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New Algorithms for Numerical Assessment of Nonlinear Integro-Differential Equations of Second-Order using Haar Wavelets

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Abstract

This paper deals with the extended design for Fredholm and Volterra integral equations and design for Fredholm and Volterra integro-differential equations of first-order to second-order nonlinear Fredholm and second-order nonlinear Volterra integro-differential equations having square integrable kernels. This approach utilizes the inherent dynamics of the Haar wavelet. The Haar wavelet is used to provide a single platform for the proposed method. The method is tested on problems from literature, and numerical results are compared with existing methods. The numerical results indicate that the accuracy of the method is reasonably high, even on a coarse grid.

Keywords: Haar wavelet, collocation method, second-order Fredholm integro-differential equation of second kind, second-order Volterra integro-differential equations of second kind

Introduction

Integro-differential equations (IDEs) have many applications in natural sciences and engineering. Numerous applications of IDEs can be found in [1-9] and the references therein. Finding accurate and efficient numerical methods for solving IDEs is the primary focus of the numerical analyses as, in most cases, analytical solutions of IDEs are hard to find. Some of the recent work in the context of different solution methods for these types of problems include the differential transform method [10,11], the Newton-Tau method [12], the Haar wavelet method [13], the hybrid Legendre polynomials and block-pulse functions approach [14], the triangular functions method [10], the single-term Walsh series method [15], the block- pulse functions method [16], the wavelet-Galerkin method [17], and the compact finite difference method [18] etc.

The motivation of the present paper is to extend the scope of our earlier work [1,2] in order to propose a new algorithm for numerical solution of second order IDEs, having reasonable accuracy and efficiency. The added advantage of the new method is that it does not involve numerical integration. The integrand is approximated with the Haar wavelet basis, and exact integration is performed for the given formulae. In the present work, we will consider two types of IDEs. The first type is second-order Fredholm integro-differential equation of second kind (FIDEST), which is given as follows;

$$u''(x)+g(x)u'(x)+w(x)u(x)=f(x)+\int_0^1 K(x,t,u(t),u'(t))dt,\ u(0)=u_0, u'(0)=u'_0 \quad , \quad (1)$$

and the second type is second-order Volterra integro-differential equation of second kind (VIDEST), given as follows;

$$u''(x) + g(x)u'(x) + w(x)u(x) = f(x) + \int_0^x K(x,t,u(t),u'(t))\,dt, \;\; u(0) = u_0, u'(0) = u'_0 \,. \tag{2}$$

The kernel function $K(x, t, u(t), u'(t))$ given in the above equations is a nonlinear function defined by $[0, 1] \times [0, 1]$, $f(x)$ and $g(x)$ are known functions defined on [0, 1], and $u(x)$ is the unknown function representing solutions of the IDEs. We assume that the kernel function $K(x, t, u(t), u'(t))$ is a square integrable function.

The organization of the rest of the paper is as follows. In Section 2, the Haar wavelet and its integrals are described. In Section 3 formulation of the method based on the Haar wavelet is defined for Fredholm and Volterra IDEs. In section 4, numerical results are reported, and conclusions are drawn in Section 5.

Haar wavelet

We use standard notations adopted in [1,2] for the scaling function, the mother wavelet, and the rest of the Haar wavelet family members. The scaling function for the Haar wavelet family is defined by the interval [0, 1), and is given as follows;

$$h_1(x) = \begin{cases} 1 & \text{for} \quad x \in [0,1) \\ 0 & \text{elsewhere.} \end{cases} \tag{3}$$

The mother wavelet for Haar wavelet family is also defined by the interval [0, 1), and is given by;

$$h_2(x) = \begin{cases} 1 & \text{for} \quad x \in [0,\tfrac{1}{2}) \\ -1 & \text{for} \quad x \in [\tfrac{1}{2},1) \\ 0 & \text{elsewhere.} \end{cases} \tag{4}$$

All other functions in the Haar wavelet family are defined by the [0, 1) interval as follows;

$$h_i(x) = \begin{cases} 1 & \text{for} \quad x \in [\alpha,\beta) \\ -1 & \text{for} \quad x \in [\beta,\gamma) \\ 0 & \text{elsewhere,} \end{cases} \tag{5}$$

where

$$\alpha = \frac{k}{m}, \;\; \beta = \frac{(k+0.5)}{m}, \;\; \gamma = \frac{(k+1)}{m}, \;\; i = 3,4,\ldots,2M.$$

The integer $m = 2j$, where $j = 0, 1, \ldots, J$, $M = 2J$, and integer $k = 0, 1, \ldots, m - 1$. The integer j indicates the level of the wavelet, and k is the translation parameter. The maximal level of resolution is the integer J. The relation between i, m, and k is given by $i = m + k + 1$. The Haar wavelet functions are orthogonal to each other because of the following relation;

$$\int_0^1 h_j(x)h_k(x)dx = 0, \quad \text{whenever} \quad j \neq k. \tag{6}$$

Any function $f(x)$ which is square integrable in the interval (0, 1) can be expressed as an infinite sum of Haar wavelet functions, as follows;

$$f(x) = \sum_{i=1}^{\infty} a_i h_i(x). \tag{7}$$

The above series terminates at finite terms if $f(x)$ is a piecewise constant, or can be approximated as a piecewise constant, during each subinterval. The following notations are introduced;

$$p_{i,1}(x) = \int_0^x h_i(z)\,dz, \tag{8}$$

and

$$C_{i,1} = \int_0^1 p_{i,1}(x)\,dx. \tag{9}$$

These integrals can be evaluated using Eq. (5), and are given as follows;

$$p_{i,1}(x) = \begin{cases} x - \alpha & \text{for} \quad x \in [\alpha, \beta) \\ \gamma - x & \text{for} \quad x \in [\beta, \gamma) \\ 0 & \text{elsewhere,} \end{cases} \tag{10}$$

and

$$C_{i,1}(x) = \frac{1}{4m^2}. \tag{11}$$

Solution procedure

In this section, we will state 2 theorems for efficient evaluation of the Haar coefficients. These theorems will be used in numerical method based on the Haar wavelet for the solution of nonlinear Fredholm and Volterra integro-differential equations. The following collocation points will be used for the Haar wavelet approximations;

$$x_p = \frac{p - 0.5}{2M}, p = 1, 2, \ldots, 2M, \tag{12}$$

$$t_q = \frac{q - 0.5}{2N}, q = 1, 2, \ldots, 2N. \tag{13}$$

One-dimensional function approximation using Haar wavelet

Any square integrable function $f(x)$ can be approximated using the Haar wavelet as follows;

$$f(x)=\sum_{i=1}^{2M}a_i h_i(x). \tag{14}$$

Substituting the collocation points given in Eq. (12), we obtain the following linear system of equations;

$$f(x_p)=\sum_{i=1}^{2M}a_i h_i(x_p),\ p=1,2,\ldots,2M. \tag{15}$$

This is a 2×2 M linear system of equations, whose solution for the unknown coefficients a_i can be calculated without solving the linear system with the help of the following theorem.

Theorem 1 The solution of the system (15) is given as follows;

$$a_1=\frac{1}{2M}\sum_{p=1}^{2M}f(x_p), \tag{16}$$

$$\alpha_i=\frac{1}{\rho}\left(\sum_{p=\alpha}^{\beta}f(x_p)-\sum_{p=\beta+1}^{\gamma}f(x_p)\right), i=2,3,\ldots,2M, \tag{17}$$

where

$$\alpha=\rho(\sigma-1)+1,$$
$$\beta=\rho(\sigma-1)+\frac{\rho}{2},$$
$$\gamma=\rho\sigma,$$
$$\rho=\frac{2M}{\tau},$$
$$\sigma=i-\tau,$$
$$\tau=2^{[\log_2(i-1)]}.$$

Proof. **See [1].**

Two-dimensional function approximation using Haar wavelet

A square integrable function $F\ (x,\ t)$ of two variables x and t can be approximated using the 2-dimensional Haar wavelet basis as;

$$F(x,t)\approx\sum_{i=1}^{2M}\sum_{j=1}^{2N}b_{i,j}h_i(x)h_j(t). \tag{18}$$

In order to calculate the unknown coefficients $b_{i,j}$, $i = 1, 2, \ldots, 2M$, $j = 1, 2, \ldots, 2N$, the collocation points defined in Eqs. (12) and (13) are substituted in Eq. (18). Hence, we obtain the following 4×4MN linear system;

$$F_{p,q} = \sum_{i=1}^{2M}\sum_{j=1}^{2n} b_{i,j} h_i(x_p) h_j(t_q),\ p = 1,2,\ldots,2M,\ q = 1,2,\ldots,2N, \tag{19}$$

where, for simplicity, we have introduced the notation $F_{p,q}$ for the value of $F(xp, tq)$. The solution of this system can be calculated using the following theorem.

Theorem 2 The solution of the system (19) is given below.

$$b_{1,1} = \frac{1}{2M \times 2N}\sum_{p=1}^{2M}\sum_{q=1}^{2N} F_{p,q}, \tag{20}$$

$$b_{i,1} = \frac{1}{\rho_1 \times 2N}\left(\sum_{p=\alpha_1}^{\beta_1}\sum_{q=1}^{2N} F_{p,q} - \sum_{p=\beta_1+1}^{\gamma_1}\sum_{q=1}^{2N} F_{p,q}\right),\ i = 2,3,\ldots,2M \tag{21}$$

$$b_{1,j} = \frac{1}{2M \times \rho_2}\left(\sum_{p=1}^{2M}\sum_{q=\alpha_2}^{\beta_2} F_{p,q} - \sum_{p=1}^{2M}\sum_{q=\beta_2+1}^{\gamma_2} F_{p,q}\right),\ j = 2,3,\ldots,2N \tag{22}$$

$$b_{i,j} = \frac{1}{\rho_1\rho_2}\left(\sum_{p=\alpha_1}^{\beta_1}\sum_{q=\alpha_2}^{\beta_2} F_{p,q} - \sum_{p=\alpha_1}^{\beta_1}\sum_{q=\beta_2+1}^{\gamma_2} F_{p,q} - \sum_{p=\beta_1+1}^{\gamma_1}\sum_{q=\alpha_2}^{\beta_2} F_{p,q} + \sum_{p=\beta_1+1}^{\gamma_1}\sum_{q=\beta_2+1}^{\gamma_2} F_{p,q}\right),\ \begin{matrix} i = 2,3,\ldots,2M \\ j = 2,3,\ldots,2N \end{matrix} \tag{23}$$

where

$$\begin{aligned} \alpha_1 &= \rho_1(\sigma_1 - 1) + 1, \\ \beta_1 &= \rho_1(\sigma_1 - 1) + \frac{\rho_1}{2}, \\ \gamma_1 &= \rho_1\sigma_1, \\ \rho_1 &= \frac{2M}{\tau_1}, \\ \tau_1 &= 2^{[\log_2(i-1)]}, \end{aligned} \tag{24}$$

and similarly,

$$\alpha_2 = \rho_2(\sigma_2 - 1) + 1,$$
$$\beta_2 = \rho_2(\sigma_2 - 1) + \frac{\rho_2}{2},$$
$$\gamma_2 = \rho_2 \sigma_2,$$
$$\rho_2 = \frac{2N}{\tau_2},$$
$$\sigma_2 = j - \tau_2,$$
$$\tau_2 = 2^{[\log_2(j-1)]}. \tag{25}$$

Proof. **See [1].**

Second-order Fredholm integro-differential equation of second kind

Consider the second-order FIDEST (1). The Kernel function $K(x, t, u(t), u'(t), u''(t))$ is approximated using the 2-dimensional Haar wavelet approximation as;

$$K(x,t,u(t),u'(t),u''(t)) = \sum_{i=1}^{2M}\sum_{j=1}^{2M} b_{i,j} h_i(x_p) h_j(t_q), \;\; p,q = 1,2,...,2M \tag{26}$$

Substituting this approximation of the Kernel function in Eq. (1), we obtain the following;

$$u''(x) + g(x)u'(x) + w(x)u(x) = f(x) + \int_0^1 \sum_{i=1}^{2M}\sum_{j=1}^{2M} b_{i,j} h_i(x) h_j(t)\, dt. \tag{27}$$

Using the property of the Haar wavelet, Eq. (27) reduces it to the following;

$$u''(x) + g(x)u'(x) + w(x)u(x) = f(x) + \sum_{i=1}^{2M} b_{i,1} h_i(x) \tag{28}$$

Substituting the collocation points given in Eq. (12), we have;

$$u''(x_r) + g(x_r)u'(x_r) + w(x_r)u(x_r) = f(x_r) + \sum_{i=1}^{2M} b_{i,1} h_i(x_r), r = 1,2,...,2M \tag{29}$$

Now using Theorem 2, the coefficients $b_{i,1}$ can be replaced with their expressions given in Eqs. (20) - (21) and, thus, we obtain the following system of nonlinear equations;

$$u''(x_r)+g(x_r)u'(x_r)+w(x_r)u(x_r)=f(x_r)$$
$$+\frac{1}{2M\times 2M}\sum_{p=1}^{2M}\sum_{q=1}^{2M}K(x_p,t_q,u(t_q),u'(t_q),u''(t_q))h_1(x_r)$$
$$+\sum_{i=2}^{2M}\frac{1}{\rho_1\times 2M}(\sum_{p=\alpha_1}^{\beta_1}\sum_{q=1}^{2M}K(x_p,t_q,u(t_q),u'(t_q),u''(t_q))-\sum_{p=\beta_1+1}^{\gamma_1}\sum_{q=1}^{2M}K(x_p,t_q,u(t_q),u'(t_q),u''(t_q)))h_i(x_r),$$
$$r=1,2,...,2M \tag{30}$$

Applying Theorem 1 we can express $u'(x_r)$, r = 1, 2, . . . , $2M$ and $u(x_r)$, r = 1, 2, . . . , $2M$ in terms of $u''(x_2)$, r = 1, 2, . . . , $2M$, which are given below.

$$u'(x_r)=u_0'+\frac{1}{2M}\sum_{q=1}^{2M}u''(x_q)p_{1,1}(x_r)+\sum_{i=2}^{2M}\frac{1}{\rho}\left(\sum_{q=\alpha}^{\beta}u''(x_q)-\sum_{q=\beta+1}^{\gamma}u''(x_q)\right)p_{i,1}(x_r), r=1,2,...,2M \tag{31}$$

and

$$u(x_r)=u_0+u_0'x_r+\frac{1}{2M}\sum_{q=1}^{2M}u''(x_q)p_{1,2}(x_r)+\sum_{i=2}^{2M}\frac{1}{\rho}\left(\sum_{q=\alpha}^{\beta}u''(x_q)-\sum_{q=\beta+1}^{\gamma}u''(x_q)\right)p_{i,2}(x_r), r=1,2,...,2M. \tag{32}$$

Hence, the only unknowns in the system given in Eq. (30) are $u''(x_r)$, r = 1, 2, . . . , $2M$. Therefore, Eq. (30) represents a 2×2 M system of nonlinear equations, which can be solved using either Newton's method or Broyden's method. The solution of this system gives the values of the second derivative $u'(x)$ at the collocation points. The solution of Eq. (1) at any point in the domain can be obtained using the following equation;

$$u(x)=u_0+u_0'x+\frac{1}{2M}\sum_{q=1}^{2M}u''(x_p)p_{1,2}(x)+\sum_{i=2}^{2M}\frac{1}{\rho}\left(\sum_{p=\alpha}^{\beta}u''(x_q)-\sum_{q=\beta+1}^{\gamma}u''(x_q)\right)p_{i,2}(x). \tag{33}$$

The Jacobian of the system given in Eq. (30) is given by;

$$J(p,r)=\begin{cases}1+\psi_1(r,r), & r=s\\ \psi_1(r,s), & r\neq s\end{cases} \tag{34}$$

where

$$\psi_1(r,s)=g(x_r)\frac{\partial u'(x_r)}{\partial u''(x_r)}+w(x_r)\frac{\partial u(x_r)}{\partial u''(x_r)}-\frac{1}{(2M)^2}\sum_{p=1}^{2M}\sum_{q=1}^{2M}\frac{\partial K(x_p,t_q,u(t_q),u'(t_q),u''(t_q))}{\partial u''(x_s)}h_1(x_r)$$
$$+\sum_{i=2}^{2M}\frac{h_i(x_r)}{2M\rho_1}(\sum_{p=\alpha_1}^{\beta_1}\sum_{q=1}^{2M}\frac{\partial K(x_p,t_q,u(t_q),u'(t_q),u''(t_q))}{\partial u''(x_s)}-\sum_{p=\beta_1+1}^{\gamma_1}\sum_{q=1}^{2M}\frac{\partial K(x_p,t_q,u(t_q),u'(t_q),u''(t_q))}{\partial u''(x_s)}), \tag{35}$$

and

$$\frac{\partial K(x_p,t_q,u(t_q),u'(t_q),u''(t_q))}{\partial u''(x_s)}$$
$$=\begin{cases}\frac{\partial K}{\partial u}(x_p,t_q,u(t_q),u'(t_q),u''(t_q))\frac{\partial u(x_p)}{\partial u''(x_s)}+\frac{\partial K}{\partial u'}(x_p,t_q,u(t_q),u'(t_q),u''(t_q))\frac{\partial u'(x_p)}{\partial u''(x_s)}+\frac{\partial K}{\partial u''}(x_p,t_q,u(t_q),u'(t_q),u''(t_q)), q=s\\ \frac{\partial K}{\partial u}(x_p,t_q,u(t_q),u'(t_q),u''(t_q))\frac{\partial u(x_p)}{\partial u''(x_s)}+\frac{\partial K}{\partial u'}(x_p,t_q,u(t_q),u'(t_q),u''(t_q))\frac{\partial u'(x_p)}{\partial u''(x_s)}, q\neq s\end{cases} \tag{36}$$

The values of the partial derivatives $\frac{\partial u(x_r)}{\partial u''(x_s)}$ and $\frac{\partial u'(x_r)}{\partial u''(x_s)}$ can be calculated using Eqs. (31) and (32), respectively, and are given as follows;

$$\frac{\partial u'(x_r)}{\partial u''(x_s)}=\frac{1}{2M}p_{1,1}(x_r)+\sum_{i=2}^{2M}\frac{1}{\rho}(I_{[\alpha,\beta]}(s)-I_{[\beta+1,\gamma]}(s))p_{i,1}(x_r),\ r,s=1,2,\ldots,2M \tag{37}$$

and

$$\frac{\partial u(x_r)}{\partial u''(x_s)}=\frac{1}{2M}p_{1,2}(x_r)+\sum_{i=2}^{2M}\frac{1}{\rho}(I_{[\alpha,\beta]}(s)-I_{[\beta+1,\gamma]}(s))p_{i,2}(x_r),\ r,s=1,2,\ldots,2M, \tag{38}$$

where I is the characteristic function, defined as follows;

$$I_{[a,b]}(x)=\begin{cases}1 & \text{if} \quad a\le x\le b\\ 0 & \text{otherwise.}\end{cases} \tag{39}$$

Volterra integro-differential equation of second-order

Consider the second-order VIDEST (2). The kernel function $K(x, t, u(t), u'(x), u''(x))$ is approximated using the 2-dimensional Haar wavelets, as given in Eq. (26). Using this approximation and the properties of the Haar wavelet, Eq. (2) can be written as;

$$u''(x)+g(x)u'(x)+w(x)u(x)=f(x)+\sum_{i=1}^{2M}\sum_{j=1}^{2M}b_{i,j}h_i(x)p_{j,1}(x) \tag{40}$$

Substituting the collocation points, we obtain the following system;

$$u''(x_r)+g(x_r)u'(x_r)+w(x_r)u(x_r)=f(x_r)+\sum_{i=1}^{2M}\sum_{j=1}^{2M}b_{i,j}h_i(x_r)p_{j,1}(x_r), r=1,2,\ldots,2M \tag{41}$$

By applying Theorem 2, the coefficients $b_{i,j}$, $i, j = 1, 2, \ldots, 2M$ can be replaced with their expressions given in Eqs. (20) - (23) and, therefore, we have;

$$u''(x_r)+g(x_r)u'(x_r)-w(x_r)u(x_r)=f(x_r)+\frac{h_1(x_r)p_{1,1}(x_r)}{(2M)^2}\sum_{p=1}^{2M}\sum_{q=1}^{2M}K_{p,q}$$
$$+\sum_{i=2}^{2M}\frac{h_i(x_r)p_{1,1}(x_r)}{2M\rho_1}\left(\sum_{p=\alpha_1}^{\beta_1}\sum_{q=1}^{2M}K_{p,q}-\sum_{p=\beta_1+1}^{\gamma_1}\sum_{q=1}^{2M}K_{p,q}\right)$$
$$+\sum_{j=2}^{2M}\frac{h_1(x_r)p_{j,1}(x_r)}{2M\rho_2}(\sum_{p=1}^{2M}\sum_{q=\alpha_2}^{\beta_2}K_{p,q}-\sum_{p=1}^{2M}\sum_{q=\beta_2+1}^{\gamma_2}K_{p,q})$$
$$+\sum_{i=2}^{2M}\sum_{j=2}^{2M}\frac{h_i(x_r)p_{j,1}(x_r)}{\rho_1\rho_2}\left(\sum_{p=\alpha_1}^{\beta_1}\sum_{q=\alpha_2}^{\beta_2}K_{p,q}-\sum_{p=\alpha_1}^{\beta_1}\sum_{q=\beta_2+1}^{\gamma_2}K_{p,q}\right.$$
$$\left.-\sum_{p=\beta_1+1}^{\gamma_1}\sum_{q=\alpha_2}^{\beta_2}K_{p,q}+\sum_{p=\beta_1+1}^{\gamma_1}\sum_{q=\beta_2+1}^{\gamma_2}K_{p,q}\right),r=1,2,...,2M, \tag{42}$$

where $K_{p,q} = K(x_p, t_q, u(t_q), u'(t_q), u''(t_q))$. Eq. (42) represents a 2×2 M system of nonlinear equations with unknowns $u''(x_r)$, $r = 1, 2, \ldots, 2M$, and can be solved in a similar way as in the case of the nonlinear Fredholm integro-differential equation of the second-order, discussed in the previous section.

Numerical examples

In this section, numerical results of four numerical experiments are presented to test the accuracy, applicability, and convergence of the proposed methods. Performance of the new methods is compared with the existing methods in literature [12,19].

The notation $E_c(N)$ is used for the maximum absolute errors at the N collocation points. The experimental rate of convergence $R_c(N)$ is calculated using the following formula;

$$R_c(N)=\frac{\log[E_c(N/2)/E_c(N)]}{\log 2}. \tag{43}$$

Remark: For problems having more than one solution, our method gives only one solution.

Test problem 1 Consider the NFIDE [12];

$$u''(x)+xu'(x)-xu(x)=e^x-\sin x+\int_0^1 \sin x e^{-2t}u^2(t)dt, \tag{44}$$

subject to initial conditions $u(0) = 1$, $u'(0) = 1$. The exact solution of the problem is $u(x) = ex$.

Table 1 shows a comparison of point-wise absolute errors of the present method with the Newton-Tau method [12]. It is clear from the table that performance of the suggested method is better than the Newton-Tau method [12]. In **Table 2**, we show the maximum absolute errors at the collocation points, as well as the experimental rate of convergence. The table shows that the maximum absolute errors decrease with the increase in the number of collocation points.

Table 1 Comparison of absolute errors for test problem 1.

x	Absolute errors (Present method) $2M = 128$	Absolute errors (NT method [12]) $m = 5, n = 5$
0.0	0.00000	0.00000
0.2	8.2864e-06	4.0000e-07
0.4	5.6785e-06	8.1000e-06
0.6	5.9638e-06	7.7300e-05
0.8	1.1696e-05	4.2480e-04
1.0	6.6724e-06	1.6413e-03

Table 2 Maximum absolute errors and rate of convergence for test problem 1.

J	$2M$	Maximum absolute errors	Rate of convergence
1	4	5.4334e-03	-
2	8	1.4609e-03	1.8950
3	16	3.7865e-04	1.9479
4	32	9.6370e-05	1.9742
5	64	2.4302e-05	1.9875
6	128	6.0977e-06	1.9947

Test problem 2 Consider the NFIDE;

$$u''(x) = e^{-4t}u^2(t)(u'(t))^2\,dt, \tag{45}$$

subject to initial conditions $u(0) = 1$, $u'(0) = 1$. The exact solution of the problem is $u(x) = ex$.

In **Table 3**, we show the maximum absolute errors at the collocation points, as well as the experimental rate of convergence. The table shows that the maximum absolute errors decrease with the increase in the number of collocation points.

Test problem 3 Consider the NFIDE [19];

$$u''(x) = \frac{7}{3}\cos x + \frac{\pi}{2}\sin x + \frac{1}{2}\int_0^{\pi}\sin(x-t)u^2(t)\,dt, \tag{46}$$

Table 3 Maximum absolute errors and rate of convergence for test problem 2.

J	2*M*	Maximum absolute errors	Rate of convergence
1	4	2.8864e-002	-
2	8	7.7597e-003	1.8952
3	16	2.0361e-003	1.9302
4	32	5.2280e-004	1.9615
5	64	1.3254e-004	1.9798
6	128	3.3371e-005	1.9898

subject to initial conditions $u(0) = 0$, $u'(0) = 0$. The exact solution of the problem is $u(x) = 1 - \cos x$. In **Table 4**, we show the maximum absolute errors at the collocation points, as well as the experimental rate of convergence. The table shows that the maximum absolute errors decrease with the increase in the number of collocation points.

Note that, in this problem, the interval of integration is not [0, 1]. However, the definition of the Haar wavelet can be modified easily, so that it is defined by any general interval [a, b]. Accordingly, the present method needs a slight modification when it is applied to problems having a different domain.

Table 4 Maximum absolute and rate of convergence for test problem 3.

J	2*M*	Maximum absolute errors	Rate of convergence
1	4	2.6521e-002	-
2	8	6.2684e-003	2.0810
3	16	1.5876e-003	1.9812
4	32	3.9782e-004	1.9967
5	64	9.8656e-005	2.0116

Test problem 4 Consider the NVIDE [19];

$$u''(x) = \sinh x + \frac{1}{2}\cosh x \sinh x - \frac{1}{2}x - \int_0^x u^2(t)dt, \tag{47}$$

subject to initial conditions $u(0) = 0$, $u'(0) = 1$. The exact solution of the problem is $u(x) = \sinh x$.

Point-wise absolute errors for a different number of collocation points using the present method is indicated in **Table 5**. The table shows that the absolute errors decrease with the increase in the number of collocation points.

Table 5 Exact solution and absolute errors for test problem 4.

x	Exact solution	Absolute errors (Present method) $2M = 4$	Absolute errors (Present method) $2M = 8$	Absolute errors (Present method) $2M = 16$
0.1	0.1002	4.5323e-004	1.4514e-004	3.3433e-005
0.2	0.2013	1.1439e-003	2.6512e-004	6.2404e-005
0.3	0.3045	1.3787e-003	3.7250e-004	9.7347e-005
0.4	0.4108	2.0402e-003	4.8024e-004	1.2354e-004
0.5	0.5211	2.3844e-003	6.0380e-004	1.5143e-004
0.6	0.6367	2.6967e-003	7.1665e-004	1.7627e-004
0.7	0.7586	3.2101e-003	7.8524e-004	1.9188e-004
0.8	0.8881	3.1008e-003	8.1890e-004	2.1048e-004
0.9	1.0265	3.4501e-003	8.3030e-004	2.1271e-004
1.0	1.1752	3.2692e-003	8.3805e-004	2.1076e-004

Conclusions

A new method based on the Haar wavelet is proposed for the numerical solution of nonlinear second- order Fredholm IDE and Volterra IDE of second kind. The 2 dimensional Haar wavelet basis is used for approximating the kernel functions of IDEs. The algorithms are validated numerically. Improved accuracy and rapid convergence of numerical solution are obtained through the new approach based on the Haar functions.

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