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Study of Functional Variable Method for Finding Exact Solutions of Nonlinear Evolution Equations

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Abstract

A direct method, called the functional variable method, has been used to construct the exact solutions of nonlinear evolution equations (NLEEs) in mathematical physics. To illustrate the validity and advantages of this method, the (2+1) dimensional Boussinesq-Kadomtsev-Petviashvili (BKP) equations and the new coupled Konno-Oono (KO) equations are considered. The obtained solutions contain an explicit function of the variables in the considered equations. It has been shown that the method provides a powerful mathematical tool for solving NLEEs in mathematical physics and engineering fields without the help of a computer algebra system.

Keywords: Functional variable method, BKP equations, KO equations, exact solution, NLEEs

Introduction

Phenomena in the real world are often described by nonlinear evolution equations (NLEEs). To understand the physical mechanism of phenomena in nature, described by NLEEs, exact solutions for the NLEEs have to be explored. Exact solutions of NLEEs play an important role in the proper understanding of qualitative features of many phenomena and processes in various areas of natural science. It is significant that many equations of physics, chemistry, and biology contain empirical parameters or empirical functions. Exact solutions allow researchers to design and run experiments, by creating appropriate natural conditions, to determine these parameters or functions. Therefore, searching for exact solutions of NLEEs play an important role in the study of physical phenomena and gradually becomes one of the most important and significant tasks. However, not all equations posed of these models are solvable. Thus, new methods for deriving exact solutions for the governing equations have to be developed. As a result, many new techniques have been successfully developed by diverse groups of mathematicians and physicists, such as, the functional variable method [1-9], the modified simple equation method [10-13], the Kudryashov method [14], the Exp-function method [15-18], the Homotopy perturbation method [19,20], the (G'/G)-expansion methods [21-25], the traveling wave hypothesis [26], the sine-cosine method [27], the $\exp(-\Phi(\xi))$-expansion method [28], transformed rational function method [29], multiple exp-function algorithm [30], generalized Hirota bilinear method [31,32], homotopy analysis method [33-38], and so on.

In soliton theory, there are many famous methods and skills to deal with the problem of solitary wave solutions for NELEs. With the development of computer science, recently, direct searching for exact solitary wave and soliton like solutions to NLEEs has attracted much attention. This is due to the availability of symbolic computation systems like Maple, Matlab or Mathematica which enable us to perform complex and tedious computations on the computer.

From our point of view, all the methods have some merits and detriments with respect to the problem considered and there is no unified method that can be used to deal with all types of NLEEs. That is why anytime an improvement is made on a particular method to allow it to recover some new solutions to the NLEEs, it is always welcomed.

Based on the observation that it has been a successful idea to generate exact solutions of nonlinear wave equations in mathematical physics by reducing partial differential equations into ordinary differential equations, recently, Zerarka *et al*. [1] proposed a new direct method called the functional variable method.

The objective of this article is to apply the functional variable method to construct the exact solutions for nonlinear evolution equations in mathematical physics via the BKP equation and KO equation.

Algorithm of the functional variable method

Suppose that a nonlinear equation, say in 2 independent variables x and t is given;

$$F\left(u,u_t,u_x,u_{tt},u_{xx},u_{xt},\cdots\cdots\right)=0, \tag{1}$$

where F is a polynomial in $u(x, t)$ and its partial derivatives.

The main steps of this method are as follows [1-9]:

Step 1 To find the traveling wave solutions of Eq. (1) we introduce the wave variable $\xi = x \pm \omega t$, so that $u(x,t)=u(\xi)$, where $\omega \in \Re - \{0\}$ is the wave velocity, to reduce Eq. (1) to the following ordinary differential equation (ODE);

$$X\left(u,u_\xi,u_{\xi\xi},u_{\xi\xi\xi},\cdots\cdots\right)=0, \tag{2}$$

where X is a polynomial in $u(\xi)$ and its derivatives, whereas $u_\xi=\frac{du}{d\xi}, u_{\xi\xi}=\frac{d^2u}{d\xi^2}$, and so on.

Step 2 We make a transformation in which the unknown function $u(\xi)$ is considered as a functional variable in the form;

$$u_\xi = Y(u), \tag{3}$$

and some successive derivatives of $u(\xi)$ are as follows;

$$\begin{aligned}
u_{\xi\xi} &= YY' = \frac{1}{2}\left(Y^2\right)', \\
u_{\xi\xi\xi} &= \frac{1}{2}\left(Y^2\right)''Y = \frac{1}{2}\left(Y^2\right)''\sqrt{Y^2}, \\
u_{\xi\xi\xi\xi} &= \frac{1}{2}\left((Y^2)'''Y^2 + \frac{1}{2}\left(Y^2\right)''\left(Y^2\right)'\right), \\
&\dots\ \dots\ \dots\ \dots\ \dots
\end{aligned} \tag{4}$$

where $Y'=\frac{dY}{du}, Y''=\frac{d^2Y}{du^2}$ and so on.

Step 3 We substitute Eqs. (3) and (4) into Eq. (2) to reduce it to the following ODE;

$$Q(u, Y, Y', Y'', \cdots\cdots\cdots) = 0\,, \tag{5}$$

Step 4 The key idea of Eq. (5) is of special importance because it admits analytical solutions for a large class of nonlinear wave equations. After integration, the Eq. (5) provides the expression of Y, and this in turn together with (3) gives the appropriate solutions to the original wave equations. In order to illustrate how the method works we examine some examples in the following section which are already treated by other methods.

Applications of the functional variable method

Example 1 The (2+1) dimensional Boussinesq-Kadomtsev-Petviashvili equation: Now we will apply the functional variable method to find exact solutions and then the solitary wave solutions to the BKP equations in the form [39];

$$\begin{aligned} u_y &= q_x \\ v_x &= q_y \\ q_t &= q_{xxx} + q_{yyy} + 6(qu)_x + 6(qv)_y \end{aligned} \tag{6}$$

Now let us suppose the traveling wave transformation equation is;

$$u(x, y, t) = u(\xi)\,,\; v(x, y, t) = v(\xi)\,,\; q(x, y, t) = q(\xi)\,,\; \xi = x + y - \omega t\,. \tag{7}$$

Eq. (7) reduces Eq. (6) to the following ODEs;

$$u_\xi = q_\xi\,, \tag{8}$$

$$v_\xi = q_\xi\,, \tag{9}$$

$$-\omega q_\xi = q_{\xi\xi\xi} + q_{\xi\xi\xi} + 6(qu)_\xi + 6(qv)_\xi\;. \tag{10}$$

Integrating Eqs. (8) - (10) once with respect to ξ, setting the constants of integration to zero, yields the following simplified forms;

$$u = q\,, \tag{11}$$

$$v = q\,, \tag{12}$$

$$-\omega q = 2q_{\xi\xi} + 6qu + 6qv\;. \tag{13}$$

Combining Eqs. (11) and (12) with Eq. (13) yields;

$$2q_{\xi\xi} = -\omega q - 12q^2\,. \tag{14}$$

Following Eq. (4), it is effortless to deduce from Eq. (14) an expression for the function $Y(q)$;

$$\left(Y^2\right)' = -\omega q - 12q^2. \tag{15}$$

After integrating Eq. (15) regarding the constant of integration to zero, yields;

$$Y(q) = \sqrt{-\frac{\omega}{2}}\, q\, \sqrt{\left(1+\frac{8}{\omega}q\right)}. \tag{16}$$

Now applying Eq. (3) into Eq. (16), we obtain;

$$q_\xi = \sqrt{-\frac{\omega}{2}}\, q\, \sqrt{\left(1+\frac{8}{\omega}q\right)}. \tag{17}$$

Separating the variables in Eq. (17) and then integrating, by setting the constant of integration to zero, we obtain;

$$\int \frac{dq}{q\sqrt{\left(1+\frac{8}{\omega}q\right)}} = \sqrt{-\frac{\omega}{2}}\, d\xi. \tag{18}$$

After completing the integration of Eq. (18), we have the following exact traveling wave solutions of BKP equations:

Case 1 If $\omega < 0$, we obtain the following hyperbolic traveling wave solutions;

$$q_1(x,y,t) = u_1(x,y,t) = v_1(x,y,t) = -\frac{\omega}{8}\operatorname{sec}h^2\left(\frac{1}{2}\sqrt{\frac{-\omega}{2}}\left(x+y-\omega t\right)\right), \tag{19}$$

$$q_2(x,y,t) = u_2(x,y,t) = v_2(x,y,t) = \frac{\omega}{8}\operatorname{csc}h^2\left(\frac{1}{2}\sqrt{\frac{-\omega}{2}}\left(x+y-\omega t\right)\right). \tag{20}$$

Case 2 If $\omega > 0$ we obtain the following hyperbolic traveling wave solutions;

$$q_3(x,y,t) = u_3(x,y,t) = v_3(x,y,t) = -\frac{\omega}{8}\sec^2\left(\frac{1}{2}\sqrt{\frac{\omega}{2}}\left(x+y-\omega t\right)\right), \tag{21}$$

$$q_4(x,y,t) = u_4(x,y,t) = v_4(x,y,t) = -\frac{\omega}{8}\csc^2\left(\frac{1}{2}\sqrt{\frac{\omega}{2}}\left(x+y-\omega t\right)\right). \tag{22}$$

Remark 1 All the obtained results has been checked with Maple by putting them back into the original equations and were found to be correct.

Example 2 The new coupled Konno-Oono equations: Now we will bring to bear the functional variable method to find exact solutions and then the solitary wave solutions of coupled Konno-Oono equations in the form [40];

$$u_{xt} - 2uv = 0, \tag{23}$$

$$v_t + 2uu_x = 0. \tag{24}$$

Now let us suppose that the traveling wave transformation equation is;

$$u(\xi) = u(x,t),\ v(\xi) = v(x,t),\ \xi = x - \omega t. \tag{25}$$

Eq. (25) reduces Eqs. (23) and (24) to the following ODEs;

$$-\omega u_{\xi\xi} - 2uv = 0, \tag{26}$$

$$-\omega v_{\xi} + 2uu_{\xi} = 0. \tag{27}$$

By integrating Eq. (27) with respect to ξ, we obtain;

$$v = \frac{1}{\omega}\left(u^2 + d\right), \tag{28}$$

where d is a constant of integration.

Eq. (28) reduces Eq. (27) to the following form;

$$\omega^2 u_{\xi\xi} + 2ud + 2u^3 = 0. \tag{29}$$

Following Eq. (4), it is easy to deduce from (29) an expression for the function $Y(u)$;

$$\left(Y^2\right)' = -\frac{2}{\omega^2}\left(2ud + 2u^3\right). \tag{30}$$

Integrating Eq. (30) and neglecting the constant of integration, we obtain;

$$Y(u) = \frac{\sqrt{-2d}}{\omega} u \sqrt{\left(1 + \frac{1}{2d}u^2\right)}. \tag{31}$$

Now combining Eq. (3) with Eq. (31) yields;

$$u_{\xi} = \frac{\sqrt{-2d}}{\omega} u \sqrt{\left(1 + \frac{1}{2d} u^2\right)}. \tag{32}$$

Separating the variables in Eq. (32) and then integrating, we obtain the solution of coupled Konno-Oono equations as follows;

$$u(\xi) = \pm\sqrt{2d}\, \operatorname{csc} h\left(\frac{\sqrt{-2d}}{\omega}(\xi + \xi_0)\right), \tag{33}$$

$$u(\xi) = \pm I\sqrt{2d}\, \sec h\left(\frac{\sqrt{-2d}}{\omega}(\xi + \xi_0)\right), \tag{34}$$

where ξ_0 is a constant of integration.

Now for $d < 0$, we can easily determine the following hyperbolic traveling wave solutions of Eq. (29);

$$u_1(x,t) = \pm\sqrt{2d}\, \operatorname{csc} h\left(\frac{\sqrt{-2d}}{\omega}(x - \omega t + \xi_0)\right), \tag{35}$$

$$u_2(x,t) = \pm I\sqrt{2d}\, \sec h\left(\frac{\sqrt{-2d}}{\omega}(x - \omega t + \xi_0)\right). \tag{36}$$

Following Eq. (28); Eqs. (35) and (36) provide the following traveling wave solutions;

$$v_1(x,t) = \frac{2d}{\omega} \operatorname{csc} h^2\left(\frac{\sqrt{-2d}}{\omega}(x - \omega t + \xi_0)\right) + \frac{d}{\omega}, \tag{37}$$

$$v_2(x,t) = -\frac{2d}{\omega} \sec h^2\left(\frac{\sqrt{-2d}}{\omega}(x - \omega t + \xi_0)\right) + \frac{d}{\omega}. \tag{38}$$

Again, for $d > 0$, we can easily determine the following periodic traveling wave solutions;

$$u_3(x,t) = \pm\sqrt{2d}\, \csc\left(\frac{\sqrt{2d}}{\omega}(x - \omega t + \xi_0)\right), \tag{39}$$

$$u_4(x,t) = \pm I\sqrt{2d}\, \sec\left(\frac{\sqrt{2d}}{\omega}(x - \omega t + \xi_0)\right). \tag{40}$$

Following Eq. (28); Eqs. (39) and (40) yields;

$$v_3(x,t) = -\frac{2d}{\omega}\csc^2\left(\frac{\sqrt{2d}}{\omega}\left(x-\omega t+\xi_0\right)\right)+\frac{d}{\omega}, \tag{41}$$

$$v_4(x,t) = -\frac{2d}{\omega}\sec^2\left(\frac{\sqrt{2d}}{\omega}\left(x-\omega t+\xi_0\right)\right)+\frac{d}{\omega}. \tag{42}$$

From the above obtained solutions we observe that $d \neq 0$.

Remark 2: All the obtained results have been checked with Maple by putting them back into the original equation and were found to be correct.

Results and discussion

In this section we will discuss the wave features of our obtained solutions.

The (2+1) dimensional Boussinesq-Kadomtsev-Petviashvili equation

The obtained solutions Eqs. (19) and (20) are solitary waves and Eqs. (21) and (22) are plane periodic waves, where the wave amplitude $A = -\frac{\omega}{8}$, wave number $k = \frac{1}{2}\sqrt{\frac{-\omega}{2}}$ and wave length $\lambda = \frac{2\pi}{k} = \frac{2\pi}{\sqrt{A}}$. It is also clear that the wavelength is inversely proportional to square root of the amplitude *A*, i.e., $\lambda \propto \frac{1}{\sqrt{A}}$. For the existence of the solitary wave $\omega < 0$. On the other hand if $\omega > 0$ it provides periodic waves which are also traveling.

The graphical demonstrations for 2 obtained solutions of BKP equations are shown in **Figures 1** and **2**.

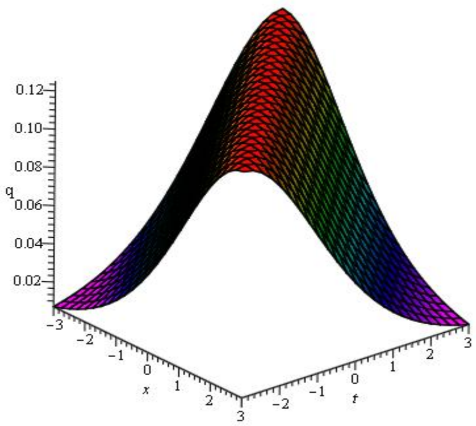

Figure 1 Bell shaped soliton profile of Eq. (19) for wave speed $\omega = -1, y = 0$ within the interval $-3 \leq x, t \leq 3$.

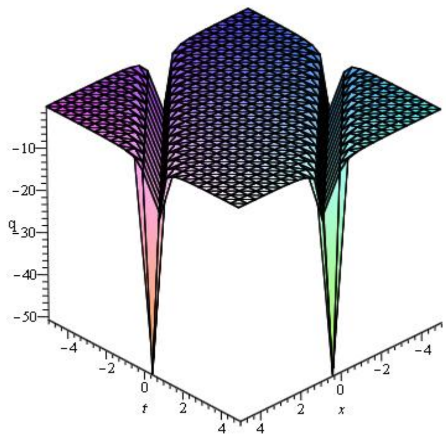

Figure 2 Periodic Profile of Eq. (21) for wave speed $\omega = 1, y = 0$ within the interval $-5 \leq x, t \leq 5$.

The coupled Konno-Oono equations

The obtained solutions Eqs. (35) - (38) are solitary waves and Eqs. (39) - (42) are plane periodic waves. For the existence of the solitary wave $d<0$. For Eq. (35) the wave amplitude $A=\sqrt{2d}$, wave number $k=\frac{\sqrt{-2d}}{\omega}$ and wavelength $\lambda=\frac{2\pi}{k}=-I\frac{2\omega\pi}{A}$ i.e., the wavelength is inversely proportional to the amplitude *A*, provides, $\lambda \propto \frac{1}{A}$. On the other hand if $d>0$ it provides periodic traveling waves.

The graphical demonstrations for 2 obtained solutions of KO equations are shown in **Figures 3** and **4**.

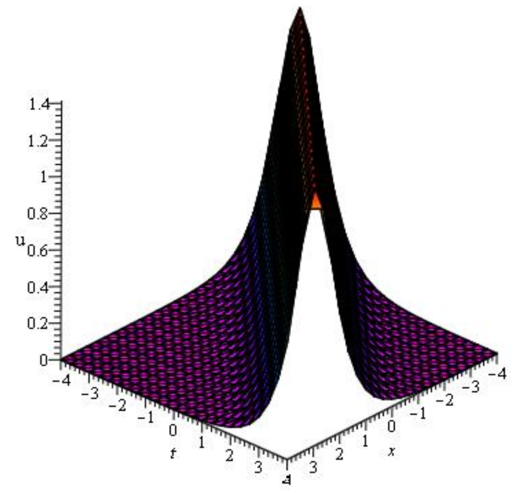

Figure 3 Bell shaped soliton profile of Eq. (36) for $d=-1$, $\omega=1$, within the interval $-4\le x,t\le 4$.

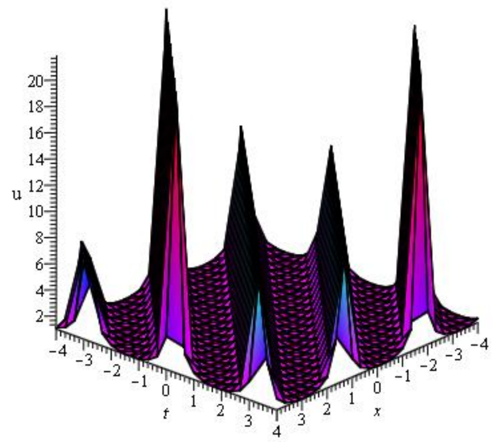

Figure 4 Periodic profile of Eq. (40) for $d=0.50$, $\omega=1$, $k=1$ within the interval $-4\le x,t\le 4$.

Comparisons

Comparisons with modified simple equation method

Khan and Akbar [13] investigated exact solutions of the coupled KO equations by means of the modified simple equation method and obtained 4 solutions for both *u* and *v* (see **Appendix A**). On the contrary by using the functional variable method in this article we have obtained 4 solutions for both *u* and *v*. The solutions of all *u* for KO equations obtained by both the MSE and functional variable methods are different, but the *v* solutions are equivalent. We conclude that we have obtained some new solutions. Comparisons between Khan and Akbar [13] solutions by the MSE method and our solutions obtained by functional variable method are shown in the following table (only for *v*):

No.	Solutions obtained by MSE method Khan and Akbar [13]	Solutions obtained by functional variable method
1.	If we set $d=-2d$ into the solution v_1 obtained by Khan and Akbar [13], then it becomes; $v_1(x,t)=\frac{2d}{\omega}\operatorname{csc}h^2\left(\frac{\sqrt{-2d}}{\omega}(x-\omega t)\right).$	If we leave out the term d/ω and set $\xi_0=0$ in our solution v_1 then; $v_1(x,t)=\frac{2d}{\omega}\operatorname{csc}h^2\left(\frac{\sqrt{-2d}}{\omega}(x-\omega t)\right).$
2.	If we set $d=-2d$ into the solution v_2 obtained by Khan and Akbar [13], then it becomes; $v_2(x,t)=-\frac{2d}{\omega}\operatorname{sec}h^2\left(\frac{\sqrt{-2d}}{\omega}(x-\omega t)\right).$	If we leave out the term d/ω and set $\xi_0=0$ in our solution v_2 then it becomes; $v_2(x,t)=-\frac{2d}{\omega}\operatorname{sec}h^2\left(\frac{\sqrt{-2d}}{\omega}(x-\omega t)\right).$
3.	If we set $d=-2d$ into the solution v_1 obtained by Khan and Akbar [13], then it becomes; $v_3(x,t)=-\frac{2d}{\omega}\csc^2\left(\frac{\sqrt{2d}}{\omega}(x-\omega t)\right).$	If we leave out the term d/ω and set $\xi_0=0$ in our solution v_3 then it becomes; $v_3(x,t)=-\frac{2d}{\omega}\csc^2\left(\frac{\sqrt{2d}}{\omega}(x-\omega t)\right).$
4.	If we set $d=-2d$ into the solution v_1 obtained by Khan and Akbar [13], then it becomes; $v_4(x,t)=-\frac{2d}{\omega}\sec^2\left(\frac{\sqrt{2d}}{\omega}(x-\omega t)\right).$	If we leave out the term d/ω and set $\xi_0=0$ in our solution v_1 then it becomes; $v_4(\xi)=-\frac{2d}{\omega}\sec^2\left(\frac{\sqrt{2d}}{\omega}(x-\omega t)\right).$

Comparisons with extended mapping method

Peng and Krishnan [41] investigated exact solutions of the coupled KO equations by means of the extended mapping method and obtained 5 jaccobi-elliptic function solutions which he converted into three hyperbolic function solutions for $m\longrightarrow 1$ (see **Appendix B**). On the contrary by using the functional variable method in this article we have obtained four solutions of KO equations for both u and v. If we set $\omega=I\sqrt{2d}$, $C_2=\omega\sqrt{-2d}$ into solution u_4 = u_5 obtained by Peng and Krishnan [41] and $\omega=-\omega$, $\xi_0=0$ into our solution u_2, we observe that our solution u_2 coincides with the solution u_4 = u_5 obtained by Peng and Krishnan [41]. Other solutions are different. It means 3 of our solutions are newly obtained by means of the functional variable method.

Conclusions

The functional variable method has been successfully used to seek exact traveling wave solutions of the BKP equations and KO equations. The reliability of the method and the reduction in the use of computational domain give this method a wider applicability. Without the help of a computer algebra system all examples in this paper show the efficiency of the functional variable method. The solution procedure is very simple and the traveling wave solutions are expressed by hyperbolic functions, and trigonometric functions. It has been shown that the method provides a very effective and powerful mathematical tool for solving nonlinear equations in mathematical physics. Comparing with the other methods in the literature, the functional variable method appears to be easier and faster, without the help of a symbolic computation system. This work confirms that the method is direct, concise and effective. The method can be used for treating many other NLEEs in mathematical physics.

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Appendix A

Khan and Akbar [13] examined the exact solutions of the KO equations by using extended mapping method and found the following 3 hyperbolic function solutions;

$$v_1(x,t) = -\frac{d}{\omega}\cos ech^2\left(\frac{\sqrt{d}}{\omega}(x-\omega t)\right),$$

$$v_2(x,t) = \frac{d}{\omega}\sec h^2\left(\frac{\sqrt{d}}{\omega}(x-\omega t)\right),$$

$$v_3(x,t) = \frac{d}{\omega}\cos ec^2\left(\frac{\sqrt{-d}}{\omega}(x-\omega t)\right),$$

$$v_4(x,t) = \frac{d}{\omega}\sec^2\left(\frac{\sqrt{-d}}{\omega}(x-\omega t)\right).$$

Appendix B

Peng and Krishnan [41] examined the exact solutions of the KO equations by using extended mapping method and found the following 3 hyperbolic function solutions for $m \longrightarrow 1$;

$$u_1 = \pm i\omega \tanh\left(\frac{C_2}{\omega^2}(x+\omega t)\right),$$

$$u_2 = u_3 = \pm\frac{\omega}{2}\left[i\tanh\left(\frac{4C_2}{\omega^2}(x+\omega t)\right) + \sec h\left(\frac{4C_2}{\omega^2}(x+\omega t)\right)\right],$$

$$u_4 = u_5(\xi) = \pm\omega \sec h\left(\frac{C_2}{\omega^2}(x+\omega t)\right).$$