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Traveling Wave Solutions for Fifth Order (1+1)-Dimensional Kaup-Kupershmidt Equation with the Help of $Exp(-\phi\eta)$-Expansion Method

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Abstract

By using the $\exp(-\phi(\eta))$ -expansion method, abundant exact traveling wave solutions for the fifth order (1+1)-dimensional Kaup-Kupershmidt equation are obtained in a uniform way. The obtained solutions in this work are imperative and significant for explanation of some practical physical phenomena. It is shown that the $\exp(-\phi(\eta))$ -expansion method, together with the first order ordinary differential equation, provides a progress mathematical tool for solving nonlinear partial differential equations. Numerical results, together with graphical representation, explicitly reveal the complete reliability and high efficiency of the proposed algorithm.

Keywords: The $\exp(-\phi(\eta))$ -expansion method, the fifth order (1+1)-dimensional Kaup-Kupershmidt equation, traveling wave solutions, nonlinear evolution equation

Introduction

Most scientific problems and physical phenomena occur nonlinearly. Nonlinear differential equations take place in a diverse range of physical phenomena, including propagation of shallow water waves, long wave and chemical reaction-diffusion models, fluid mechanics, physics, astrophysics, solid state physics, chemistry, various branches of biology, astronomy, hydrodynamics, nuclear physics, and applied and engineering sciences. In recent years, the exact solutions of nonlinear partial differential equations (PDEs) have been investigated by many researchers (see [1-42]) who were concerned with nonlinear physical phenomena, and many powerful and efficient methods have been used by them. Among non-integrable nonlinear differential equations, there is a wide class of equations that are referred to as partially integrable, because these equations become integrable for some values of their parameters. Recently, many kinds of powerful methods have been proposed to find exact solutions of nonlinear PDEs, e.g. the homotopy analysis method [1,2], the 3-wave method [3], the extended homoclinic test approach [4], the improved F-expansion method [5], the projective Riccati equation method [6], the Weierstrass elliptic function method [7], the Jacobi elliptic function expansion method [8,9], and the tanh-function method [10-13]. For integrable nonlinear differential equations, the inverse scattering transform method [14], the Hirota method [15], the Backlund transform method [16] and the Exp-function method [17-20], the truncated Painlevé expansion method [21], the extended tanh-method [22,23], the homogeneous balance method [24-26], and other methods [27-33], are used for searching for the exact solutions. Zhao and Li [34] proposed a direct and concise method, called the $\exp(-\Phi(\xi))$ -expansion, for solving nonlinear evolution equations to find new types of solution.

The objective of this article is to implement the $\exp(-\varphi(\eta))$-expansion method to construct the exact solutions for nonlinear evolution equations in mathematical physics via the fifth order (1+1)-dimensional Kaup-Kupershmidt equation for the first time.

Description of the $\exp(-\phi(\eta))$ -expansion method

In this section, we describe the main steps of the $\exp(-\phi(\eta))$ -expansion method for finding the traveling wave solutions of nonlinear evolution equations [42]. Consider that a nonlinear equation in 2 independent variables x and t is given by;

$$P(U,U_x,U_t,U_{xx},U_{xt},U_{tt},\ldots\ldots)=0 \tag{1}$$

where $U=U(x,t)$ is an unknown function, P is a polynomial in $U=U(x,t)$, and its various partial derivatives in which the highest order derivatives and nonlinear terms are involved.

Step 1 Combining the independent variables x and t into one variable $\eta = x - wt$, we suppose that;

$$U(x,t)=u(\eta),\ \eta = x - wt, \tag{2}$$

the traveling wave variable (2) permits us to reduce Eq. (1) to an ODE for $u=u(\eta)$;

$$P(u,u',u'',\ldots\ldots\ldots)=0 \tag{3}$$

Step 2 Suppose that the solution of ODE (3) can be expressed by a polynomial in $\exp(-\phi(\eta))$ as follows;

$$u=\sum_{i=0}^{m} a_i \exp(-\phi(\eta))^i \tag{4}$$

where $\phi'(\eta)$ satisfies the ODE in the form;

$$\phi'(\eta)=\exp(-\phi(\eta))+\mu\exp(\phi(\eta))+\lambda, \tag{5}$$

then the solutions of ODE (5) are;

when $\lambda^2-4\mu>0, \mu\neq 0$, then $$\phi(\eta)=\ln\left(\frac{-\sqrt{\lambda^2-4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\eta+E)\right)-\lambda}{2\mu}\right) \tag{6}$$

When $\lambda^2-4\mu>0, \mu=0$, then $$\phi(\eta)=-\ln\left(\frac{\lambda}{\exp(\lambda(\eta+E))-1}\right) \tag{7}$$

When $\lambda^2-4\mu=0, \mu\neq 0, \lambda\neq 0$, then $$\phi(\eta)=\ln\left(-\frac{2(\lambda(\eta+E)+2)}{\lambda^2(\eta+E)}\right) \tag{8}$$

When $\lambda^2 - 4\mu = 0, \mu = \lambda = 0$, then $\phi(\eta) = \ln(\eta + E)$ (9)

When $\lambda^2 - 4\mu < 0$, then $\phi(\eta) = \ln\left(\dfrac{\sqrt{4\mu - \lambda^2}\tan(\frac{\sqrt{4\mu - \lambda^2}}{2}(\xi + E)) - \lambda}{2\mu}\right)$ (10)

$a_i, w, \lambda;\ i = 0, \ldots\ldots, m$ and μ are constants to be determined later, $a_m \neq 0$, and the positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in ODE (3).

Step 3 By substituting (4) into Eq. (3) and using the ODE (5), collecting all terms with the same order of $\exp(-\phi(\eta))$ together, the left hand side of Eq. (3) is converted into another polynomial in $\exp(-\phi(\eta))$. Equating each coefficient of this polynomial to zero yields a set of algebraic equations for $a_i, \ldots, w, \lambda;\ i = 0, \ldots\ldots, m$ and μ.

Step 4 Assuming that the constants $a_i, \ldots, w, \lambda;\ i = 0, \ldots\ldots, m$ and μ can be obtained by solving the algebraic equations in step 3, since the general solutions of ODE (5) are well known to us, then substituting $a_i, \ldots, w;\ i = 0, \ldots\ldots, m$, along with the general solutions of Eq. (5), into (4) completes the determination of the solution of Eq. (1).

New exact solutions to the fifth order (1+1)-dimensional Kaup-Kupershmidt equation

In this section, the $\exp(-\phi(\eta))$-expansion method is employed to construct some new traveling wave solutions for the fifth order (1+1)-dimensional Kaup-Kupershmidt equation, which is a very important non-linear evolution equation (NLEE) in mathematical physics and engineering. The Kaup-Kupershmidt equation is the nonlinear fifth-order partial differential equation. It is the first equation in a hierarchy of integrable equations with Lax operator. It has properties similar (but not identical) to those of the better-known KdV hierarchy. Fifth-order KdV type equations occur naturally in modeling many different wave phenomena, such as gravity-capillary waves, the propagation of shallow water waves over a flat surface, and magneto-sound propagation in plasmas [35]. Although the fifth order (1+1)-dimensional Kaup-Kupershmidt equation is completely integrable [36] and has bilinear representations [37,38]. Salas *et al*. [39] used the projective Riccati equations method and the Cole-Hopf transformation to find the traveling wave solutions. Goodarzian *et al*. [40] applied the Exp-function method to the Kaup-Kupershmidt equation to find exact solutions. Feng and Li [41] used the Fan sub-equation method to construct exact traveling wave solutions of the (1+1)-dimensional Kaup-Kupershmidt equation. Shakeel and Mohyud-Din [42] used the alternative (G'/G)-expansion method with generalized Riccati equation for finding some exact traveling wave solutions of the fifth order (1+1)-dimensional Kaup-Kupershmidt equation. Let us now consider the fifth order (1+1)-dimensional Kaup-Kupershmidt equation;

$$U_t + U_{xxxxxx} + 10UU_{xxx} + 25U_xU_{xx} + 20U^2U_x = 0 \tag{11}$$

Upon using the transformation;

$$U(x,t) = u(\eta);\ \eta = x - wt \tag{12}$$

where w is speed of travel, Eq. (11) is transferred to;

$$-wu'+u^{(5)}+10uu'''+25u'u''+20u^2u'=0 \tag{13}$$

Integrating Eq. (13) with respect to, η we have;

$$C-wu+u^{(4)}+10uu''+\frac{15}{2}(u')^2+\frac{20}{3}u^3=0 \tag{14}$$

where the prime denotes differentiation with respect to η. By balancing the orders of u' and u^2 in Eq. (14), we have $m=2$. So, Eq. (14) has the following solution;

$$u(\eta)=a_0+a_1\exp(-\phi(\eta))+a_2(\exp(-\phi(\eta)))^2, \tag{15}$$

where $U(x,t)=u(\eta),\ \eta=x-wt$ and $a_2\neq 0$
Substitute (5) and (15) into (14), letting the coefficient of $(\exp(-\varphi(\eta)))^i$, $(i=0,1,2,...,6)$ be zero, yields a set of algebraic equations about a_i, w as follows;

$$C+16a_2\mu^3+\frac{15}{2}a_1^2\mu^2+a_1\mu\lambda^3+20a_0a_2\mu^2+14a_2\mu^2\lambda^2$$
$$+8a_1\mu^2\lambda+10a_0a_1\mu\lambda-wa_0+\frac{20}{3}a_0^3=0$$
$$a_1\lambda^4+60a_0a_2\mu\lambda+10a_0a_1\lambda^2+20a_0a_1\mu+120a_2\lambda\mu^2-wa_1+50a_1a_2\mu^2$$
$$+22a_1\mu\lambda^2+25a_1^2\lambda+16a_1\mu^2+20a_0^2a_1+30a_2\mu\lambda^3=0$$
$$136a_2\mu^2-wa_2+20a_0a_1^2+16a_2\lambda^4+60a_1\mu\lambda+\frac{35}{2}a_1^2\lambda^2+40a_0a_2\lambda^2+80a_0a_2\mu+$$
$$20a_0^2a_2+30a_0a_1\lambda+50a_2^2\mu^2+35a_1^2\mu+130a_1a_2\mu\lambda+15a_1\lambda^3+232a_2\mu\lambda^2=0$$
$$80a_1a_2\lambda^2+\frac{20}{3}a_1^3+120a_2^2\mu\lambda+440a_2\mu\lambda+45a_1^2\lambda+40a_1\mu+100a_0a_2\lambda$$
$$+130a_2\lambda^3+20a_0a_1+40a_0a_1a_2+50a_1\lambda^2+160a_1a_2\mu=0$$
$$70a_2^2\lambda^2+60a_1\lambda+20a_1^2a_2+330a_2\lambda^2+190a_1a_2\lambda+\frac{55}{2}a_1^2$$
$$+60a_0a_2+140a_2^2\mu+240a_2\mu+20a_2^2=0$$
$$336a_2\lambda+24a_1+160a_2^2\lambda+110a_1a_2\mu+20a_1a_2^2=0$$

Solving the above sets of algebraic equations, we obtain 2 sets of solutions as follows;

Set 1 $C=-\frac{1}{3}\mu^3+\frac{1}{4}\lambda^2\mu^2-\frac{1}{16}\lambda^4\mu+\frac{1}{192}\lambda^6,\quad w=\mu^2+\frac{1}{16}\lambda^4-\frac{1}{2}\mu\lambda^2,$

$$a_0=-\frac{\lambda^2}{8}-\mu,\quad a_1=-\frac{3}{2}\lambda \quad \text{and} \quad a_1=-\frac{3}{2}$$

Set 2 $C=\frac{832}{3}\mu^3-208\lambda^2\mu^2+52\lambda^4\mu-\frac{13}{3}\lambda^6,\ \ w=176\mu^2+11\lambda^4-88\mu\lambda^2,$

$a_0=-\lambda^2-8\mu,\quad a_1=-12\lambda$ and $a_1=-12$

Substituting solution set-1 into (15), we have;

$$u=-\frac{\lambda^2}{8}-\mu-\frac{3\lambda}{2}\exp(-\phi(\eta))-\frac{3}{2}\left(\exp(-\phi(\eta))\right)^2 \tag{16}$$

where $\eta=x-\left(\mu^2+\frac{1}{16}\lambda^4-\frac{1}{2}\mu\lambda^2\right)t$.

Respectively substituting (6), (7), (8), (9) and (10) into formula (16), we have 5 traveling wave solutions of the fifth order (1+1)-dimensional Kaup-Kupershmidt Eq. (11), as follows:
when $\lambda^2-4\mu>0,\mu\neq 0$, then;

$$U_{1_1}(x,t)=-\frac{\lambda^2}{8}-\mu+\frac{3\lambda}{2}\left(\frac{2\mu}{\sqrt{\lambda^2-4\mu}\tanh(\frac{\sqrt{\lambda^2-4\mu}}{2}(\eta+E))+\lambda}\right)$$
$$-\frac{3}{2}\left(\frac{2\mu}{\sqrt{\lambda^2-4\mu}\tanh(\frac{\sqrt{\lambda^2-4\mu}}{2}(\eta+E))+\lambda}\right)^2 \tag{17}$$

$\eta=x-\left(\mu^2+\frac{1}{16}\lambda^4-\frac{1}{2}\mu\lambda^2\right)t$, where E is an arbitrary constant.

When $\lambda^2-4\mu>0,\mu=0$, then;

$$U_{1_2}(x,t)=-\frac{\lambda^2}{8}-\frac{3\lambda}{2}\left(\frac{\lambda}{\exp(\lambda(\eta+E))-1}\right)-\frac{3}{2}\left(\frac{\lambda}{\exp(\lambda(\eta+E))-1}\right)^2, \tag{18}$$

$\eta=x-\left(\mu^2+\frac{1}{16}\lambda^4-\frac{1}{2}\mu\lambda^2\right)t,$ where E is an arbitrary constant.
When $\lambda^2-4\mu=0,\mu\neq 0,\lambda\neq 0$, we obtain the traveling solution;

$$U_{1_3}(x,t)=-\frac{\lambda^2}{8}-\mu+\frac{3\lambda}{2}\left(\frac{\lambda^2(\eta+E)}{\lambda(\eta+E)+2}\right)-\frac{3}{2}\left(\frac{\lambda^2(\eta+E)}{\lambda(\eta+E)+2}\right)^2, \tag{19}$$

$\eta = x - \left(\mu^2 + \frac{1}{16}\lambda^4 - \frac{1}{2}\mu\lambda^2\right)t,$ where E is an arbitrary constant.

When $\lambda^2 - 4\mu = 0, \mu = 0, \lambda = 0$, we obtain the non-traveling solution;

$$U_{1_4}(x,t) = -\frac{3}{2}\left(\frac{2}{\eta + E}\right)^2, \tag{20}$$

$\eta = x,$ where E is an arbitrary constant.

When $\lambda^2 - 4\mu < 0$, then;

$$U_{1_5}(x,t) = -\frac{\lambda^2}{8} - \mu - \frac{3\lambda}{2}\left(\frac{2\mu}{\sqrt{4\mu - \lambda^2}\tan(\frac{\sqrt{4\mu-\lambda^2}}{2}(\eta + E)) - \lambda}\right) - \frac{3}{2}\left(\frac{2\mu}{\sqrt{4\mu - \lambda^2}\tan(\frac{\sqrt{4\mu-\lambda^2}}{2}(\eta + E)) - \lambda}\right)^2, \tag{21}$$

$\eta = x - \left(\mu^2 + \frac{1}{16}\lambda^4 - \frac{1}{2}\mu\lambda^2\right)t,$ where E is an arbitrary constant.

Again, substituting solution set-2 into (15), we have;

$$u = -\lambda^2 - 8\mu - 12\lambda\exp(-\phi(\eta)) - 12\left(\exp(-\phi(\eta))\right)^2 \tag{22}$$

where $\eta = x - \left(\mu^2 + \frac{1}{16}\lambda^4 - \frac{1}{2}\mu\lambda^2\right)t$

Respectively substituting (6), (7), (8), (9), and (10) into formula (22), we have 5 traveling wave solutions of the fifth order (1+1)-dimensional Kaup-Kupershmidt Eq. (11), as follow:
when $\lambda^2 - 4\mu > 0, \mu \neq 0$, then;

$$U_{2_1}(x,t) = -\lambda^2 - 8\mu + 12\lambda\left(\frac{2\mu}{\sqrt{\lambda^2 - 4\mu}\tanh(\frac{\sqrt{\lambda^2-4\mu}}{2}(\eta + E)) + \lambda}\right) - 12\left(\frac{2\mu}{\sqrt{\lambda^2 - 4\mu}\tanh(\frac{\sqrt{\lambda^2-4\mu}}{2}(\eta + E)) + \lambda}\right)^2 \tag{23}$$

$\eta = x - \left(176\mu^2 + 11\lambda^4 - 88\mu\lambda^2\right)t$, where E is an arbitrary constant.

When $\lambda^2 - 4\mu > 0, \mu = 0$, then;

$$U_{2_2}(x,t) = -\lambda^2 - 12\lambda\left(\frac{\lambda}{\exp(\lambda(\eta+E))-1}\right) - 12\left(\frac{\lambda}{\exp(\lambda(\eta+E))-1}\right)^2, \tag{24}$$

$\eta = x - \left(11\lambda^4\right)t$, where E is an arbitrary constant.

When $\lambda^2 - 4\mu = 0, \mu \neq 0, \lambda \neq 0$, we obtain the traveling solution;

$$U_{2_3}(x,t) = -\lambda^2 - 8\mu + 12\lambda\left(\frac{\lambda^2(\eta+E)}{\lambda(\eta+E)+2}\right) - 12\left(\frac{\lambda^2(\eta+E)}{\lambda(\eta+E)+2}\right)^2, \tag{25}$$

$\eta = x - \left(176\mu^2 + 11\lambda^4 - 88\mu\lambda^2\right)t$, where E is an arbitrary constant.

When $\lambda^2 - 4\mu = 0, \mu = 0, \lambda = 0$, we obtain the non-traveling solution;

$$U_{2_4}(x,t) = -12\left(\frac{2}{\eta+E}\right)^2, \tag{26}$$

$\eta = x$, where E is an arbitrary constant.

When $\lambda^2 - 4\mu < 0$, then;

$$U_{2_5}(x,t) = -\lambda^2 - 8\mu - 12\lambda\left(\frac{2\mu}{\sqrt{4\mu-\lambda^2}\ \tan(\frac{\sqrt{4\mu-\lambda^2}}{2}(\eta+E)) - \lambda}\right)$$

$$-12\left(\frac{2\mu}{\sqrt{4\mu-\lambda^2}\ \tan(\frac{\sqrt{4\mu-\lambda^2}}{2}(\eta+E)) - \lambda}\right)^2 \tag{27}$$

$\eta = x - \left(176\mu^2 + 11\lambda^4 - 88\mu\lambda^2\right)t$, where E is an arbitrary constant.

Graphical representation of solutions

The graphical illustrations of the solutions are given below in **Figures 1 - 10**, with the aid of Maple.

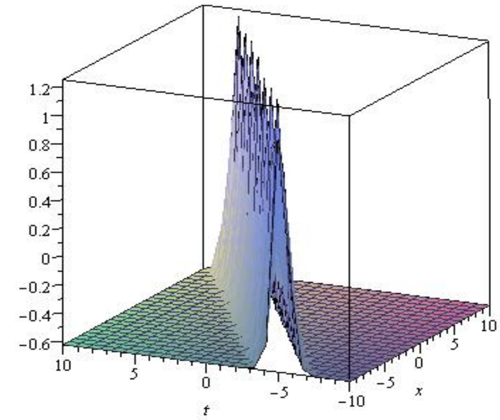

Figure 1 Traveling wave solution $U_{1_1}(\eta)$ when $\mu = 1,\ \lambda = 3,\ E = 1$ and $-10 \le x, t \le 10$.

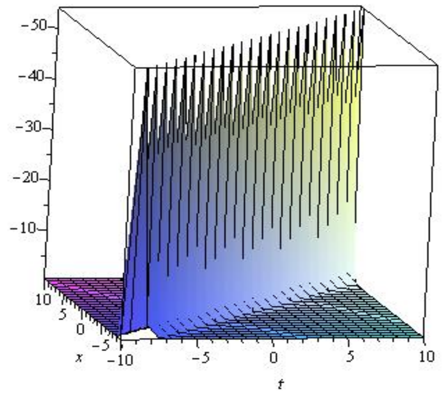

Figure 2 Traveling wave solution $U_{1_2}(\eta)$ when $\mu = 0,\ \lambda = 2,\ E = 1$ and $-10 \le x, t \le 10$.

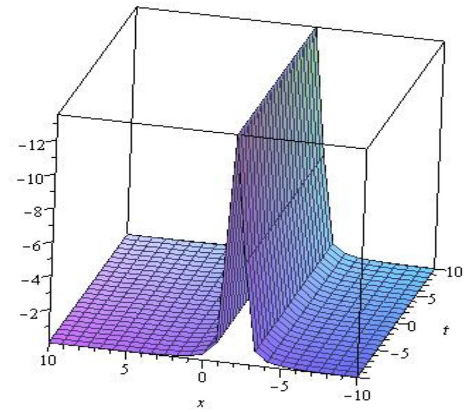

Figure 3 Traveling wave solution $U_{1_3}(\eta)$ when $\mu = 1,\ \lambda = 2,\ E = 1$ and $-10 \le x, t \le 10$.

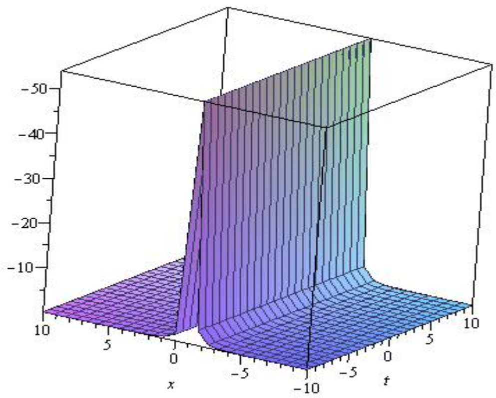

Figure 4 Traveling wave solution $U_{1_4}(\eta)$ when $\mu = 0,\ \lambda = 0,\ E = 1$ and $-10 \le x, t \le 10$.

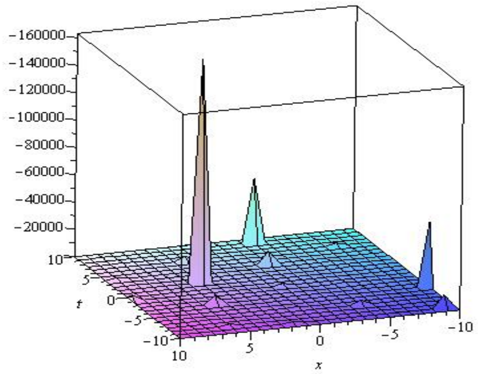

Figure 5 Traveling wave solution $U_{1_5}(\eta)$ when $\mu = 1,\ \lambda = 1,\ E = 1$ and $-10 \le x, t \le 10$.

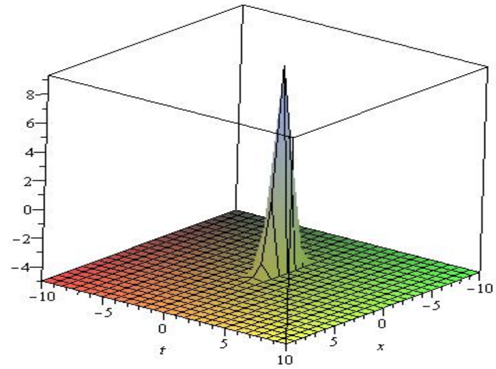

Figure 6 Traveling wave solution $U_{2_1}(\eta)$ when $\mu = 1,\ \lambda = 3,\ E = 1$ and $-10 \le x, t \le 10$.

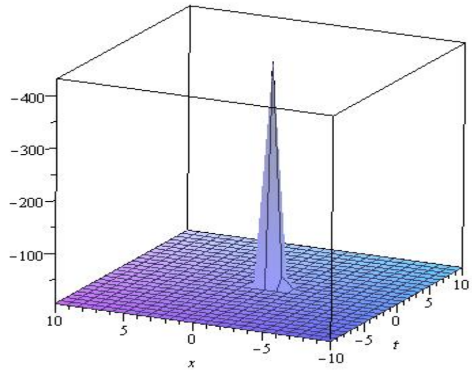

Figure 7 Traveling wave solution $U_{2_2}(\eta)$ when $\mu = 0,\ \lambda = 2,\ E = 1$ and $-10 \le x, t \le 10$.

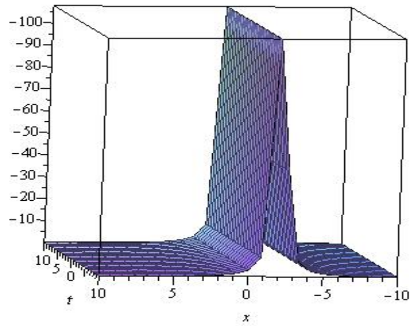

Figure 8 Traveling wave solution $U_{2_3}(\eta)$ when $\mu = 1,\ \lambda = 2,\ E = 1$ and $-10 \le x, t \le 10$.

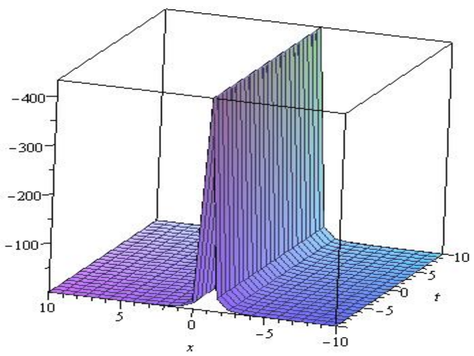

Figure 9 Traveling wave solution $U_{2_4}(\eta)$ when $\mu = 0,\ \lambda = 0,\ E = 1$ and $-10 \le x, t \le 10$.

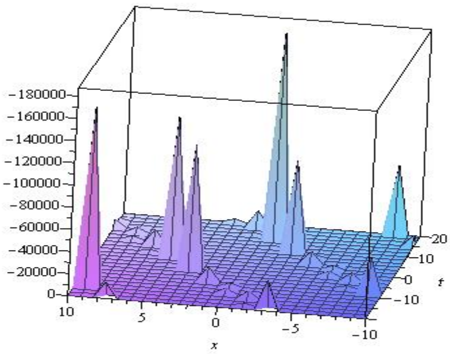

Figure 10 Traveling wave solution $U_{2_5}(\eta)$ when $\mu = 1,\ \lambda = 1,\ E = 1$ and $-10 \le x, t \le 10$.

Conclusions

In this work, we have applied the $\exp(-\phi(\eta))$ -expansion method to handle the Kaup-Kupershmidt equation. In fact, we have presented ten new solutions for the fifth order (1+1)-dimensional Kaup-Kupershmidt equation. The results of the current work illustrates that the $\exp(-\phi(\eta))$ -expansion method is indeed a powerful analytical technique for most types of nonlinear problems, and several such problems in scientific studies and engineering may be solved by this method. This study shows that the method is quite efficient and well suited practically to be used in finding the exact solutions of NLEEs.

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