WALAILAK JOURNAL
http://wjst.wu.ac.th Applied Mathematics

A New Class of Integral Relations Involving a General Class of Polynomials and I-Functions

Shilpi JAIN[1] and Praveen AGARWAL[2,*]

[1]*Department of Mathematics, Poornima College of Engineering, Jaipur 302002, India*
[2]*Department of Mathematics, Anand International College of Engineering, Jaipur 303012, India*

([*]Corresponding author: goyal.praveen2011@gmail.com)

Received: 7 March 2013, Revised: 24 July 2014, Accepted: 25 August 2014

Abstract

The aim of the present paper is to evaluate new integral relations involving a general class of polynomials and I-functions. The values of the relations are obtained in terms of the Psi functions $\psi(z)$ (the logarithmic derivative of $\Gamma(z)$). These integral relations are unified in nature and act as a key formula from which we can obtain their special cases. For the sake of illustration, we record here some special cases of our main formulas which are also new and known. The formulas established here are basic in nature and are likely to find useful applications in the field of science and engineering.

Keywords: I-function, general class of polynomials, Psi (or digamma) functions $\psi(z)$

Introduction

Many important functions in applied sciences are defined via improper integrals or series (or infinite products). The general names of these important functions are called special functions. A general class of polynomials and I-functions are important special functions and their closely related ones are widely used in physics and engineering; therefore, they are of interest to physicists and engineers as well as mathematicians. In recent years, numerous integral formulas involving a variety of special functions have been developed by many authors (see, e.g., [1-7] and see also, [8]). Also many integral formulas associated with the general class of polynomials have been presented (see, e.g., [9-10]; see also [11]). Those integrals involving the general class of polynomials are not only of great interest in pure mathematics, but they are often of extreme importance in many branches of theoretical and applied physics and engineering. Here we aim at presenting four generalized integral formulas, which are expressed in terms of the Psi functions $\psi(z)$ by inserting the general class of polynomials and I-functions. Some interesting special cases of our main results are also considered.

For our purpose, we begin by recalling some known functions and earlier works. The general class of polynomials $S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}[x]$ will be defined and represented as follows [12];

$$S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}[x] = \sum_{l_1=0}^{[n_1/m_1]} \cdots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} x^{l_i}, \tag{1}$$

where $m_1,\ldots,m_r \in \mathbb{N} := \{1,2,3,\ldots\}$ and $n_1,\ldots,n_r \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The coefficients A_{n_i,l_i} $(n_i, l_i \geq 0)$ are arbitrary constants, real or complex.

A lot of research work has recently come up on the study and development of a function that is more general than the Fox's H-function, popularly known as an I-function.

The I-function is defined and represented as follows [13];

$$I^{m,n}_{p_i,q_i;r}\left[z\left|\begin{matrix}\left(a_j,\alpha_j\right)_{1,n}\left(a_{ji},\alpha_{ji}\right)_{n+1,p_i}\\ \left(b_j,\beta_j\right)_{1,m}\left(b_{ji},\beta_{ji}\right)_{m+1,q_i}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L\phi\left(\xi\right)z^{\xi}d\xi, \tag{2}$$

where

$$\phi\left(\xi\right)=\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}\left[\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}\xi)\right]}, \tag{3}$$

and m,n,p_i,q_i are integers satisfying $1\le n\le p_i, 1\le m\le q_i (i=1,...,r), r$ being finite. $\alpha_j,\beta_j,\alpha_{ji},\beta_{ji}$ are positive integers and a_j,b_j,a_{ji},b_{ji} are complex numbers. The I-function is a generalized form of the well known Fox's H-function [14]. Here the I-function is studied under the following conditions of existence;

(i) $A_i>0;\ |\arg z|<\frac{A_i\pi}{2}.$

and

(ii) $A_i\ge 0;\ |\arg z|\le\frac{A_i\pi}{2};\ \mathrm{Re}(B+1)<0,$

where

$$A_i=\sum_{j=1}^{n}\alpha_j-\sum_{j=n+1}^{p_i}\alpha_{ji}+\sum_{j=1}^{m}\beta_j-\sum_{j=m+1}^{q_i}\beta_{ji},\ \forall i=(1,...,r) \tag{4}$$

and

$$B=\sum_{j=1}^{m}b_j+\sum_{j=m+1}^{q_i}b_{ji}-\sum_{j=1}^{n}a_j-\sum_{j=n+1}^{p_i}a_{ji}+\frac{1}{2}\left(p_i-q_i\right),\forall i=(1,...,r). \tag{5}$$

By summing up the residues at the simple pole of the integrand in (2), we have after a little simplification [15];

$$I_{p_i,q_i,r}^{m,n}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n}(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m}(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]$$
$$=\sum_{k=0}^{\infty}\sum_{h=1}^{m}\frac{(-1)^k\prod_{j=1}^{m}\Gamma\left(b_j-\beta_j\delta^*\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+\alpha_j\delta^*\right)(z)^{\delta^*}}{\beta_h k!\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma\left(1-b_{ji}+\beta_{ji}\delta^*\right)\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}-\alpha_{ji}\delta^*\right)\right\}}, \tag{6}$$

and this should be used $\delta^*=\frac{b_h+k}{\beta_h}$.

Main results

In this section, we establish four generalized integral formulas, which are expressed in terms of $\psi(z)$, by inserting the general class of polynomials (1) and the I-function (2) with suitable arguments into the integrand of (15).

For our purpose, we begin by assuming that $\psi(z)$ denote the Psi (or digamma) function $\Gamma(z)$ defined by $\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$.

Also, let E(x) stand for;

$$\left\{2+\lambda_1(1-x)+\lambda_2(1+x)\right\}, \tag{7}$$

and

$$G(x)=\frac{(1-x)^{s-1}(1+x)^{t-1}}{\{E(x)\}^{s+t}}S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}\left[c\prod_{i=1}^{r}\frac{(1-x)^{s_i}(1+x)^{t_i}}{\{E(x)\}^{s_i+t_i}}\right]I_{p_i,q_i,r}^{m,n}\left[y\frac{(1-x)^{\mu}(1+x)^{\nu}}{\{E(x)\}^{\mu+\nu}}\right]. \tag{8}$$

We have;

Integral formula 1

$$\int_{-1}^{1}G(x)\log\left(\frac{1-x}{E(x)}\right)dx$$
$$=\frac{1}{2}\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_il_i}}{l_i!}A_{n_i,l_i}\left[\sum_{k=0}^{\infty}\sum_{h=1}^{m}\frac{(-1)^k\prod_{j=1}^{m}\Gamma\left(b_j-\beta_j\delta^*\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+\alpha_j\delta^*\right)}{\beta_h k!\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma\left(1-b_{ji}+\beta_{ji}\delta^*\right)\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}-\alpha_{ji}\delta^*\right)\right\}}(c)^{l_i}(y)^{\delta^*}\times\right.$$
$$\left.\left\{\prod_{i=1}^{r}(1+\lambda_1)^{-u}(1+\lambda_2)^{-v}B\left(s+s_il_i+\mu\delta^*,t+t_il_i+v\delta^*\right)\left[-log(1+\lambda_1)+\psi\left(s+s_il_i+\mu\delta^*\right)-\psi\left(s+t+(s_i+t_i)l_i+(\mu+\nu)\delta^*\right)\right]\right\}\right]. \tag{9}$$

Integral formula 2

$$\int_{-1}^{1} G(x) log\left(\frac{1+x}{E(x)}\right) dx$$

$$= \frac{1}{2} \sum_{l_1=0}^{[n_1/m_1]} \cdots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} \left[\sum_{k=0}^{\infty} \sum_{h=1}^{m} \frac{(-1)^k \prod_{j=1}^{m} \Gamma\left(b_j - \beta_j \delta^*\right) \prod_{j=1}^{n} \Gamma\left(1 - a_j + \alpha_j \delta^*\right)}{\beta_h k! \sum_{i=1}^{r} \left\{ \prod_{j=m+1}^{q_i} \Gamma\left(1 - b_{ji} + \beta_{ji} \delta^*\right) \prod_{j=n+1}^{p_i} \Gamma\left(a_{ji} - \alpha_{ji} \delta^*\right) \right\}} (c)^{l_i} (y)^{\delta^*} \times \right.$$

$$\left. \left\{ \prod_{i=1}^{r} \left(1+\lambda_1\right)^{-u} \left(1+\lambda_2\right)^{-v} B\left(s + s_i l_i + \mu\delta^*, t + t_i l_i + v\delta^*\right) \left[-log\left(1+\lambda_2\right) + \psi\left(t + t_i l_i + v\delta^*\right) - \psi\left(s + t + (s_i + t_i) l_i + (\mu + v)\delta^*\right) \right] \right\} \right]. \tag{10}$$

Integral formula 3

$$\int_{-1}^{1} G(x) log\left(\frac{1-x^2}{\{E(x)\}^2}\right) dx$$

$$= \frac{1}{2} \sum_{l_1=0}^{[n_1/m_1]} \cdots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} \left[\sum_{k=0}^{\infty} \sum_{h=1}^{m} \frac{(-1)^k \prod_{j=1}^{m} \Gamma\left(b_j - \beta_j \delta^*\right) \prod_{j=1}^{n} \Gamma\left(1 - a_j + \alpha_j \delta^*\right)}{\beta_h k! \sum_{i=1}^{r} \left\{ \prod_{j=m+1}^{q_i} \Gamma\left(1 - b_{ji} + \beta_{ji} \delta^*\right) \prod_{j=n+1}^{p_i} \Gamma\left(a_{ji} - \alpha_{ji} \delta^*\right) \right\}} (c)^{l_i} (y)^{\delta^*} \times \right.$$

$$\left\{ \prod_{i=1}^{r} \left(1+\lambda_1\right)^{-u} \left(1+\lambda_2\right)^{-v} B\left(s + s_i l_i + \mu\delta^*, t + t_i l_i + v\delta^*\right) \times \right.$$

$$\left. \left. \left[-log\left\{\left(1+\lambda_1\right)\left(1+\lambda_2\right)\right\} + \psi\left(s + s_i l_i + \mu\delta^*\right) + \psi\left(t + t_i l_i + v\delta^*\right) - 2\psi\left(s + t + (s_i + t_i) l_i + (\mu + v)\delta^*\right) \right] \right\} \right]. \tag{11}$$

Integral formula 4

$$\int_{-1}^{1} G(x) log\left(\frac{1-x}{1+x}\right) dx$$

$$= \frac{1}{2} \sum_{l_1=0}^{[n_1/m_1]} \cdots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} \left[\sum_{k=0}^{\infty} \sum_{h=1}^{m} \frac{(-1)^k \prod_{j=1}^{m} \Gamma\left(b_j - \beta_j \delta^*\right) \prod_{j=1}^{n} \Gamma\left(1 - a_j + \alpha_j \delta^*\right)}{\beta_h k! \sum_{i=1}^{r} \left\{ \prod_{j=m+1}^{q_i} \Gamma\left(1 - b_{ji} + \beta_{ji} \delta^*\right) \prod_{j=n+1}^{p_i} \Gamma\left(a_{ji} - \alpha_{ji} \delta^*\right) \right\}} (c)^{l_i} (y)^{\delta^*} \times \right.$$

$$\left. \left\{ \prod_{i=1}^{r} \left(1+\lambda_1\right)^{-u} \left(1+\lambda_2\right)^{-v} B\left(s + s_i l_i + \mu\delta^*, t + t_i l_i + v\delta^*\right) \left[-log\left(\frac{1+\lambda_1}{1+\lambda_2}\right) + \psi\left(s + s_i l_i + \mu\delta^*\right) - \psi\left(t + t_i l_i + v\delta^*\right) \right] \right\} \right]. \tag{12}$$

The following interesting integral will be required to establish the results from (9) to (12);

$$\int_{-1}^{1}\frac{(1-x)^{s-1}(1+x)^{t-1}}{\{E(x)\}^{s+t}}S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}\left[c\prod_{i=1}^{r}\frac{(1-x)^{s_i}(1+x)^{t_i}}{\{E(x)\}^{s_i+t_i}}\right]I_{p_i,q_i,r}^{m,n}\left[y\frac{(1-x)^{\mu}(1+x)^{\nu}}{\{E(x)\}^{\mu+\nu}}\right]dx$$

$$=\frac{1}{2}\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_i l_i}}{l_i!}A_{n_i,l_i}c^{l_i}(1+\lambda_1)^{-s-s_i l_i}(1+\lambda_2)^{-t-t_i l_i}I_{p_i+2,q_i+1;r}^{m,n+2}\left[y(1+\lambda_1)^{-\mu}(1+\lambda_2)^{-\nu}\middle|\begin{matrix}T_1\\T_2\end{matrix}\right],$$

(13)

where

$$T_1\equiv(1-s-s_i l_i,\mu);(1-t-t_i l_i,\nu);(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}$$

and

$$T_2\equiv(bj,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i};(1-s-t-(s_i+t_i)l_i,\mu+\nu)$$

The result (13) is also written in the following form with the help of Eq. (6) as follows;

$$\int_{-1}^{1}\frac{(1-x)^{s-1}(1+x)^{t-1}}{\{E(x)\}^{s+t}}S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}\left[c\prod_{i=1}^{r}\frac{(1-x)^{s_i}(1+x)^{t_i}}{\{E(x)\}^{s_i+t_i}}\right]I_{p_i,q_i,r}^{m,n}\left[y\frac{(1-x)^{\mu}(1+x)^{\nu}}{\{E(x)\}^{\mu+\nu}}\right]dx$$

$$=\frac{1}{2}\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_i l_i}}{l_i!}A_{n_i,l_i}\left[\sum_{k=0}^{\infty}\sum_{h=1}^{m}\frac{(-1)^k\prod_{j=1}^{m}\Gamma(b_j-\beta_j\delta^*)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\delta^*)}{\beta_h k!\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}\delta^*)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}\delta^*)\right\}}(c)^{l_i}(y)^{\delta^*}\right.$$

$$\left.\times\left\{\prod_{i=1}^{r}(1+\lambda_1)^{-s-s_i l_i-\mu\delta^*}(1+\lambda_2)^{-t-t_i l_i-\nu\delta^*}\frac{\Gamma(s+s_i l_i+\mu\delta^*)\Gamma(t+t_i l_i+\nu\delta^*)}{\Gamma(s+t+(s_i+t_i)l_i+(\mu+\nu)\delta^*)}\right\}\right].$$

(14)

The above result will be convergent under the following conditions;

(i) $A_i>0,\ \ |\arg z|<\dfrac{A_i\pi}{2}.$

(ii) $A_i\geq 0,\ \ |\arg z|\leq\dfrac{A_i\pi}{2},\ \ \mathrm{Re}(B+1)<0.$

(iii) $\mathrm{Re}(s)+\mu\min\left(\dfrac{b_j}{\beta_j}\right)>0.$

and

(iv) $\operatorname{Re}(t)+\nu \min\left(\frac{b_j}{\beta_j}\right)>0.$

All the parameters $s, t, s_i, t_i, l_i, \mu, \nu$ are positive and A_i and B are defined by (4) and (5).

Proof: To evaluate the above integral, we express $S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}[x]$ in its series form with the help of (1) and $\mathrm{I}[x]$ in terms of a Mellin-Barnes type contour integral given by (2), change the order of integration and summation and integrate the x-integral with the help of the known result [16];

$$\int_a^b \frac{(t-a)^{\alpha-1}(b-t)^{\beta-1}}{\{b-a+\lambda(t-a)+\mu(b-t)\}^{\alpha+\beta}}dt$$
$$=\frac{(1+\lambda)^{-\alpha}(1+\mu)^{-\beta}}{b-a}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{15}$$

where $a \neq b$, $\operatorname{Re}(\alpha)>0$, $\operatorname{Re}(\beta)>0$, $b-a+\lambda(t-a)+\mu(b-t)\neq 0$.

Finally, interpreting the ξ contour integral in terms of the I-function, we arrive at the right hand side of (13). If we express the I-function involved in the right hand side of (13) in series form with the help of (6), we easily arrive at (14). This completes the proof of (13).

Proof of the main results

To prove the first integral formula (IF), by taking the partial derivative of both sides of (14) with respect to s. IF 2 is similarly established by taking the partial derivative of both sides of (14) with respect to t. To establish IF 3 and 4, we use the IF 1 and 2, first adding the IF 1 and 2, then we get the IF 3. IF 4 is similarly established by subtracting IF 2 from IF 1.

Special cases of the main integral formulas

In this section, we consider other variations of Integral formulas 1 to 4. In fact, on account of the most general nature of the I-function and $S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}[x]$ occurring in our main integrals given by (9) to (12), a large number of integrals involving simpler functions of one variable can easily be obtained as their special cases. For an illustration, we just give several special cases.

Example 1

If we set $r=1$ in (9), (10), (11) and (12), respectively, then the I-function reduces to the familiar Fox's H-function, we get the following results after a little simplification, which are believed to be new;

$$\int_{-1}^{1}\frac{(1-x)^{s-1}(1+x)^{t-1}}{\{E(x)\}^{s+t}}\log\left(\frac{1-x}{E(x)}\right)S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}\left[c\prod_{i=1}^{r}\frac{(1-x)^{s_i}(1+x)^{t_i}}{\{E(x)\}^{s_i+t_i}}\right]\mathrm{H}_{p,q}^{m,n}\left[y\frac{(1-x)^{\mu}(1+x)^{\nu}}{\{E(x)\}^{\mu+\nu}}\right]\mathrm{d}x$$

$$=\frac{1}{2}\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_i l_i}}{l_i!}A_{n_i,l_i}\left[\sum_{k=0}^{\infty}\sum_{h=1}^{m}\frac{(-1)^k\prod_{j=1}^{m}\Gamma(b_j-\beta_j\delta^*)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\delta^*)}{\beta_h k!\left\{\prod_{j=m+1}^{q}\Gamma(1-b_{ji}+\beta_{ji}\delta^*)\prod_{j=n+1}^{p}\Gamma(a_{ji}-\alpha_{ji}\delta^*)\right\}}(c)^{l_i}(y)^{\delta^*}\times\right.$$
$$\left.\left\{\prod_{i=1}^{r}(1+\lambda_1)^{-u}(1+\lambda_2)^{-v}B(s+s_il_i+\mu\delta^*,t+t_il_i+v\delta^*)\left[-\log(1+\lambda_1)+\psi(s+s_il_i+\mu\delta^*)-\psi(s+t+(s_i+t_i)l_i+(\mu+v)\delta^*)\right]\right\}\right]. \quad (16)$$

$$\int_{-1}^{1}\frac{(1-x)^{s-1}(1+x)^{t-1}}{\{E(x)\}^{s+t}}\log\left(\frac{1+x}{E(x)}\right)S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}\left[c\prod_{i=1}^{r}\frac{(1-x)^{s_i}(1+x)^{t_i}}{\{E(x)\}^{s_i+t_i}}\right]\mathrm{H}_{p,q}^{m,n}\left[y\frac{(1-x)^{\mu}(1+x)^{\nu}}{\{E(x)\}^{\mu+\nu}}\right]\mathrm{d}x$$

$$=\frac{1}{2}\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_i l_i}}{l_i!}A_{n_i,l_i}\left[\sum_{k=0}^{\infty}\sum_{h=1}^{m}\frac{(-1)^k\prod_{j=1}^{m}\Gamma(b_j-\beta_j\delta^*)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\delta^*)}{\beta_h k!\left\{\prod_{j=m+1}^{q}\Gamma(1-b_{ji}+\beta_{ji}\delta^*)\prod_{j=n+1}^{p}\Gamma(a_{ji}-\alpha_{ji}\delta^*)\right\}}(c)^{l_i}(y)^{\delta^*}\times\right.$$
$$\left.\left\{\prod_{i=1}^{r}(1+\lambda_1)^{-u}(1+\lambda_2)^{-v}B(s+s_il_i+\mu\delta^*,t+t_il_i+v\delta^*)\left[-\log(1+\lambda_2)+\psi(t+t_il_i+v\delta^*)-\psi(s+t+(s_i+t_i)l_i+(\mu+v)\delta^*)\right]\right\}\right]. \quad (17)$$

$$\int_{-1}^{1}\frac{(1-x)^{s-1}(1+x)^{t-1}}{\{E(x)\}^{s+t}}\log\left(\frac{1-x^2}{\{E(x)\}^2}\right)S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}\left[c\prod_{i=1}^{r}\frac{(1-x)^{s_i}(1+x)^{t_i}}{\{E(x)\}^{s_i+t_i}}\right]\mathrm{H}_{p,q}^{m,n}\left[y\frac{(1-x)^{\mu}(1+x)^{\nu}}{\{E(x)\}^{\mu+\nu}}\right]\mathrm{d}x$$

$$=\frac{1}{2}\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_i l_i}}{l_i!}A_{n_i,l_i}\left[\sum_{k=0}^{\infty}\sum_{h=1}^{m}\frac{(-1)^k\prod_{j=1}^{m}\Gamma(b_j-\beta_j\delta^*)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\delta^*)}{\beta_h k!\left\{\prod_{j=m+1}^{q}\Gamma(1-b_{ji}+\beta_{ji}\delta^*)\prod_{j=n+1}^{p}\Gamma(a_{ji}-\alpha_{ji}\delta^*)\right\}}(c)^{l_i}(y)^{\delta^*}\times\right.$$
$$\left.\left\{\begin{array}{l}\prod_{i=1}^{r}(1+\lambda_1)^{-u}(1+\lambda_2)^{-v}B(s+s_il_i+\mu\delta^*,t+t_il_i+v\delta^*)\times\\ \left[-\log\{(1+\lambda_1)(1+\lambda_2)\}+\psi(s+s_il_i+\mu\delta^*)+\psi(t+t_il_i+v\delta^*)-2\psi(s+t+(s_i+t_i)l_i+(\mu+v)\delta^*)\right]\end{array}\right\}\right]. \quad (18)$$

$$\int_{-1}^{1}\frac{(1-x)^{s-1}(1+x)^{t-1}}{\{E(x)\}^{s+t}}\log\left(\frac{1-x}{1+x}\right)S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}\left[c\prod_{i=1}^{r}\frac{(1-x)^{s_i}(1+x)^{t_i}}{\{E(x)\}^{s_i+t_i}}\right]\mathrm{H}_{p,q}^{m,n}\left[y\frac{(1-x)^{\mu}(1+x)^{\nu}}{\{E(x)\}^{\mu+\nu}}\right]\mathrm{d}x$$

$$=\frac{1}{2}\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_i l_i}}{l_i!}A_{n_i,l_i}\left[\sum_{k=0}^{\infty}\sum_{h=1}^{m}\frac{(-1)^k\prod_{j=1}^{m}\Gamma(b_j-\beta_j\delta^*)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\delta^*)}{\beta_h k!\left\{\prod_{j=m+1}^{q}\Gamma(1-b_{ji}+\beta_{ji}\delta^*)\prod_{j=n+1}^{p}\Gamma(a_{ji}-\alpha_{ji}\delta^*)\right\}}(c)^{l_i}(y)^{\delta^*}\times\right. \quad (19)$$
$$\left.\left\{\prod_{i=1}^{r}(1+\lambda_1)^{-u}(1+\lambda_2)^{-v}B(s+s_il_i+\mu\delta^*,t+t_il_i+v\delta^*)\left[-\log\left(\frac{1+\lambda_1}{1+\lambda_2}\right)+\psi(s+s_il_i+\mu\delta^*)-\psi(t+t_il_i+v\delta^*)\right]\right\}\right].$$

Example 2

If we take $\lambda_1 = \lambda_2 = 0$, c=1, $m_1, ..., m_r = 1, n_1, ..., n_r = n$, $s_i = 1$, $t_i = 0$ and $A(n_1, l_1, ..., n_r, l_r) = \binom{n+\alpha}{n} \frac{(\alpha+\beta+n+1)_l}{(\alpha+1)_l}$ in (9), then the polynomial $S_n^1\left[\frac{1-x}{2}\right]$ occurring therein breaks up in to the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ [12] and the integral (9) takes the following form after a little simplification;

$$\int_{-1}^{1} \frac{(1-x)^{s-1}(1+x)^{t-1}}{\{2\}^{s+t}} log\left(\frac{1-x}{2}\right) P_n^{(\alpha,\beta)}(x)\, I_{p,q}^{m,n}\left[y\frac{(1-x)^{\mu}(1+x)^{\nu}}{\{2\}^{\mu+\nu}}\right] dx$$

$$= \frac{1}{2}\sum_{l=0}^{n} \frac{(-n)_l(\alpha+\beta+n+1)_l}{l!(\alpha+1)_l\, 2^l} \binom{n+\alpha}{n} \left[\sum_{k=0}^{\infty}\sum_{h=1}^{m} \frac{(-1)^k \prod_{j=1}^{m}\Gamma(b_j-\beta_j\delta^*)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\delta^*)}{\beta_h k! \sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}\delta^*)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}\delta^*)\right\}} (y)^{\delta^*} \times\right.$$

$$\left.\left\{\prod_{i=1}^{r} B\left(s+l_i+\mu\delta^*, t+\nu\delta^*\right)\left[\psi\left(s+l_i+\mu\delta^*\right)-\psi\left(s+t+l_i+(\mu+\nu)\delta^*\right)\right]\right\}\right].$$

(20)

Example 3

If we put $r = 1$, $n = p_i = 0$, $m = 1$, $q_i = 2$, $b_1 = 0$, $\beta_1 = 1$, $b_{m+1,1} = -\lambda'$, $\beta_{m+1,1} = \mu'$ in (2) I-function reduces to Wright's generalized Bessel function, i.e. $I_{0,2;1}^{1,0}\left[z\left|\begin{matrix}(............)\\(0,1),(-\lambda',\mu')\end{matrix}\right.\right] = J_{\lambda'}^{\mu'}(z)$ then result (9) reduces to the following form after a little simplification, which is also believed to be new;

$$\int_{-1}^{1} \frac{(1-x)^{s-1}(1+x)^{t-1}}{\{2\}^{s+t}} log\left(\frac{1-x}{2}\right) P_n^{(\alpha,\beta)}(x)\, J_{\lambda'}^{\mu'}\left(y\frac{(1-x)^{\mu}(1+x)^{\nu}}{\{E(x)\}^{\mu+\nu}}\right) dx$$

$$= \frac{1}{2}\sum_{l=0}^{n} \frac{(-n)_l(\alpha+\beta+n+1)_l}{l!(\alpha+1)_l\, 2^l}\binom{n+\alpha}{n}\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(1+\lambda'+\mu'k)}(c)^{l_i}(y)^k\left\{\prod_{i=1}^{r}B\left(s+l_i+\mu k, t+\nu k\right)\left[-\psi\left(s+l_i+\mu k\right)-\psi\left(s+t+(l_i+(\mu+\nu)k\right)\right]\right\}\right].$$

(21)

Example 4

If we put $r = 1$, $n = p_i = p$, $m = 1$, $q_i = q + 1$, $b_1 = 0$, $\beta_1 = 1$, $a_j = 1 - a_j$, $b_{ji} = 1 - b_j$, $\beta_{ji} = \beta_j$ in (2) I-function reduces to Wright's generalized Hypergeometric function, i.e. $I_{p,q+1;1}^{1,p}\left[z\left|\begin{matrix}(1-a_j-\alpha_j)_{1,p}\\(0,1),(1-b_j,\beta_j)_{1,p}\end{matrix}\right.\right] = {}_p\psi_q\left[\begin{matrix}(a_j,\alpha_j)_{1,p};\\(b_j,\beta_j)_{1,p};\end{matrix} -z\right]$ then the result (9) reduces to the following form after a little simplification, which is also believed to be new;

$$\int_{-1}^{1} \frac{(1-x)^{s-1}(1+x)^{t-1}}{\{2\}^{s+t}} \log\left(\frac{1-x}{2}\right) P_n^{(\alpha,\beta)}(x) \; {}_p\psi_q\left[\begin{matrix}(a_j,\alpha_j)_{1,p};\\(b_j,\beta_j)_{1,p};\end{matrix} -y\frac{(1-x)^{\mu}(1+x)^{\nu}}{\{2\}^{\mu+\nu}}\right]\mathrm{d}x$$

$$= \frac{1}{2}\sum_{l=0}^{n} \frac{(-n)_l(\alpha+\beta+n+1)_l}{l!(\alpha+1)_l\, 2^l}\binom{n+\alpha}{n}\left[\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{\infty}\Gamma(a_j+\alpha_j k)}{\Gamma(b_j+\beta_j k)k!}(c)^{l_i}(y)^k\left\{\prod_{i=1}^{r} B(s+l_i+\mu k, t+\nu k)\left[\psi(s+l_i+\mu k)-\psi(s+t+l_i+(\mu+\nu)k)\right]\right\}\right]. \tag{22}$$

Concluding remarks

We can consider another variation of the results derived in the preceding sections. The I-function due to Saxena [13] can be regarded as an extreme generalization of the Fox H-functions. Further, it can be easily seen that the I-function in (2) is a special case of the Fox *H*-function. Therefore, the results presented in this paper are easily converted in terms of a similar type of new interesting integrals with different arguments after some suitable parametric replacements. We are also trying to find certain possible applications of the results presented here to some other research areas.

Acknowledgements

The authors should express their deepest thanks for the referees' valuable comments and essential suggestions to improve this paper as in the present form.

References

[1] P Agarwal. *A Study of New Trends and Analysis of Special Function*. LAP Lambert Academic Publishing, 2013, p. 1-164.

[2] P Agarwal. Certain multiple integral relations involving generalized Mellin-Barnes type of contour integral. *Acta Univ. Apulensis Math. Inform.* 2013; **33**, 257-68.

[3] P Agarwal. On a unified integral involving the product of Srivastava's polynomials and generalized Mellin-Barnes type contour integral. *Adv. Mech. Eng. Appl.* 2012; **2**, 158-61.

[4] P Agarwal and CL Koul. Integrals involving product of H [x,y] and the generalized polynomials $S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}[x]$. *Proc. Nat. Acad. Sci. India* 2004; **74**, 507-14.

[5] P Agarwal. On multiple integral relations involving generalized Mellin-Barnes type of contour integral. *Tamsui Oxf. J. Math. Sci*. 2011; **27**, 449-62.

[6] J Choi and P Agarwal. Certain classes of infinite series deducible from Mellin-Barnes type of contour integrals. *J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math.* 2013; **20**, 233-42.

[7] P Agarwal. On a new theorem involving generalized Mellin-Barnes type of contour integral and Srivastava polynomials. *J. Appl. Math. Stat. Inform.* 2012; **8**, 29-35.

[8] P Agarwal, Certain properties of the generalized Gauss hypergeometric functions, *Appl. Math. Inf. Sci.* 2014; **8(5),** 2315-2320.

[9] P Agarwal. New unified integral involving a Srivastava polynomials and $\overline{H}$-function. *J. Fract. Calc. Appl.* 2012; **3**, 1-7.

[10] P Agarwal and S Jain. New integral formulas involving polynomials and **Ī**-function. *J. Appl. Math. Stat. Inform.* 2012; **8**, 79-88.

[11] P Agarwal and S Jain. On unified finite integrals involving a multivariable polynomial and a generalized Mellin Barnes type of contour integral having general argument. *Nat. Acad. Sci. Lett.* 2009; **32**, 281-6.

[12] HM Srivastava. A multilinear generating function for the Konhauser sets of bi-orthogonal polynomials suggested by the Laguerre polynomials. *Pacific J. Math.* 1985; **117**, 183-91.

[13] VP Saxena. A formal solution of certain new pair of dual integral equations involving H-functions. *Proc. Nat. Acad. Sci. India* 1982; **A52**, 366-75.
[14] HM Srivastava, KC Gupta and SP Goyal. *The H-Function of One and Two Variables with Applications*. South Asian Publishers, New Delhi, India, 1982.
[15] VP Saxena. *The I-Function*. Anamaya Publishers, New Delhi, India, 2008.
[16] IS Gradshteyan and IM Ryzhik. *Tables of Integrals*. Academic Press, New York, 1980.