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A Comparison between Solving of Two-Dimensional Nonlinear Fredholm Integral Equations of the Second Kind by the Optimal Homotopy Asymptotic Method and Homotopy Perturbation Method

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Abstract

In this article, we present the optimal homotopy asymptotic method (OHAM) and homotopy perturbation method (HPM) for solving 2-dimensional nonlinear Fredholm integral equations of the second kind. A comparison is made between these methods to solve 2-dimensional nonlinear Fredholm integral equations of the second kind. The results show that the presented methods are very powerful and simple techniques in solving 2-dimensional nonlinear Fredholm integral equations of the second kind.

Keywords: Optimal homotopy asymptotic method, homotopy perturbation method, 2-dimensional nonlinear Fredholm integral equations of the second kind

Introduction

Two-dimensional Fredholm integral equations play an important role in many branches of mathematics, physics, mechanics and engineering. Therefore, many different techniques are used to solve 2-dimensional linear and nonlinear Fredholm integral equations. Abdou *et al.* [1] studied the approximate solution to a class of nonlinear 2-dimensional Hammerstein integral equations by the Adomian decomposition and homotopy analysis methods. Chebyshev polynomials were applied by Avazzadeh and Heydari [2] to approximate 2-dimensional linear and nonlinear integral equations of the second kind. Also, Babolian *et al.* [3] used rationalized Haar functions to obtain the solution of 2-dimensional nonlinear integral equations of the second kind. Guoqiang and Jiong [4] presented the Nystrom method to solve 2-dimensional nonlinear Fredholm integral equations. Heydari *et al.* [5] introduced integral mean value theorem to solve Fredholm integral equations of the second kind and high dimensional problems. Xie and Lin [6] introduced a collocation method to solve 2-dimensional Fredholm integral equations of the second kind.

The homotopy perturbation method (HPM) was established and improved by He [7-12] based on a combination of the homotopy technique from topology and the perturbation method. Many researchers have studied this method to linear and nonlinear problems. For example, to nonlinear oscillators with discontinuities [9], boundary value problems [12], Volterra's integro-differential equation [13], systems of nonlinear coupled equations [14] and inverse space-dependent heat source [15].

Recently, the optimal homotopy asymptotic method (OHAM) was introduced and developed by Marinca *et al.* [16-19]. This method was applied in solving many types of linear and nonlinear problems for integral and different equations such as Hashmi *et al.* [20] who used the OHAM for solutions of weakly singular Volterra integral equations. Anakira *et al.* [21] applied the OHAM to find the algorithm of approximate analytic solution of delay differential equations. Almousa and Ismail [22] employed this method for finding approximate numerical solutions of linear Fredholm integral equations of the first kind and Mabood *et al.* [23,24] used this method for heat transfer.

The aim of this article is to present the OHAM and HPM for solving 2-dimensional nonlinear Fredholm integral equations of the second kind. Consider the 2-dimensional nonlinear Fredholm integral equation of the second kind as follows [4];

$$g(x,t) = f(x,t) + \int_a^b \int_c^d k(x,t,s,y,g(s,y))dsdy,\ (x,t) \in [a,b] \times [c,d] \tag{1}$$

where $g(x,t)$ is a continuous function, $k(x,t,s,y,g(s,y))$ and $f(x,t)$ are continuous functions on $\Omega = [a,b] \times [c,d]$.

Analysis of the HPM for 2-dimensional nonlinear Fredholm integral equations of the second kind

To explain the HPM, we consider Eq. (1) as;

$$N(g) = g(x,t) - f(x,t) - \int_a^b \int_c^d k(x,t,s,y,g(s,y))dsdy = 0, \tag{2}$$

where N is an integral operator. Using the HPM technique, we have a homotopy $v(x,y,p): \Omega \times [0,1] \to R$ for an embedding parameter $p \in [0,1]$ which satisfies;

$$H(v,p) = (1-p)[v(x,t,p) - g_0(x,t)] + p\left[v(x,t,p) - f(x,t) - \int_a^b \int_c^d k(x,t,s,y,v(s,y,p))dsdy\right]. \tag{3}$$

By choosing a convex homotopy $H(v,p) = 0$, we obtain;

$$(1-p)[v(x,t,p) - g_0(x,t)] = -p\left[v(x,t,p) - f(x,t) - \int_a^b \int_c^d k(x,t,s,y,v(s,y,p))dsdy\right]. \tag{4}$$

From Eq. (4), when $p = 0$ and $p = 1$ it holds that;

$$H(v,0) = v(x,t,0) - g_0(x,t) = 0$$

i.e.;

$$v(x,t,0) = g_0(x,t), \tag{5}$$

and

$$H(v,1) = v(x,t,1) - f(x,t) - \int_a^b \int_c^d k(x,t,s,y,v(s,y,1))\, dsdy = 0$$

i.e.;

$$v(x,t,1) = f(x,t) + \int_a^b \int_c^d k(x,t,s,y,v(s,y,1))\, dsdy, \tag{6}$$

respectively. We can obtain the HPM solution of Eq. (3) in the form of power series;

$$v(x,t,p) = \sum_{m=0}^{\infty} v_m(x,t)p^m. \tag{7}$$

When the series (7) of $v(x,t,p)$ converges at $p = 1$, then;

$$g(x,t) = \lim_{p\to 1} v(x,t,p) = \sum_{m=0}^{\infty} v_m(x,t). \tag{8}$$

Inserting Eq. (7) into Eq. (4), one can obtain;

$$\sum_{m=0}^{\infty} v_m(x,t)p^m = g_0(x,t) - pg_0(x,t) + pf(x,t) + p\int_a^b \int_c^d k(x,t,s,y,\sum_{m=1}^{\infty} v_{m-1}(s,y)p^m)dsdy. \quad (9)$$

For simplicity we choose $v_0(x,t) = g_0(x,t) = f(x,t)$, and replace $v_0(x,t)$ into Eq. (7) and then equate the coefficients of like powers of p.

Analysis of the OHAM for 2-dimensional nonlinear Fredholm integral equations of the second kind

To explain the OHAM, we rewrite Eq. (1) as follows;

$$g(x,t) - f(x,t) - \int_a^b \int_c^d k(x,t,s,y,g(s,y))dsdy = 0. \quad (10)$$

According to the OHAM, a set of equations for an embedding parameter $p \in [0,1]$ is given by;

$$L\big(g(x,t,p)\big) = g(x,t)$$

$$N\big(g(x,t,p)\big) = -\int_a^b \int_c^d k(x,t,s,y,g(s,y))dsdy,$$

which satisfies;

$$(1-p)\big[L\big(g(x,t,p)\big) - f(x,t)\big] = H(p)\big[L(g(x,t,p)) - f(x,t) + N\big(g(x,t,p)\big)\big], \quad (11)$$

where $g(x,t,p): \Omega \times [0,1] \to R$ and $H(p) = \sum_{j=1}^{m} c_j p^j$ is a nonzero auxiliary function for $p \neq 0$ and $H(0) = 0$, where $c_j, j = 1,2,\ldots$ are constants. When $p = 0$ and $p = 1$, then;

$$g(x,t,0) = g_0(x,t), \qquad g(x,t,1) = g(x,t) \quad (12)$$

respectively. By using Taylor's series, we expand the solution about p as follows;

$$g\big(x,t,p,c_j\big) = g_0(x,t) + \sum_{m=1}^{\infty} g_m\big(x,t,c_j\big)p^m, j = 1,2,\ldots \quad (13)$$

When $p = 1$, then Eq. (13) becomes;

$$g\big(x,t,1,c_j\big) = g_0(x,t) + \sum_{m=1}^{\infty} g_m\big(x,t,c_j\big), j = 1,2,\ldots. \quad (14)$$

Using Eq. (13) into Eq. (11), we obtain the zeroth order, first order and *m*th order problems as follows;

$$O(p^0): g_0(x,t) = f(x,t). \quad (15)$$

$$O(p^1): g_1(x,t) = -c_1 \int_a^b \int_c^d k(x,t,s,y,g_0(s,y))dsdy. \quad (16)$$

$$O(p^m): g_m(x,t) = g_{m-1}(x,t) - c_m \int_a^b \int_c^d k(x,t,s,y,g_0(s,y))dsdy$$

$$+ \sum_{i=1}^{m-1} c_i\, g_{m-i}(x,t) + \sum_{i=1}^{m-1} c_i\, N_{m-i}\big(g_0(x,t), g_1(x,t), \ldots, g_{m-1}(x,t)\big),\ m = 2,3,\ldots, \quad (17)$$

where $N_m(g_0(x,t), g_1(x,t), \ldots, g_m(x,t))$ are the coefficient of p^m in the expansion of $N(g(x,t,p))$ about p;

$$N\left(g(x,t,p,c_j)\right) = N_0\left(g_0(x,t)\right) + \sum_{m=1}^{\infty} N_m\left(g_0(x,t), g_1(x,t), \ldots, g_m(x,t)\right)p^m, j = 1,2, \ldots \tag{18}$$

The result of the *m*th-order approximations are given by;

$$g^m(x,t,c_j) = g_0(x,t) + \sum_{k=1}^{m} g_k(x,t,c_j), j = 1,2, \ldots, m. \tag{19}$$

Using Eq. (19) into Eq. (1), we can obtain the residual for $j = 1,2, \ldots$;

$$\mathcal{R}(x,t,c_j) = g^m(x,t,c_j) - f(x,t) - \int_a^b \int_c^d k(x,t,s,y,g^m(s,y))dsdy. \tag{20}$$

If $\mathcal{R}(x,t,c_j) = 0$, then $g^m(x,t,c_j)$ will be the exact solution. For the determination of $c_1, c_2, c_3, \ldots$, we can apply least squares method as follows;

$$J(c_j) = \int_a^b \int_c^d \mathcal{R}^2(x,t,c_j)dxdt, \tag{21}$$

By using Galerkin's method, we have;

$$\frac{dJ}{dc_m} = \int_a^b \int_c^d \mathcal{R}(x,t,c_j)\frac{\partial \mathcal{R}}{\partial c_m}dxdt, \tag{22}$$

and

$$\frac{dJ}{dc_1} = \frac{dJ}{dc_2} = \cdots = \frac{dJ}{dc_m} = 0, \tag{23}$$

with the values $c_1, c_2, c_3, \ldots, c_m$, the *m*th-order solution is well determined.

Numerical examples and discussion

In this section, some examples of 2-dimensional nonlinear Fredholm integral equations of the second kind are solved to show the applicability and accuracy of both the OHAM and HPM for solving this type of integral equations and comparing the results in OHAM with HPM. Maple software with long format and double accuracy was used to carry out the computations.

Example 1 Let us consider the 2-dimensional nonlinear Fredholm integral equation of the second kind with the exact solution $g(x,t) = \frac{1}{(1+x+t)^2}$ [3].

$$g(x,t) = \frac{1}{(1+x+t)^2} - \frac{x}{6(1+t)} + \int_0^1 \int_0^1 \frac{x}{1+t}(1+y+s)(g(y,s))^2 dyds, \tag{24}$$

To derive the solution Eq. (24) by using the OHAM, let;

$$L\left(g(x,t,p)\right) = g(x,t) \tag{25}$$

$$N\left(g(x,t,p)\right) = -\int_0^1 \int_0^1 \frac{x}{1+t}(1+y+s)(g(y,s))^2 dyds, \tag{26}$$

which satisfies;

$$(1-p)\left(L\big(g(x,t,p)\big)-\left(\frac{1}{(1+x+t)^2}-\frac{x}{6(1+t)}\right)\right)$$
$$=H(p)\left[L\big(g(x,t,p)\big)-\left(\frac{1}{(1+x+t)^2}-\frac{x}{6(1+t)}\right)-N\big(g(x,t,p)\big)\right]. \quad (27)$$

By using Eqs. (15) - (17), we obtain a series of problems;

$$O(p^0): g_0(x,t)=\left(\frac{1}{(1+x+t)^2}-\frac{x}{6(1+t)}\right). \quad (28)$$

$$O(p^1): g_1(x,t)=-c_1\int_0^1\int_0^1\frac{x}{1+t}\,(1+y+s)(g_0(y,s))^2dyds. \quad (29)$$

$$O(p^2): g_2(x,t)=(1+c_1)g_1(x,t)$$
$$-2c_1\int_0^1\int_0^1\frac{x}{1+t}\,(1+y+s)\;g_0(y,s)g_1(y,s)dyds\;-c_2\int_0^1\int_0^1\frac{x}{1+t}\,(1+y+s)(g_0(y,s))^2dyds. \quad (30)$$

Hence, the solutions are;

$$O(p^0): g_0(x,t)=\left(\frac{1}{(1+x+t)^2}-\frac{x}{6(1+t)}\right).$$

$$O(p^1): g_1(x,t)=\frac{-0.1199239063c_1x}{1+t}.$$

$$O(p^2): g_2(x,t)=\frac{-0.1199239063c_1(1+c_1)x}{1+t}+\frac{0.02651698096{c^2}_1x}{1+t}-\frac{0.1199239063c_2x}{1+t}.$$

Now, adding the above equations $g_0(x,t)$, $g_1(x,t)$ and $g_2(x,t)$, we have;

$$g(x,t)=\frac{1}{(1+x+t)^2}-\frac{x}{6(1+t)}-\frac{0.1199239063c_1x}{1+t}-\frac{0.1199239063c_1(1+c_1)x}{1+t}+\frac{0.02651698096{c^2}_1x}{1+t}-\frac{0.1199239063c_2x}{1+t}. \quad (31)$$

By using Galerkin's method, we can find the values c_1 and c_2, yielding;

$c_1=-1.389770162$, $c_2=-1.504385978$.

Using the above values c_1 and c_2 in Eq. (31), the second order approximate OHAM solution is given by;

$$g(x,t)=\frac{1}{(1+x+t)^2}-\frac{x}{6(1+t)}+\frac{0.1666666667}{1+t}. \quad (32)$$

To derive solution Eq. (24) by using the HPM, let a convex homotopy be;

$$H(v,p)=(1-p)[v(x,t,p)-g_0(x,t)]$$
$$+p\left[v(x,t,p)-\left(\frac{1}{(1+x+t)^2}-\frac{x}{6(1+t)}\right)-\int_0^1\int_0^1\frac{x}{1+t}\,(1+y+s)(v(y,s,p))^2dyds\right]=0. \quad (33)$$

Substituting Eq. (7) into Eq. (33), one can get;

$O(p^0): v_0(x,t) = \frac{1}{(1+x+t)^2}.$

$O(p^1): v_1(x,t) = -\frac{x}{6(1+t)} + \int_0^1 \int_0^1 \frac{x}{1+t}\,(1+y+s)(v_0(y,s))^2 dyds = 0.$

$O(p^2): v_2(x,t) = 2\int_0^1 \int_0^1 \frac{x}{1+t}\,(1+y+s)\; v_0(y,s)v_1(y,s)dyds = 0.$

$O(p^3): v_3(x,t) = \int_0^1 \int_0^1 \frac{x}{1+t}\,(1+y+s)\;\left(2v_0(y,s)v_2(y,s) + \left(v_1(y,s)\right)^2\right)dyds = 0.$

Again, continuing the procedure, one can obtain;

$v_4(x,t) = v_5(x,t) = v_6(x,t) = \cdots = 0.$

Thus, the solution is given by;

$$g(x,t) = \sum_{m=0}^{\infty} v_m(x,t) = \frac{1}{(1+x+t)^2} + 0 + 0 + \cdots = \frac{1}{(1+x+t)^2}\,. \tag{34}$$

This is the exact solution.

Table 1 shows the comparison of the absolute errors obtained by HPM and OHAM. These results show the efficiency of the methods for 2-dimensional nonlinear Fredholm integral equation of the second kind.

Table 1 Comparison of absolute errors of Example 1.

(x,t)	Errors of HPM	Errors of OHAM		
		Zeroth order	First order	Second order
(0.0,0.0)	0	0	0	0
(0.1,0.1)	0	0.01515151515	0	0
(0.2,0.2)	0	0.02777777778	1×10^{-11}	1×10^{-11}
(0.3,0.3)	0	0.03846153846	1×10^{-11}	1×10^{-11}
(0.4,0.4)	0	0.04761904762	1×10^{-11}	1×10^{-11}
(0.5,0.5)	0	0.05555555556	0	0
(0.6,0.6)	0	0.06250000000	1×10^{-11}	1×10^{-11}
(0.7,0.7)	0	0.06862745098	2×10^{-11}	2×10^{-11}
(0.8,0.8)	0	0.07407407407	1×10^{-11}	1×10^{-11}
(0.9,0.9)	0	0.07894736842	1×10^{-11}	1×10^{-11}
(1.0,1.0)	0	0.08333333333	2×10^{-11}	2×10^{-11}

Example 2 Let us consider the 2-dimensional nonlinear Fredholm integral equation of the second kind with the exact solution $g(x,t) = x\cos t$ [2].

$$g(x,t) = x\cos(t) - \frac{1}{8} - \frac{7}{24}\sin(1)\cos(1) - \frac{1}{12}\cos^2(1) + \int_0^1 \int_0^1 (s+y)(g(y,s))^2 dyds, \tag{35}$$

To derive the solution Eq. (35) by using the OHAM, let;

$$L\big(g(x,t,p)\big) = g(x,t) \tag{36}$$

$$N\big(g(x,t,p)\big) = -\int_0^1\int_0^1 (s+y)(g(y,s))^2 dyds, \tag{37}$$

which satisfies;

$$(1-p)\left[L\big(g(x,t,p)\big) - \left(x\cos(t) - \tfrac{1}{8} - \tfrac{1}{12}\cos^2(1) + \tfrac{7}{24}\sin(1)\cos(1)\right)\right]$$
$$= H(p)\left[L\big(g(x,t,p)\big) - (x\cos(t) - \tfrac{1}{8} - \tfrac{1}{12}\cos^2(1) + \tfrac{7}{24}\sin(1)\cos(1)) - N\big(g(x,t,p)\big)\right]. \tag{38}$$

By using Eqs. (15) - (17), we obtain a series of problems;

$$O(p^0): g_0(x,t) = \left(x\cos(t) - \tfrac{1}{8} + \tfrac{7}{24}\sin(1)\cos(1) - \tfrac{1}{12}\cos^2(1)\right). \tag{39}$$

$$O(p^1): g_1(x,t) = -c_1\int_0^1\int_0^1 (s+y)(g_0(y,s))^2 dyds. \tag{40}$$

$$O(p^2): g_2(x,t) = (1+c_1)g_1(x,t)$$
$$-2c_1\int_0^1\int_0^1 (s+y)g_0(y,s)g_1(y,s)dyds - c_2\int_0^1\int_0^1 (s+y)(g_0(y,s))^2 dyds. \tag{41}$$

Hence, the solutions are;

$$O(p^0): g_0(x,t) = \left(x\cos(t) - \tfrac{1}{8} + \tfrac{7}{24}\sin(1)\cos(1) - \tfrac{1}{12}\cos^2(1)\right).$$

$$O(p^1): g_1(x,t) = -0.0956258238c_1.$$

$$O(p^2): g_2(x,t) = -0.0956258238\,(1+c_1)c_1 + 0.03623145488\,{c_1}^2 - 0.0956258238\,c_2.$$

Now, adding the above equations $g_0(x,t)$, $g_1(x,t)$ and $g_2(x,t)$, we have;

$$g(x,t)$$
$$= x\cos(t) - \tfrac{1}{8} + \tfrac{7}{24}\sin(1)\cos(1) - \tfrac{1}{12}\cos^2(1) - 0.0956258238c_1 + 0.03623145488\ {c_1}^2$$
$$- 0.0956258238(1+c_1)c_1 - 0.0956258238\,c_2. \tag{42}$$

By using Galerkin's method, we can find the values c_1 and c_2, yielding;

$c_1 = -2.948294491$, $c_2 = -2.450686585$.

Using the above values c_1 and c_2 in Eq. (42), the second order approximate OHAM solution is given by;

$$g(x,t) = x\cos(t) + 4.5\times 10^{-10}. \tag{43}$$

To derive the solution Eq. (35) by using the HPM, let a convex homotopy as;

$$H(v,p)$$
$$= (1-p)[v(x,t,p) - g_0(x,t)] + p\ \left[v(x,t,p) - \left(x\cos(t) - \tfrac{1}{8} - \tfrac{7}{24}\sin(1)\cos(1) - \tfrac{1}{12}\cos^2(1)\right)\right.$$
$$\left. - \int_0^1\int_0^1 (s+y)(v(y,s,p))^2 dyds\right] = 0. \tag{44}$$

Substituting Eq. (7) into Eq. (44), one can get;

$O(P^0): v_0(x,t) = x\cos(t).$

$O(p^1): v_1(x,t)$
$$= -\frac{1}{8} - \frac{7}{24}\sin(1)\cos(1) - \frac{1}{12}\cos^2(1) + \frac{1}{12}\cos^2(1) + \int_0^1\int_0^1 (s+y)(v_0(y,s))^2\, dyds = 0.$$

$O(p^2): v_2(x,t) = 2\int_0^1\int_0^1 (s+y)\ v_0(y,s)v_1(y,s)dyds = 0.$

$O(p^3): v_3(x,t) = \int_0^1\int_0^1 (s+y)\ \left(2v_0(y,s)v_2(y,s) + \left(v_1(y,s)\right)^2\right)dyds = 0.$

Again, continuing the procedure, one can obtain;

$v_4(x,t) = v_5(x,t) = v_6(x,t) = \cdots = 0.$

Thus, the solution is given by;

$$g(x,t) = \sum_{m=0}^{\infty} v_m(x,t) = x\cos(t) + 0 + 0 + \cdots = x\cos(t)\,. \tag{45}$$

This is the exact solution.

Table 2 shows a comparison of the absolute errors obtained by HPM and OHAM. **Table 3** shows some numerical results of Example 2 calculated according to HPM and OHAM. This example shows that the OHAM and HPM solutions are very close to the exact solution.

Table 2 Comparison of absolute errors of Example 2.

(x,t)	Errors of HPM	Errors of OHAM		
		Zeroth order	First order	Second order
(0.0,0.0)	0	0.2819330900	4.5×10^{-10}	4.5×10^{-10}
(0.1,0.1)	0	0.2819330900	4.5×10^{-10}	4.5×10^{-10}
(0.2,0.2)	0	0.2819330900	4.5×10^{-10}	4.5×10^{-10}
(0.3,0.3)	0	0.2819330900	4.5×10^{-10}	4.5×10^{-10}
(0.4,0.4)	0	0.2819330900	4.5×10^{-10}	4.5×10^{-10}
(0.5,0.5)	0	0.2819330900	4.5×10^{-10}	4.5×10^{-10}
(0.6,0.6)	0	0.2819330900	4.5×10^{-10}	4.5×10^{-10}
(0.7,0.7)	0	0.2819330900	4.5×10^{-10}	4.5×10^{-10}
(0.8,0.8)	0	0.2819330900	4.5×10^{-10}	4.5×10^{-10}
(0.9,0.9)	0	0.2819330900	4.5×10^{-10}	4.5×10^{-10}
(1.0,1.0)	0	0.2819330900	4.5×10^{-10}	4.5×10^{-10}

Table 3 Numerical results of Example 2.

(x,t)	Exact solution	Second order HPM solution	Second order OHAM solution	First order OHAM solution
(0.0,0.0)	0	0	-4.5×10^{-10}	-4.5×10^{-10}
(0.1,0.1)	0.09950041653	0.09950041653	0.09950041698	0.09950041608
(0.2,0.2)	0.19601331560	0.19601331560	0.19601331600	0.19601331520
(0.3,0.3)	0.28660094670	0.28660094670	0.28660094720	0.28660094620
(0.4,0.4)	0.36842439760	0.36842439760	0.36842439800	0.36842439720
(0.5,0.5)	0.43879128100	0.43879128100	0.43879128140	0.43879128060
(0.6,0.6)	0.49520136890	0.49520136890	0.49520136940	0.49520136840
(0.7,0.7)	0.53538953110	0.53538953110	0.53538953160	0.53538953060
(0.8,0.8)	0.55736536740	0.55736536740	0.55736536780	0.55736536700
(0.9,0.9)	0.55944897150	0.55944897150	0.55944897200	0.55944897100
(1.0,1.0)	0.54030230590	0.54030230590	0.54030230640	0.54030230540

Example 3 Let us consider the 2-dimensional nonlinear Fredholm integral equation of the second kind with the exact solution $g(x,t) = x\cos t$ [5].

$$g(x,t) = x\cos(t) + \frac{1}{20}(\cos^4(1) - 1) - \frac{1}{12}\sin(1)(\cos^2(1) + 2) + \int_0^1\int_0^1 (y\sin(s) + 1)(g(y,s))^3 dy\ ds, \tag{46}$$

To derive the solution Eq. (46) by using the OHAM, let;

$$L\big(g(x,t,p)\big) = g(x,t) \tag{47}$$

$$N\big(g(x,t,p)\big) = -\int_0^1\int_0^1 (y\sin(s) + 1)\ )(g(y,s))^3 dyds, \tag{48}$$

which satisfies;

$$(1-p)\Big[L\big(g(x,t,p)\big) - \Big(x\cos(t) + \frac{1}{20}(\cos^4(1) - 1) - \frac{1}{12}\sin(1)(\cos^2(1) + 2)\Big)\Big]$$
$$= H(p)\Big[L\big(g(x,t,p)\big) - \Big(x\cos(t) + \frac{1}{20}(\cos^4(1) - 1) - \frac{1}{12}\sin(1)(\cos^2(1) + 2)\Big) - N\big(g(x,t,p)\big)\Big]. \tag{49}$$

By using Eqs. (15) - (17), we obtain a series of problems;

$$O(p^0): g_0(x,t) = \Big(x\cos(t) + \frac{1}{20}(\cos^4(1) - 1) - \frac{1}{12}\sin(1)(\cos^2(1) + 2)\Big). \tag{50}$$

$$O(p^1): g_1(x,t) = -c_1\int_0^1\int_0^1 (y\sin(s) + 1)(g_0(y,s))^3 dyds. \tag{51}$$

$$O(p^2): g_2(x,t) = (1 + c_1)g_1(x,t)$$
$$-3c_1\int_0^1\int_0^1 (y\sin(s) + 1)g_0{}^2(y,s)g_1(y,s)dy\,ds - c_2\int_0^1\int_0^1 (y\sin(s) + 1)(g_0(y,s))^3 dyds. \tag{52}$$

Hence, the solutions are;

$O(p^0): g_0(x,t) = \left(x\cos(t) + \frac{1}{20}(\cos^4(1) - 1) - \frac{1}{12}\sin(1)(\cos^2(1) + 2)\right).$

$O(p^1): g_1(x,t) = -0.07089002607c_1.$

$O(p^2): g_2(x,t) = -0.07089002607c_1\,(1 + c_1) + 0.03032616165\,{c_1}^2 - 0.07089002607\,c_2.$

Now, adding the above equations $g_0(x,t)$, $g_1(x,t)$ and $g_2(x,t)$, we have;

$$g(x,t) = x\cos(t) + \frac{1}{20}(\cos^4(1) - 1) - \frac{1}{12}\sin(1)\,(\cos^2(1) + 2) - 0.07089002607c_1 + 0.03032616165\,{c_1}^2 - 0.07089002607c_1\,(1 + c_1) - 0.07089002607c_2\,. \quad (53)$$

By using Galerkin's method, we can find the values c_1 and c_2, yielding;

$c_1 = -1.184796567,\;\; c_2 = -1.345964673.$

Using the above values c_1 and c_2 in Eq. (53), the second order approximate OHAM solution is given by;

$$g(x,t) = x\cos(t) - 3.9 \times 10^{-10}. \quad (54)$$

To derive the solution Eq. (46) by using the HPM, let a convex homotopy as;

$$H(v,p) = (1-p)[v(x,t,p) - g_0(x,t)] + p\left[v(x,t,p) - \left(\frac{1}{20}(\cos^4(1) - 1) - \frac{1}{12}\sin(1)(\cos^2(1) + 2)\right) - \int_0^1\int_0^1 (y\sin(s) + 1)(v(y,s,p))^3 dyds\right] = 0. \quad (55)$$

Substituting Eq. (7) into Eq. (55), one can get;

$O(p^0): v_0(x,t) = x\cos(t).$

$O(p^1): v_1(x,t) = \frac{1}{20}(\cos^4(1) - 1) - \frac{1}{12}\sin(1)(\cos^2(1) + 2) + \int_0^1\int_0^1 (y\sin(s) + 1)(g_0(y,s))^3\,dyds = 0.$

$O(p^2): v_2(x,t) = 3\int_0^1\int_0^1\,(y\sin(s) + 1))\,\,(v_0(y,s))^2 v_1(y,s)dyds = 0.$

$O(p^3): v_3(x,t) = 3\int_0^1\int_0^1 (\,y\sin(s) + 1)\,\left(\left(v_0(y,s)\right)^2 v_2(y,s) + v_0(y,s)\left(v_1(y,s)\right)^2\right)dyds = 0.$

Again, continuing the procedure, one can obtain;

$v_4(x,t) = v_5(x,t) = v_6(x,t) = \cdots = 0.$

Thus, the solution is given by;

$$g(x,t) = \sum_{m=0}^{\infty} v_m(x,t) = x\cos(t) + 0 + 0 + \cdots = x\cos(t)\,. \quad (56)$$

This is the exact solution.

Table 4 shows some numerical results of Example 3 calculated according to the OHAM and HPM. It is clear that the solutions obtained by using HPM and OHAM are nearly identical to the exact solution.

Table 4 Comparison of absolute errors of Example 3.

(x,t)	Exact solution	Absolute errors of methods		
		Second order HPM	First order OHAM	Second order OHAM
(0.0,0.0)	0	0	0.1224644938	3.9×10^{-10}
(0.1,0.1)	0.09950041653	0	0.1224644938	3.9×10^{-10}
(0.2,0.2)	0.1960133156	0	0.1224644938	3.9×10^{-10}
(0.3,0.3)	0.2866009467	0	0.1224644938	3.9×10^{-10}
(0.4,0.4)	0.3684243976	0	0.1224644938	3.9×10^{-10}
(0.5,0.5)	0.4387912810	0	0.1224644938	3.9×10^{-10}
(0.6,0.6)	0.4952013689	0	0.1224644938	3.9×10^{-10}
(0.7,0.7)	0.5353895311	0	0.1224644938	3.9×10^{-10}
(0.8,0.8)	0.5573653674	0	0.1224644938	3.9×10^{-10}
(0.9,0.9)	0.5594489715	0	0.1224644938	3.9×10^{-10}
(1.0,1.0)	0.5403023059	0	0.1224644938	3.9×10^{-10}

Conclusions

In this article, the OHAM and HPM have been described to solve 2-dimensional nonlinear Fredholm integral equations of the second kind. Three examples were given to show the applicability and accuracy of the presented methods for solving this type of equation. The results obtained by HPM were compared with OHAM. It is clear that the solutions obtained by using HPM and OHAM are nearly identical to the exact solution and it is shown that the presented methods are accurate, effective and simple for solving these types of equations.

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