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The Characterizations of the Curves in a 3-Dimensional Lightlike Cone

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Abstract

In this paper, m-curvature functions are defined for curves in a 3-dimensional null cone. Furthermore, a new characterization is given for the curvature functions.

Keywords: Null cone, m-curvature functions, frenet frame, curvature, curve

Introduction

As is well known, from the viewpoint of general relativity and the related theories, physical events space is represented by a semi-Riemannian manifold. A semi-Riemannian manifold has three casual types of submanifolds; spacelike, timelike, and lightlike, depending on the character of the induced metric on the tangent space. Due to the degeneracy of the metric, the study of lightlike submanifolds have attracted the attention of many scientists.

Lightlike hypersurfaces are studied in general relativity by producing models of different types of horizons (event horizons, Cauchy's horizons, Kruskal's horizons,...etc.) The relation between general relativity theory and geometry provides a theoritical framework in which the fundamental principles of physics may appear to be unified, as in the Kaluza-Klein scheme. Especially, event horizons and their properities are manifested in the proof of numerous theorem concerning black holes and singularities. There are also researches in the theory of electromagnetism [1,2].

On the other hand, according to general relativity, light moves along a light cone, so everything we see lies on our past light cone. To see us as we are now, the observer has to view our future light cone. As we move in time along our world line, we drag our light cones with us, so that they sweep over the spacetime. The motion of a massive body is always timelike, and the motion of massless particles is always lightlike [3].

As indicated above, it is interesting to deal with the curve concept in a lightlike cone. This is the main motivation of the present paper and also a supplement to our previous research note. In this paper, we gave a characterization for curves in a 3-dimensional lightlike cone in Minkowski space. More precisely, we obtained some characterizations for curvature functions m_i of the curve.

There are many brilliant works on timelike, spacelike, and lightlike curves in Minkowski space, the pseudo-Riemannian sphere, and pseudo-Riemannian hyperbolic space [4-13]. However, to the author's knowledge, there are few articles dedicated to studying the notion of curves in a lightlike cone in Minkowski space (see, for example [14-16]).

The main goal of this paper is to give some characterizations for the curves for special cases in a lightlike cone.

Curves in the lightlike cone

In the following, we use notations and concepts from [14-16], unless otherwise stated.

Let E_1^4 denote 4-dimensional Minkowski space-time, i.e. the Euclidean space E^4, with the standard flat metric given by;

$$<,> = dx_1{}^2 + dx_2{}^2 + dx_3{}^2 - dx_4{}^2, \tag{1}$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of E_1^4. Since $< , >$ is an indefinite metric, recall that a vector v in E_1^4 can have one of 3 casual characters: it can be spacelike if $< \mathrm{v}, \mathrm{v} > \; > 0$ or v = **0**, timelike if $< \mathrm{v}, \mathrm{v} > \; < 0$, and null (lightlike) if $< \mathrm{v}, \mathrm{v} > \; = 0$ and v ≠ **0**, respectively. The norm of a vector v is given by $\| \mathrm{v} \| = \sqrt{|\langle \mathrm{v}, \mathrm{v} \rangle|}$. Therefore, v is a unit vector if $< \mathrm{v}, \mathrm{v} >= \mp 1$. Next, the vectors v and w are said to be orthogonal if < v, w >=0. Similarly, an arbitrary curve x = x(s) can be locally spacelike, timelike, or null (lightlike) if all of its velocity x′(s) are spacelike, timelike, or null (lightlike), respectively. Also x = x(s) is a unit speed curve if < x′(s), x′(s) >= ∓1.

Recall some of the most important hyperquadrics in E_1^4.

Let $S_1^3(c,r)$ and $H_1^3(c,r)$ denote the pseudo-Riemannian sphere and pseudo-Riemannian hyperbolic space of radius r and center c, defined by;

$$S_1^3(c,r) = \{x \in E_1^4 : \langle x-c, x-c \rangle = r^2\}; \tag{2}$$

$$H_1^3(c,r) = \{x \in E_1^4 : \langle x-c, x-c \rangle = -r^2\}, \tag{3}$$

respectively. Finally, a lightlike cone (or null cone or quadric cone) $Q_1^3(c)$ with vertex at point c in E_1^4 is defined by;

$$Q_1^3(c) = \{x \in E_1^4 : \langle x-c, x-c \rangle = 0\}. \tag{4}$$

If c = **0**, we simply denote $Q_1^3(c)$ by Q^3.

Let $x = x(s) : I \to Q^3 \subset E_1{}^4$ be a curve in a 3-dimensional lightlike cone Q^3 of Minkowski 4-space E_1^4 with an arc length parameter s. We have;

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0. \tag{5}$$

Then, from;

$$x_1^2 - (\mathrm{i}x_2)^2 = -(x_3^2 - x_4^2) \tag{6}$$

we get;

$$\frac{x_1 + \mathrm{i}x_2}{x_3 + x_4} = -\frac{x_3 - x_4}{x_1 - \mathrm{i}x_2} \quad \text{or} \quad \frac{x_1 + \mathrm{i}x_2}{x_3 - x_4} = -\frac{x_3 + x_4}{x_1 - \mathrm{i}x_2}. \tag{7}$$

Without loss of generality, for a curve $x = x(s) : I \to Q^3 \subset E_1{}^4$ with $x = x(s) = (x_1, x_2, x_3, x_4)$, we may assume that;

$$\frac{x_1 + \mathrm{i}x_2}{x_3 + x_4} = -\frac{x_3 - x_4}{x_1 - \mathrm{i}x_2} \tag{8}$$

and using;

$$y(s) = -x''(s) - \left(\frac{1}{2}\right) < x''(s), x''(s) > x(s) \tag{9}$$

we can see that;

$$< y(s), y(s) > = < x(s), x(s) > = < y(s), x'(s) > = 0, \; < x(s), y(s) > = 1. \tag{10}$$

Supposing α(s) = x′(s), and choosing β(s), as in the following form;

$$\det(x(s), \alpha(s), \beta(s), y(s)) = 1. \tag{11}$$

Then, from (9) we have;

$$\begin{aligned} \alpha'(s) &= x''(s) = -\left(\frac{1}{2}\right) < x''(s), x''(s) > x(s) - y(s) \\ &= \kappa(s)\, x(s) - y(s). \end{aligned} \tag{12}$$

Therefore, the Frenet Formulas of curve $x = x(s) : I \to Q^3 \subset E_1{}^4$ can be written as;

$$\begin{aligned} x'(s) &= \alpha(s) \\ \alpha'(s) &= \kappa(s)x(s) - y(s) \\ \beta'(s) &= \tau(s)x(s) \\ y'(s) &= -\kappa(s)\alpha(s) - \tau(s)\beta(s). \end{aligned} \tag{13}$$

The Frame field {x(s), α(s), y(s), β(s)} is called the cone Frenet frame of the curve x(s).
The functions κ(s) and τ(s) are defined as;

$$\kappa(s) = -\left(\frac{1}{2}\right) < x''(s), x''(s) > \tag{14}$$

$$(\tau(s))^2 = < x''(s), x''(s) > -4(\kappa(s))^2 \tag{15}$$

from [15], for any asymtotic orthonormal frame {x(s), α(s), y(s), β(s)} of the curve $x = x(s) : I \to Q^3 \subset E_1{}^4$ with;

$$\begin{aligned} &< x(s), x(s) > = < y(s), y(s) > = < x(s), \alpha(s) > = < x(s), \beta(s) > = 0 \\ &< y(s), \alpha(s) > = < y(s), \beta(s) > = < \alpha(s), \beta(s) > = 0 \end{aligned} \tag{16}$$

$$< x(s), y(s) > = < \alpha(s), \alpha(s) > = < \beta(s), \beta(s) > = 1 \tag{17}$$

the Frenet Formulas read as (13).

We know $\lambda(s) \equiv 0$ if and only if y(s) satisfies (9). Therefore, some authors called the frame satisfying (9) the Cartan frame. Note that equations (9) and (10) are satisfied in any dimension.

Definition 1 The functions κ(s) and τ(s) used in (13) are called the (first) cone curvature and cone torsion (or second cone curvature) of the curve x(s) in $Q^3 \in E_1{}^4$ space, respectively.

Definition 2 A curve x(s) which satisfies that the functions $(\kappa(s)/\tau(s)) = const.$ is called a general helix.

If both κ(s) and τ(s) are constants along x(s), then x(s) is called a circular helix.

If $\kappa(s) = \tau(s) = 0$, then x(s) is called a null cubic.

The characterization of curves in the lightlike cone

Theorem 1 Let $x = x(s): I \to Q^3 \subset E_1{}^4$ be a curve in a 3-dimensional lightlike cone Q^3 of Minkowski 4-space $E_1{}^4$ with arc length parameter s. Let us suppose that c is an arbitrary constant, and;

$$c - x(s) = m_1(s)\alpha(s) + m_2(s)(x(s) + y(s)) + m_3(s)\beta(s) \tag{18}$$

is a lightlike vector in $Q^3 \subset E_1{}^4$ space, then $m_i(s)$ $(1 \leq i \leq 3)$ curvature functions of $x(s)$ are given by;

$$m_1(s) \ = \ 0, \tag{19}$$

$$m_2(s) \ = \ (1/(\kappa(s) - 1)), \tag{20}$$

$$m_3(s) \ = \ ((\sqrt{2}i)/(\kappa(s) - 1)). \tag{21}$$

Proof. Since;

$$c - x(s) \tag{22}$$

is a lightlike vector, we can write;

$$< c - x(s), c - x(s) > = 0. \tag{23}$$

Differentiating the above equation with respect to s and making use of (13), we have;

$$< \alpha(s), c - x(s) > = 0. \tag{24}$$

Taking the derivative of Eq. (24) with respect to s and using (13), we obtain;

$$\kappa(s) < x(s), c - x(s) > - < y(s),\ c - x(s) > = 1. \tag{25}$$

A direct computation shows that the values of the differentiable functions $m_i(s)$ $(1 \leq i \leq 3)$ are as follows;

$$\begin{aligned} m_1(s) \ &= \ 0, \\ m^2(s) \ &= < c - x(s), y(s) > = < c - x(s), x(s) >, \\ m_3(s) \ &= < c - x(s), \beta(s) >. \end{aligned} \tag{26}$$

Considering (26) in (25), we get;

$$m_2(s) = (1/(\kappa(s) - 1)). \tag{27}$$

On the other hand, from (18) and (23), the following is obtained.

$$2{m_2}^2(s) + {m_3}^2(s) = 0 \tag{28}$$

Substituting (27) in (28), we obtain;

$$m_3(s) = \left(\frac{\sqrt{2}i}{\kappa(s)-1}\right) \tag{29}$$

which completes the proof.

Theorem 2 Let $x = x(s): I \to Q^3 \subset E_1^4$ be a curve in a three dimensional lightlike cone Q^3 of Minkowski 4-space E_1^4 with arc length parameter *s*. Then, one may write $m_3(s)$ in a different form, as follows;

$$m_3(s) = \left(\frac{-\kappa'(s)}{\tau(s)(\kappa(s)-1)}\right) \tag{30}$$

where $\tau(s) \neq 0$ and $\kappa(s) \neq 1$.

Proof. From (25), the differentiation gives us the following equation;

$$\kappa'(s) < x(s), c - x(s) > +\tau(s) < \beta(s), c - x(s) >= 0. \tag{31}$$

By making use of (26), (27) and (31), we obtain (30).

Theorem 3 Let $x = x(s): I \to Q^3 \subset E_1^4$ be a curve in a 3-dimensional lightlike cone $Q^3$of Minkowski 4-space E_1^4 with arc length parameter *s*. Then, the cone curvature and cone torsion satisfies the following relation;

$$\kappa(s)=\sqrt{2i\int \tau(s)ds} + c, \quad c \in R. \tag{32}$$

Proof: From (28) and (31), we get (32).

Theorem 4 Let $x = x(s): I \to Q^3 \subset E_1^4$ be a curve in a 3-dimensional lightlike cone Q^3 of Minkowski 4-space E_1^4 with arc length parameter *s*. If the cone curvature $\kappa(s) \neq \sqrt{2i\int \tau(s)ds}+c$ and the cone torsion $\tau(s) \neq 0$, then;

$$m_2(s) = C_1 e^{-\int A(s)ds}, \qquad C_1 \in R \tag{33}$$

where

$$A(s) = \frac{\kappa'(s)\kappa''(s)\tau(s) - \left(\kappa'(s)\right)^2 \tau'(s)}{\left(\kappa'(s)\right)^2 \tau(s) + 2(\tau(s))^3} \tag{34}$$

Proof: Differentiating (28) gives us;

$$2m_2(s)m_2(s)' + m_3(s)m_3'(s) = 0 \tag{35}$$

Then, combining the value of (31) in the above equation, we get;

$$\left[\left[\frac{\left(\kappa'(s)\right)^2 + 2\tau^2(s)}{\tau^2(s)}\right] m_2'(s) + \left[\frac{\kappa'(s)\kappa''(s)\tau(s) - \left(\kappa'(s)\right)^2 \tau'(s)}{\left(\tau(s)\right)^3}\right] m_2(s)\right] m_2(s) \tag{36}$$

or from $m_2(s) \neq 0$ and (34);

$$m_2'(s) + A(s)m_2(s) = 0 \tag{37}$$

The solution of the previous differential equation is;

$$m_2(s) = C_1 e^{-\int A(s)ds}, \qquad C_1 \in R \tag{38}$$

Appendix

1. Let us suppose $x = x(s): I \rightarrow Q^3 \subset E_1^4$ is a curve in a 3-dimensional lightlike cone Q^3 of Minkowski 4-space E_1^4 with arc length parameter *s*. If *c* is an arbitrary constant and;

$$c - x(s) = m_1(s)\alpha(s) + m_2(s)x(s) + m_3(s)(y(s) + \beta(s)) \quad (39)$$

is a lightlike vector in $Q^3 \subset E_1^4$,then the following statements hold.

i) The $m_i(s)(1 \leq i \leq 3)$ curvature functions of $x(s)$ satisfy the following equalities;

$$\begin{aligned} m_2(s) &= 0, \\ m_2(s) &= \frac{(-1)}{1+2\kappa(s)}, \\ m_3(s) &= \frac{2}{1+2\kappa(s)} \end{aligned} \quad (40)$$

or

ii) If $m_3(s) \neq 0$, then $\kappa(s) = -\int \tau(s)ds - C, C \in R$.

If $m_3(s) \neq 0$, then $m_3(s)$ is given by;

$$C_2 e^{-\int B(s)ds}, \ \ C_2 \in R \quad (41)$$

where $B(s) = \frac{2\kappa'(s)}{1+2\kappa(s)}$

iii) If $m_3(s) = 0$, then c is the center point of $Q^3 \subset {E_1}^4$.

2. Let us suppose that $x = x(s): I \rightarrow Q^3 \subset {E_1}^4$ is a curve in a 3-dimensional lightlike cone Q^3 of Minkowski 4-space ${E_1}^4$ with arc length parameter *s*. If *c* is an arbitrary constant and;

$$c - x(s) = m_1(s)\alpha(s) + m_2(s)y(s) + m_3(s)(x(s) + \beta(s)) \quad (42)$$

İs a lightlike vector in $Q^3 \subset {E_1}^4$, then the following statements hold.

i) The $m_i(s)$ $(1 \leq i \leq 3)$ curvature functions of $x(s)$ satisfy the following equalities;

$$\begin{aligned} m_1(s) &= 0, \\ m_2(s) &= \frac{(-1)}{1+2\kappa(s)}, \\ m_3(s) &= \frac{-2}{1+2\kappa(s)} \end{aligned} \quad (43)$$

or

ii) If $m_2(s) \neq 0$, then $\kappa(s) = -2\int\tau(s)ds - C, C \in R$.

If $m_2(s) \neq 0$, then $m_2(s) \neq 0$, is given by;

$$m_2(s) = C_3 e^{-\int D(s)ds}, \ \ C_3 \in R. \quad (44)$$

where $D(s) = \frac{\kappa'(s)}{2+\kappa(s)}$

iii) If $m_2(s) = 0$, then $m_1(s) = m_3(s) = 0$.

3. Let us suppose that $x = x(s): I \to Q^3 \subset E_1{}^4$ is a curve in a 3-dimensional lightlike cone Q^3 of Minkowski 4-space $E_1{}^4$ with arc length parameter s. If c is an arbitrary constant and;

$$c - x(s) = m_1(s)(\alpha(s) + x(s) + y(s)) + m_2(s)\beta(s) \tag{45}$$

is a lightlike vector in $Q^3 \subset E_1{}^4$, then the following statements hold.

i) The $m_i(s)$ $(1 \le i \le 2)$ curvature functions of $x(s)$ satisfy the following equalities;

$$m_1(s) = \frac{1}{\kappa(s)-1} \text{ and } m_2(s) = \frac{\sqrt{2}i}{\kappa(s)-1} \tag{46}$$

ii) The $m_2(s)$ curvature functions of $x(s)$ can be written as the following;

$$m_2(s) = \frac{-\kappa'(s)}{\tau(s)(\kappa(s)-1)} \tag{47}$$

iii) The cone curvature is defined by $\kappa(s) = \sqrt{2}i\int \tau(s)ds - C, C \in R$.

Remark. If we take;

$$c - x(s) = m_1(s)\alpha(s) + m_2(s)(x(s) + y(s)) + \beta(s)) \tag{48}$$

as a lightlike vector, then we obtain;

$$m_1(s) = m_2(s) = 0. \tag{49}$$

Conclusions

We have defined m-curvature functions to obtain some characterizations for a curve in a 3-dimensional null cone. Each solution contains a new result. These results will be useful for many researchers; for example, in physics, a lightlike cone has an extensive usage. Furthermore, from the viewpoint of general relativity and related theories, the physical events space is represented by Minkowski space-time.

Conflict of interests

The authors declare that we do not have any conflicts of interests.

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