminorm on* X *if it has the following four properties*:

(S_1) $\| x_1, x_2, \dots, x_n \| = 0$ if $x_1, x_2, \dots, x_n$ are linearly dependent;

(S_2) $\| x_1, x_2, \dots, x_n \|$ is invariant under any permutation of $x_1, x_2, \dots, x_n$;

(S_3) $\| x_1, \dots, x_{n-1}, cx_n \| = |c| \| x_1, \dots, x_{n-1}, x_n \|$ for any real c;

(S_4) $\| x_1, \dots, x_{n-1}, y+z \| \le \| x_1, \dots, x_{n-1}, y \| + \| x_1, \dots, x_{n-1}, z \|$.

An n-seminorm is called a n-norm if $\| x_1, x_2, \ldots, x_n \| > 0$ whenever $x_1, x_2, \ldots, x_n$ are linearly independent.

Definition 2 *A fuzzy subset N of* $X^n \times \mathbb{R}$ *is called a fuzzy n-norm on* X *if and only if*:

(F_1) For all $t \le 0$, $N(x_1, x_2, \ldots, x_n, t) = 0$;

(F_2) For all $t > 0$, $N(x_1, x_2, \ldots, x_n, t) = 1$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent;

(F_3) $N(x_1, x_2, \ldots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$;

(F_4) For all $t > 0$ and $c \in \mathbb{R}$, $c \ne 0$,

$$N(x_1, x_2, \ldots, cx_n, t) = N(x_1, x_2, \ldots, x_n, \frac{t}{|c|});$$

(F_5) For all $s, t \in \mathbb{R}$,

$$N(x_1, \ldots, x_{n-1}, y+z, s+t) \ge \min\{N(x_1, \ldots, x_{n-1}, y, s), N(x_1, \ldots, x_{n-1} z, t)\}.$$

(F_6) $N(x_1, x_2, \ldots, x_n, t)$ is a non-decreasing function of $t \in \mathbb{R}$ and $\lim_{t\to\infty} N(x_1, x_2, \ldots, x_n, t) = 1$.

Definition 3 *[2] Let* (X, N) *be a fuzzy normed space; a subset* A *of* X *is said to be l-fuzzy closed if for any sequence* $\{x_n\}$ *and for each* $\alpha \in (0,1)$, *and* $x \in A$;

$$\lim_{n\to\infty} N(x_n - x, t) \ge \alpha \tag{1}$$

for all $t > 0$.

Definition 4 *[5] Let* (X, N) *is a fuzzy n-normed space, that is,* X *is real vector space, and* N *is fuzzy n-norm on* X. *We form the family of n-seminorms* $\| \bullet, \bullet, \ldots, \bullet \|_\alpha$, $\alpha \in (0,1)$, *and this family generates a family* $\mathcal{F}$ *of seminorms*
$\| x_1, \ldots, x_{n-1}, \bullet \|_\alpha$, where x_1, x_{n-1}, x and (0,1). The family $\mathcal{F}$ generates a locally convex topology on X; a basis of neighborhoods at the origin is given by; $\{x \in X : p_i(x) \le \varepsilon_i \text{ for } i = 1, 2, \ldots, \mathrm{n}\}$,
where $p_i \in \mathcal{F}$ and $\varepsilon_i > 0$ for $i = 1, 2 \ldots, n$. We call this the locally convex topology generated by the fuzzy n-norm N.

Definition 5 *[2] Let* (X, N) *be a fuzzy normed space; a mapping* $T : (X, N_1) \to (Y, N_2)$ *is said to be fuzzy continuous at* $x_0 \in X$, *if for a given* $\varepsilon > 0$ *and* $\alpha \in (0,1)$ *there exist* $\delta = \delta(\alpha, \varepsilon) > 0$ *and* $\beta = \beta(\alpha, \varepsilon) \in (0,1)$ *such that for each* $\varepsilon > 0$, *there exists* $\delta > 0$ *and*

$$N_1(x-x_0,\delta)>\beta \Rightarrow N_2(T(x)-T(x_0),\varepsilon)>\alpha$$
$$\text{for all } x\in X. \tag{2}$$

If T is fuzzy continuous at each point of X, then T is said to be sectional fuzzy continuous on X.

Definition 6 *[2] Let (X,N) be a fuzzy normed space; a mapping $T:(X,N_1)\to(Y,N_2)$ is said to be sectional fuzzy continuous at $x_0\in X$, if there exists $\alpha_0\in(0,1)$ such that for each $\varepsilon>0$, there exists $\delta>0$ and*

$$N_1(x-x_0,\delta)\ge\alpha_0 \Rightarrow N_2(T(x)-T(x_0),\varepsilon)\ge\alpha_0$$
$$\text{for all } x\in X. \tag{3}$$

If T is sectional fuzzy continuous at each point of X, then T is said to be sectional fuzzy continuous on X.

Definition 7 *[3] Let (X,N) be a fuzzy normed space; a subset A of X is said to be l-fuzzy compact if for any sequence $\{x_n\}$ and for each $\alpha\in(0,1)$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x\in A$ (both depending on $\{x_n\}$ and α) such that*;

$$\lim_{k\to\infty} N(x_{n_k}-x,t)\ge\alpha \tag{4}$$

for all $t>0$.

Schauder fixed point theorem

In this section, we establish Schauder fixed point theorems in the locally convex topology generated by fuzzy *n*-normed spaces.

Definition 8 *A subset A of X is said to be l-fuzzy closed in the locally convex topology generated by N if for any sequence $\{x_n\}$ and for each $\alpha\in(0,1)$, and $x\in A$*;

$$\lim_{n\to\infty} N(a_1,\ldots,a_{n-1},x_n-x,t)\ge\alpha \tag{5}$$

for all $a_1,\ldots,a_{n-1}\in X$ and all $t>0$.

Definition 9 *A mapping $T:(X,N_1)\to(Y,N_2)$ is said to be fuzzy continuous at $x_0\in X$ in the locally convex topology generated by N, if for a given $\varepsilon>0$ and $\alpha\in(0,1)$ there exist $\delta=\delta(\alpha,\varepsilon)>0$ and $\beta=\beta(\alpha,\varepsilon)\in(0,1)$ such that for each $\varepsilon>0$, there exists $\delta>0$ and*

$$N_1(a_1,\ldots,a_{n-1},x-x_0,\delta)>\beta \Rightarrow N_2(a_1,\ldots,a_{n-1},T(x)-T(x_0),\varepsilon)>\alpha$$
$$\text{for all } a_1,\ldots,a_{n-1},x,x_0\in X. \tag{6}$$

If T is fuzzy continuous at each point of X, then T is said to be sectional fuzzy continuous on X.

Definition 10 *A mapping* $T:(X,N_1)\to(Y,N_2)$ *is said to be sectional fuzzy continuous at* $x_0\in X$ *, in the locally convex topology generated by* N *if there exists* $\alpha_0\in(0,1)$ *such that for each* $\varepsilon>0,$ *there exists* $\delta>0$ *and*

$$N_1(a_1,\ldots,a_{n-1},x-x_0,\delta)\geq\alpha_0 \Rightarrow N_2(a_1,\ldots,a_{n-1},T(x)-T(x_0),\varepsilon)\geq\alpha_0$$
$$\text{for all } a_1,\ldots,a_{n-1},x,x_0\in X. \tag{7}$$

If T is sectional fuzzy continuous at each point of X, then T is said to be sectional fuzzy continuous on X.

Definition 11 *A subset* A *of* X *is said to be l-fuzzy compact in the locally convex topology generated by* N *if for any sequence* $\{x_n\}$ *and for each* $\alpha\in(0,1)$*, there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *and* $x\in A$ *(both depending on* $\{x_n\}$ *and* α *) such that*;

$$\lim_{k\to\infty} N(a_1,\ldots,a_{n-1},x_{n_k}-x,t)\geq\alpha \tag{8}$$

for all $a_1,\ldots,a_{n-1}\in X$ and all $t>0$.

Lemma 12 *A subset* A *of* X *is l-fuzzy compact in the locally convex topology generated by* N *if and only if* A *is compact w.r.t.* $\|\|_\alpha$ $(\alpha\ n-\text{norm of } N)$ *for each* $\alpha\in(0,1)$.

Proof. First suppose that A is *l*-fuzzy compact. Take $\alpha_0\in(0,1)$. Let $\{x_n\}$ be a sequence in A. Thus, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and x in A (both depend on α_0) such that;

$$\lim_{k\to\infty} N(a_1,\ldots,a_{n-1},x_{n_k}-x,t)\geq\alpha_0 \tag{9}$$

for all $a_1,\ldots,a_{n-1}\in X$ and all $t>0$. This implies that for a given $\varepsilon>0$ with $\alpha_0-\varepsilon>0$ and for a given $t>0,$ there exists a positive integer $K(\varepsilon,t)$ such that;

$$N(a_1,\ldots,a_{n-1},x_{n_k}-x,t)>\alpha_0-\varepsilon \ \text{for all } n\geq K(\varepsilon,t). \tag{10}$$

which implies that;

$$\| a_1, a_2, \ldots, a_n \|_{\alpha_0 - \varepsilon} \le t \text{ for all } n \ge K(\varepsilon, t). \tag{11}$$

This implies that A is compact. Since $\alpha_0 \in (0,1)$ and $\varepsilon > 0$ are arbitrary, it follows that A is compact w.r.t. $\|\|_\alpha$ for each $\alpha \in (0,1)$. Conversely, suppose that A is compact w.r.t. $\|\|_\alpha$ for each $\alpha \in (0,1)$. Let $\{x_n\}$ be a sequence in A. Take $\alpha_0 \in (0,1)$. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and x in A (both depend on α_0) such that;

$$\lim_{k \to \infty} \| a_1, a_2, \ldots, a_{n-1}, x_{n_k} - x \|_{\alpha_0} = 0. \tag{12}$$

for all $a_1, a_2, \ldots, a_{n-1} \in X$. This implies that for a given $\varepsilon > 0$, there exists a positive integer $K(\varepsilon)$ such that;

$$\| a_1, a_2, \ldots, a_{n-1}, x_{n_k} - x \|_{\alpha_0} < \varepsilon \text{ for all } k \ge K(\varepsilon). \tag{13}$$

From this we conclude that;

$$N(a_1, \ldots, a_{n-1}, x_{n_k} - x, \varepsilon) > \alpha_0 \text{ for all } k \ge K(\varepsilon) \tag{14}$$

for all $a_1, a_2, \ldots, a_{n-1} \in X$. Since ε is arbitrary, so;

$$\lim_{k \to \infty} N(a_1, \ldots, a_{n-1}, x_{n_k} - x, t) > \alpha_0 \text{ for all } t > 0. \tag{15}$$

Since $\alpha_0 \in (0,1)$ is arbitrary, it follows that A is *l*-fuzzy compact.

Lemma 13 *A mapping* $T:(X, N_1) \to (Y, N_2)$ *is sectional fuzzy continuous in the locally convex topology generated by* N *if and only if* $T:(X, \|\|_\alpha^1) \to (Y, \|\|_\alpha^2)$ *is continuous for some* $\alpha \in (0,1)$.

Proof. First we suppose that, $T:(X, N_1) \to (Y, N_2)$ is sectional fuzzy continuous. Thus, there exists $\alpha_0 \in (0,1)$ such that for each $\varepsilon > 0$, there exists $\delta > 0$ and

$$N_1(a_1, \ldots, a_{n-1}, x - y, \delta) \ge \alpha_0 \Rightarrow N_2(a_1, \ldots, a_{n-1}, T(x) - T(y), \varepsilon) \ge \alpha_0$$
$$\text{for all } a_1, \ldots, a_{n-1}, x, y \in X. \tag{16}$$

Choose η_0 such that $\delta_1 = \delta - \eta_0 > 0$. Let $\| a_1, a_2, \ldots, a_{n-1}, x - y \|_{\alpha_0}^1 \le \delta_1$. Then

$\| a_1,a_2,\ldots,a_{n-1},x-y \|^1_{\alpha_0} \le \delta.$ This leads to $N_1(a_1,\ldots,a_{n-1},x-y,\delta) \ge \alpha_0,$ since $T:(X,N_1)\to(Y,N_2)$ is sectional fuzzy continuous, this implies that $N_2(a_1,\ldots,a_{n-1},T(x)-T(y),\varepsilon) \ge \alpha_0$ for all $a_1,\ldots,a_{n-1},x,y \in X,$ and hence $\| a_1,a_2,\ldots,a_{n-1},T(x)-T(y) \|^2_{\alpha_0} \le \varepsilon.$ Thus $T:(X,N_1)\to(Y,N_2)$ is continuous w.r.t. $\|\|^1_\alpha$ and $\|\|^2_\alpha$. Conversely, suppose that $T:(X,N_1)\to(Y,N_2)$ is continuous w.r.t. $\|\|^1_\alpha$ and $\|\|^2_\alpha$. Thus;

$$\| a_1,a_2,\ldots,a_{n-1},x-y \|^1_{\alpha_0} \le \delta \Rightarrow \| a_1,a_2,\ldots,a_{n-1},T(x)-T(y) \|^2_{\alpha_0} \le \frac{\varepsilon}{2} \tag{17}$$

for all $a_1,\ldots,a_{n-1},x,y \in X$. Let $N_1(a_1,\ldots,a_{n-1},x-y,\delta) \ge \alpha_0,$

so $\| a_1,a_2,\ldots,a_{n-1},x-y \|^1_{\alpha_0} \le \delta$, which implies that $\| a_1,a_2,\ldots,a_{n-1},T(x)-T(y) \|^2_{\alpha_0} \le \varepsilon.$ Therefore;

$$N_2(a_1,\ldots,a_{n-1},T(x)-T(y),\varepsilon) \ge \alpha_0 \text{ for all } a_1,\ldots,a_{n-1},x,y \in X. \tag{18}$$

Thus, the mapping $T:(X,N_1)\to(Y,N_2)$ is sectional fuzzy continuous.

Theorem 14 (*Schauder fixed point theorem*). *Let K be a nonempty convex, l-fuzzy compact subset in the locally convex topology generated by N and $T:K\to K$ be sectional fuzzy continuous. Then T has a fixed point.*

Proof. By using theorem. For every $\alpha \in (0,1)$, $\| \bullet,\bullet,\ldots,\bullet \|_\alpha$ is an *n*-seminorm on X. As K is an *l*-fuzzy compact of X, thus K is a compact subset of $(X,\|\|_\alpha)$ for each $\alpha \in (0,1)$ (by Lemma 1), since $T:K\to K$ be sectional fuzzy continuous, it follows by Lemma 2 $T:K\to K$ is continuous w.r.t. $\|\|_{\alpha_0}$ for some $\alpha_0 \in (0,1)$. Therefore, we get K is a nonempty convex and compact subset of a normed linear space $(X,\|\|_{\alpha_0})$ and $T:K\to K$ is a continous mapping. By Schauder fixed point theorem [18], it follows that *T* has a fixed point.

Conclusions

We investigated the concepts of sectional fuzzy continuous mappings and *l*-fuzzy compact sets in locally convex topology generated by fuzzy *n*-normed spaces as an extension of the fuzzy normed space. In this new frame, we established the Schauder-type and other fixed point theorems, as well as some results in locally convex topology generated by fuzzy *n*-normed spaces, which are useful tools in the development of the fuzzy set theory.

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