

An Implicit Numerical Method for Semilinear Space-Time Fractional Diffusion Equation

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Abstract

The aim of the study is to obtain the solution of semilinear space-time fractional diffusion equation for the first initial boundary value problem (IBVP), by applying an implicit method. The main idea of the method is to convert the problem into an algebraic system which simplifies the computations. We discuss the stability, convergence and error analysis of the implicit finite difference scheme with suitable example using MATLAB.

Keywords: Riemann-Liouville fractional derivative, Caputo fractional derivative, implicit finite difference method, stability, convergence

Introduction

Fractional differential equations play an important role in the study of various physical, chemical and biological phenomena. Therefore, many researchers are attracted from the fields of theory, methods and applications of fractional differential equations. Therefore, there is a need to study reliable and efficient techniques to obtain either exact or approximate solutions of fractional differential equations. The researchers have developed few numerical techniques and obtained approximate solutions of both linear and nonlinear fractional differential equations.

Liu *et al.* [1] considered time fractional advection-dispersion equations and obtained its complete solution. Eidelman and Kochubei [2] investigated the Cauchy problem for the time-fractional diffusion equations with variable coefficients, and constructed the fundamental solution. Lin and Xu [3], Liu *et al.* [4] and Yuste [5] proposed an explicit finite difference scheme for the time fractional diffusion equation. Zhuang and Liu [6], Zhuang *et al.* [7,8], Chen *et al.* [9], Murio [10] and Birajdar & Dhaigude [11], developed an implicit finite difference approximation for the time fractional diffusion equation and also discussed its stability and convergence. Liu *et al.* [12] employed an implicit finite difference scheme for space-time fractional diffusion equation. Whereas Zhuang and Liu [13] developed an implicit difference scheme for the 2 dimensional space-time fractional diffusion equation.

Nonlinear partial differential equations have lots of applications in various branches of sciences [14,15]. In fact, published papers on the numerical methods for the nonlinear fractional partial differential equations are limited. This motivates us to consider an effective numerical method for such problems. Choi *et al.* [16] and Liu *et al.* [17] developed a numerical technique for fractional diffusion equation with a nonlinear source term. Zhang and Liu [18] considered a Riesz space fractional diffusion equation with a nonlinear source term. Also Yang *et al.* [19] provided a numerical solution of a fractional Fokker Plank equation with a nonlinear source term and proved its stability as well as convergence. Recently, Dhaigude and Birajdar [20-22] developed a discrete Adomian decomposition method for a nonlinear system of fractional partial differential equations.

We consider the space-time semilinear fractional diffusion equation;

$$\frac{\partial^\alpha u}{\partial t^\alpha} = a(x,t)D_x^\beta u(x,t) + f(u), \quad a(x,t) > 0, 0 < \alpha \leq 1, 1 < \beta \leq 2, \quad (1)$$

with an initial condition;

$$u(x,0) = g(x) \quad 0 < x < l \quad (2)$$

and boundary conditions;

$$u(0,t) = 0 = u(l,t) \quad 0 < t \leq T, \quad (3)$$

it is called the first initial boundary value problem (IBVP) for a space-time semilinear fractional diffusion equation. Note that $\frac{\partial^\alpha (\cdot)}{\partial t^\alpha}$ is Caputo fractional derivative of order α and is defined [23] as;

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\xi)}{\partial \xi} \frac{d\xi}{(t-\xi)^\alpha}, & 0 < \alpha < 1 \\ \frac{\partial u}{\partial t}, & \alpha = 1, \end{cases}$$

and the Riemann-Liouville fractional derivative of order $\beta (1 < \beta \leq 2)$ is defined [24] as;

$$D_x^\beta u(x,t) = \begin{cases} \frac{1}{\Gamma(2-\beta)} \frac{\partial^2}{\partial x^2} \int_0^t \frac{u(x,\eta) d\eta}{(t-\eta)^{\beta-1}}, & 1 < \beta < 2 \\ \frac{\partial^2 u}{\partial x^2}, & \beta = 2, \end{cases}$$

when $\alpha = 1$ and $\beta = 2$ in (1), it reduces to the first IBVP for a reaction-diffusion equation.

Here our aim is to find the numerical solution of IBVP for space-time fractional diffusion Eqs. (1) - (3) by using an implicit finite difference method. We replace the time derivative by a Caputo fractional derivative and the space derivative by a Riemann-Liouville fractional derivative respectively.

The plan of the paper is as follows. In section 2 we develop an implicit difference scheme for first IBVP. Stability of the implicit difference scheme is proved in section 3. Section 4 shows that the implicit difference scheme is convergent. Finally, a test problem is given as an application of the method.

Implicit finite difference scheme

Consider that the first IBVP for a space-time semilinear fractional diffusion equation is;

$$\frac{\partial^\alpha u}{\partial t^\alpha} = a(x,t)D_x^\beta u(x,t) + f(u) \quad 0 < x < l, 0 < t \leq T, 0 < \alpha \leq 1, 1 < \beta \leq 2 \quad (4)$$

with an initial condition;

$$u(x,0) = g(x) \tag{5}$$

and boundary conditions;

$$u(0,t) = 0 = u(l,t) \tag{6}$$

Our aim is to find the discrete solution of the fractional IBVP (4) - (6). We divide the whole domain into equal parts of rectangles. Define $t_k = k\tau, k = 0,1,2,\dots,n, x_i = ih, i = 0,1,2,\dots,m$ where

$$\tau = \frac{T}{m} \text{ \& } h = \frac{l}{m}$$

are the temporal and spatial steps respectively.

Let u_i^k be the numerical approximation to $u(x_i, t_k)$. First, we approximate the time derivative as follows.

$$\begin{aligned} \frac{\partial^\alpha u}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x, \xi)}{\partial \xi} \frac{d\xi}{(t-\xi)^\alpha} \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^k \int_{(j)\tau}^{(j+1)\tau} \frac{\partial u(x, \xi)}{\partial \xi} \frac{d\xi}{(t_{k+1}-\xi)^\alpha} \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^k \frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{\tau} \int_{(j)\tau}^{(j+1)\tau} \frac{d\xi}{(t_{k+1}-\xi)^\alpha} \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^k \frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{\tau} \int_{(k-j)\tau}^{(k-j+1)\tau} \frac{d\eta}{\eta^\alpha} \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^k \frac{u(x_i, t_{k+1-j}) - u(x_i, t_{k-j})}{\tau} \int_{(j)\tau}^{(j+1)\tau} \frac{d\eta}{\eta^\alpha} \end{aligned}$$

$$\begin{aligned} \frac{\partial^\alpha u}{\partial t^\alpha} &= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} [u(x_i, t_{k+1}) - u(x_i, t_k)] + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \\ &\quad \sum_{j=1}^k [u(x_i, t_{k+1-j}) - u(x_i, t_{k-j})] [(j+1)^{1-\alpha} - j^{1-\alpha}] \end{aligned}$$

where $b_j = (j+1)^{1-\alpha} - j^{1-\alpha}, i = 0,1,2,\dots,m; j = 0,1,2,\dots,k$, we have;

$$\begin{aligned} \frac{\partial^\alpha u(x_i, t_{k+1})}{\partial t^\alpha} &= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} [u(x_i, t_{k+1}) - u(x_i, t_k)] + \\ &\quad \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^k [u(x_i, t_{k+1-j}) - u(x_i, t_{k-j})] b_j + O(\tau^{1+\alpha}). \end{aligned} \tag{7}$$

For every $\beta(0 \leq n-1 < n)$ the Riemann-Liouville derivative exists and coincides with the Grunwald-Letnikov derivative. The relationship between the Riemann-Liouville and Grunwald-Letnikov definitions also has another consequence which is important for the numerical approximations of the fractional order differential equations. This allows the use of the Riemann-Liouville definitions during the problem formulation and then the Grunwald-Letnikov definitions for obtaining the numerical solution. For $D_x^\beta u(x, t)$, we adopt the shifted Grunwald formula at all time levels for approximating the second order space derivative.

$$D_x^\beta u(x_i, t_{k+1}) = \frac{1}{h^\beta} \sum_{j=0}^{i+1} g_j u(x_i - (j-1)h, t_{k+1}) + O(h) \tag{8}$$

where the Grunwald weights are defined by;

$$g_0 = 1, \quad g_j = (-1)^j \frac{(\beta)(\beta-1)(\beta-2)\dots(\beta-j+1)}{j!}, \quad j = 1, 2, 3, \dots \tag{9}$$

and the nonlinear function approximate as;

$$f(x_i, t_k, u(x_i, t_{k+1})) = f(x_i, t_k, u(x_i, t_k)) + O(\tau) \tag{10}$$

We rewrite the Eq. (7) as follows;

$$\begin{aligned} \sum_{j=1}^k [u(x_i, t_{k+1-j}) - u(x_i, t_{k-j})] b_j &= \sum_{j=1}^k u(x_i, t_{k+1-j}) b_j - \sum_{j=1}^k u(x_i, t_{k-j}) b_j \\ &= b_1 u(x_i, t_k) + \sum_{j=2}^k u(x_i, t_{k+1-j}) b_j - \sum_{j=1}^{k-1} u(x_i, t_{k-j}) b_j - b_k u(x_i, t_0) \\ &= b_1 u(x_i, t_k) + \sum_{j=1}^{k-1} u(x_i, t_{k-j}) b_{j+1} - \sum_{j=1}^{k-1} u(x_i, t_{k-j}) b_j - b_k u(x_i, t_0) \\ &= b_1 u(x_i, t_k) + \sum_{j=1}^{k-1} (b_{j+1} - b_j) u(x_i, t_{k-j}) - b_k u(x_i, t_0) \end{aligned} \tag{11}$$

Now using Eqs. (7) - (11);

$$\begin{aligned} u(x_i, t_{k+1}) &= u(x_i, t_k) + r \sum_{j=0}^{i+1} g_j u(x_{i-j+1}, t_{k+1}) - b_1 u(x_i, t_k) + \\ &\quad \sum_{j=1}^{k-1} (b_j - b_{j+1}) u(x_i, t_{k-j}) + b_k u(x_i, t_k) + r_1 f(u(x_i, t_k), x_i, t_k) + R_i^{k+1} \end{aligned} \tag{12}$$

where $r = r(i, k) = \frac{a_i^k \tau^\alpha \Gamma(2-\alpha)}{h^\beta}$, $r_1 = \tau^\alpha \Gamma(2-\alpha)$

$$|R_i^{k+1}| \leq c_1 \tau^\alpha (\tau^{1+\alpha} + h^\beta + \tau) \tag{13}$$

In general, $k \geq 1, i = 1, 2, \dots, m-1$ we can write as;

$$AU^{k+1} = (1-b_1)U^k + \sum_{j=1}^{k-1} (b_j - b_{j+1})U_i^{k-j} + b_k U^0 + r_1 F^k, k > 0 \tag{18}$$

where $F^k = [f(u_1^k), f(u_2^k), \dots, f(u_{m-1}^k)]^T$; $U^{k+1} = [u_1^{k+1}, u_2^{k+1}, \dots, u_{m-1}^{k+1}]^T$.

Stability

Lemma 1 In Eq. (14), the coefficients $b_k (k = 0, 1, \dots)$ and $g_j (j = 0, 1, 2, \dots)$ satisfy:

1. $b_j > b_{j+1}, j = 0, 1, 2, \dots$;
2. $b_0 = 1, b_j > 0, j = 0, 1, \dots$;
3. $g_1 = -\beta, g_j \geq 0 (j \neq 1), \sum_{j=0}^{\infty} g_j = 0$;
4. For any positive integer n, we have $\sum_{j=0}^n g_j < 0$.

Let \bar{u}_i^k be the approximate solution of the implicit finite difference scheme (4) - (6), and let \bar{f}_i^k be the approximations of f_i^k . Setting $\varepsilon_i^k = u_i^k - \bar{u}_i^k$, we obtain the roundoff error equation.

$$(1 + \beta r)\varepsilon_i^1 - r \sum_{j=0, j \neq 1}^{i+1} g_j \varepsilon_{i+1-j}^1 = \varepsilon_i^0 - r_1(f_i^0 - \bar{f}_i^0), k = 0$$

$$(1 + \beta r)\varepsilon_i^{k+1} - r \sum_{j=0, j \neq 1}^{i+1} g_j \varepsilon_{i+1-j}^{k+1} = (1 - b_1)\varepsilon_i^k + r_1(f_i^k - \bar{f}_i^k) + \sum_{j=1}^{k-1} (b_j - b_{j+1})\varepsilon_i^{k-j} + b_k \varepsilon_i^0, k > 1$$

for $i = 1, 2, \dots, m-1, k = 0, 1, 2, \dots, n$, assuming $\|E^k\|_{\infty} = \max_{1 \leq i \leq m-1} |\varepsilon_i^k|$.

We now analyze the stability via the method of induction. When $k = 1$, assume that $|\varepsilon_l^1| = \max\{|\varepsilon_1^1|, |\varepsilon_2^1|, \dots, |\varepsilon_{m-1}^1|\}$ i.e.;

$$|\varepsilon_l^1| \leq (1 + \beta r)|\varepsilon_l^1| - r \sum_{j=0, j \neq 1}^{i+1} g_j |\varepsilon_l^1|$$

$$\leq (1 + \beta r)|\varepsilon_l^1| - r \sum_{j=0, j \neq 1}^{i+1} g_j |\varepsilon_{l-j+1}^1|$$

$$\leq (1 + \beta r)\varepsilon_l^1 - r \sum_{j=0, j \neq 1}^{i+1} g_j \varepsilon_{l-j+1}^1$$

$$= |\varepsilon_{l-1}^1 + r_1(f_i^0 - \bar{f}_i^0)| \tag{19}$$

$$\leq |\varepsilon_l^0| + r_1 L |\varepsilon_l^0|$$

$$\leq (1+r_1L) |\varepsilon_i^0|$$

$$\|E^1\|_\infty \leq C \|\varepsilon^0\|_\infty \quad (\because C = 1+r_1L) \tag{20}$$

Let $\|E^{k+1}\|_\infty = \|\varepsilon_i^{k+1}\| = \max\{|\varepsilon_1^{k+1}|, |\varepsilon_2^{k+1}|, \dots, |\varepsilon_{m-1}^{k+1}|\}$ and assume that $\|E^j\|_\infty \leq C \|\varepsilon^0\|_\infty, j = 1, 2, \dots, k$, we have;

$$|\varepsilon_i^{k+1}| \leq (1+\beta r) |\varepsilon_i^{k+1}| - r \sum_{j=0, j \neq i}^{i+1} g_j |\varepsilon_i^{k+1}|$$

$$\leq (1+\beta r) |\varepsilon_i^{k+1}| - r \sum_{j=0, j \neq i}^{i+1} g_j |\varepsilon_i^{k+1}|$$

$$= |(1-b_1)\varepsilon_i^k + r_1(f_i^k - \bar{f}_i^k) + \sum_{j=1}^{k-1} (b_j - b_{j+1})\varepsilon_i^{k-j} + b_k \varepsilon_i^0|$$

$$\leq (1-b_1) |\varepsilon_i^k| + r_1 |f_i^k - \bar{f}_i^k| + \sum_{j=1}^{k-1} (b_j - b_{j+1}) |\varepsilon_i^{k-j}| + b_k |\varepsilon_i^0|$$

$$\leq (1-b_1) |\varepsilon_i^k| + r_1 L |\varepsilon_i^k| + (b_1 - b_k) |\varepsilon_i^k| + b_k |\varepsilon_i^0|$$

$$\leq (1+r_1L) |\varepsilon_i^0|$$

$$\|E^{k+1}\|_\infty \leq C_0 \|\varepsilon^0\|_\infty, \quad (\because C_0 = C + r_1L). \tag{21}$$

Hence, the following theorem is obtained.

Theorem 1 Suppose that $\varepsilon_i^k (i = 1, 2, \dots, m-1, k = 1, 2, \dots, n)$ is the solution of the roundoff error Eqs. (4) - (6) and the nonlinear source term satisfies the Lipschitz condition, then there is a positive constant C_0 such that $\|\varepsilon^k\|_\infty \leq C_0 \|\varepsilon^0\|_\infty, k = 1, 2, \dots, n$.

Convergence

In this section, we analyze the convergence of the implicit finite difference scheme (4) - (6). Let $u(x_i, t_k)$ be the exact solution of the IBVP (4) - (6) at mesh point (x_i, t_k) and let u_i^k be the numerical solution of (4) - (6) computed using the implicit finite difference scheme.

Define $e_i^k = u(x_i, t_k) - u_i^k$ and $E^k = (e_1^k, e_2^k, \dots, e_{m-1}^k)^T$;

$$(1+\beta r)e_i^1 - r \sum_{j=0, j \neq i}^{i+1} g_j e_{i+1-j}^1 = e_i^0 - r_1(f_i^0 - \bar{f}_i^0) + R_i^1, k = 0$$

$$(1+\beta r)e_i^{k+1} - r \sum_{j=0, j \neq i}^{i+1} g_j e_{i+1-j}^{k+1} = (1-b_1)e_i^k + r_1(f_i^k - \bar{f}_i^k) + \sum_{j=1}^{k-1} (b_j - b_{j+1})e_i^{k-j} + b_k e_i^0 + R_i^{k+1}, k > 1$$

where $i = 1, 2, \dots, m-1, k = 0, 1, 2, \dots, n$

$$|R_i^{k+1}| \leq c_1 \tau^\alpha (\tau^{1+\alpha} + h^\beta + \tau) \text{ for } i = 1, 2, \dots, m-1, k = 0, 1, 2, \dots, n.$$

Using mathematical induction and Lemma 1, we give the convergence analysis as follows. For $k = 1$, assuming that $\|e^1\|_\infty = |e_i^1| = \max_{1 \leq i \leq M-1} |e_i^1|$;

$$\begin{aligned} |e_i^1| &\leq (1 + \beta r) |e_i^1| - r \sum_{j=0, j \neq 1}^{i+1} g_j |e_i^1| \\ &\leq (1 + \beta r) |e_i^1| - r \sum_{j=0, j \neq 1}^{i+1} g_j |e_{i-j+1}^1| \\ &\leq (1 + \beta r) |e_i^1| - r \sum_{j=0, j \neq 1}^{i+1} g_j |e_{i-j+1}^1| \\ &= |e_i^1 + r_1(f_i^0 - \bar{f}_i^0) + R_i^1| \\ &\leq |e_i^0| + r_1 L |e_i^0| + |R_i^1| \\ \|E^1\|_\infty &\leq |R_i^1|. \end{aligned} \tag{22}$$

Using $e^0 = 0$ and $|R_i^1| \leq c_1 \tau^\alpha (\tau^{1+\alpha} + h^\beta + \tau)$, we obtain;

$$\|e^1\|_\infty \leq b_0^{-1} c_1 \tau^\alpha (\tau^{1+\alpha} + h^\beta + \tau).$$

Suppose that it holds for j , $\|e^j\|_\infty \leq b_{j-1}^{-1} c_1 \tau^\alpha (\tau^{1+\alpha} + h^\beta + \tau), j = 1, 2, \dots, k$ and $|e_i^{k+1}| = \max\{|e_1^{k+1}|, |e_2^{k+1}|, \dots, |e_{m-1}^{k+1}|\}$. Note that $b_j^{-1} \leq b_k^{-1}$;

$$\begin{aligned} |e_i^{k+1}| &\leq (1 + \beta r) |e_i^{k+1}| - r \sum_{j=0, j \neq 1}^{i+1} g_j |e_i^{k+1}| \\ &\leq (1 + \beta r) |e_i^{k+1}| - r \sum_{j=0, j \neq 1}^{i+1} g_j |e_{i-j+1}^{k+1}| \\ &= |(1 - b_1) e_i^k + r_1(f_i^k - \bar{f}_i^k) + \sum_{j=1}^{k-1} (b_j - b_{j+1}) e_i^{k-j} + b_k e_i^0 + R_i^{k+1}| \\ &\leq (1 - b_1) |e_i^k| + r_1 |(f_i^k - \bar{f}_i^k)| + \sum_{j=1}^{k-1} (b_j - b_{j+1}) |e_i^{k-j}| + |R_i^{k+1}| \\ &\leq (1 - b_1) \|e^k\|_\infty + r_1 L \|e^k\|_\infty + (b_1 - b_k) \|e^k\|_\infty + |R_i^{k+1}| \\ &\leq b_k^{-1} \{(1 - b_1) + r_1 L + (b_1 - b_k) + b_k\} c_1 \tau^\alpha (\tau^{1+\alpha} + h^\beta + \tau) \\ &\leq b_k^{-1} \{1 + r_1 L\} c_1 \tau^\alpha (\tau^{1+\alpha} + h^\beta + \tau) \end{aligned}$$

$$\|e^{k+1}\|_{\infty} \leq C_0 k^{\alpha} \tau^{\alpha} (\tau^{1+\alpha} + h^{\beta} + \tau), (\because (1+r_1L)c_1 = C_0) \tag{23}$$

If $k\tau \leq T$ is finite then, the following theorem is obtained.

Theorem 2 Let u_i^k be the approximate value of $u(x_i, t_k)$ computed by using an implicit finite difference scheme and source term to satisfy the Lipschitz condition. Then there is a positive constant C_0 such that $|u_i^k - u(x_i, t_k)| \leq C_0(\tau + h)$

Test problem

Example 1 Consider the space-time semilinear fractional diffusion equation

$$\frac{\partial^{0.9} u}{\partial t^{0.9}} = \frac{\partial^{1.8} u}{\partial x^{1.8}} + u^2 + f, \quad 0 < x < \pi, 0 < t \leq T$$

with initial condition, $u(x, 0) = \sin(x)$;

and the boundary conditions $u(0, t) = 0 = u(\pi, t)$;

where $f = t^{0.1} \sin(x) E_{1,1,1}(t) - e^t \sin(x + 0.9\pi) - \sin^2 x e^{2t}$ and exact solution is $u(x, t) = e^t \sin(x)$.

The solution of Example 1 is given as follows with its error analysis (**Table 1**).

Table 1 Comparison between exact solution and obtained solution at $t = 0.01$

$u(x,t)$	I.F.D.M.	Exact solution	Absolute error	Relative error	% Error
$u(\frac{\pi}{6}, 0.01)$	0.5064	0.5050	0.0014	0.0028	0.2772
$u(\frac{\pi}{3}, 0.01)$	0.8751	0.8747	0.0004	4.573×10^{-4}	0.04578
$u(\frac{\pi}{2}, 0.01)$	1.0103	1.0101	0.00024	2.376×10^{-4}	0.0238
$u(\frac{2\pi}{3}, 0.01)$	0.8754	0.8747	0.0007	8.0027×10^{-4}	0.08
$u(\frac{5\pi}{6}, 0.01)$	0.5064	0.5050	0.0014	0.0028	0.2772

I.F.D.M. = Implicit Finite Difference Scheme

The solution of Example 1 at different time levels and its graphical representation is given as follows (**Figure 1**).

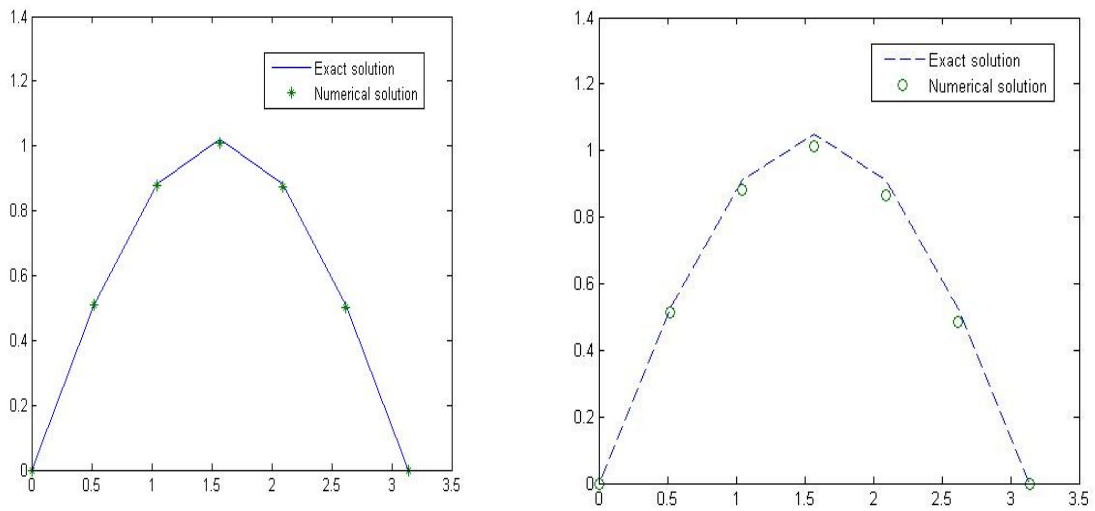


Figure 1 Comparison between the exact solution and the numerical solution when $t = 0.02$ and $t = 0.05$.

Example 2 Consider the space-time semilinear fractional diffusion equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = x \frac{\partial^\beta u}{\partial x^\beta} + \sin(u), \quad 0 < x < 1, 0 < t \leq T, \quad 0 < \alpha \leq 1, 1 < \beta \leq 2$$

with initial conditions, $u(x,0) = x(1-x)$;

and boundary conditions $u(0,t) = 0 = u(1,t)$;

The solution of Example 2 is given as follows with different values of alpha and beta (**Figure 2**).

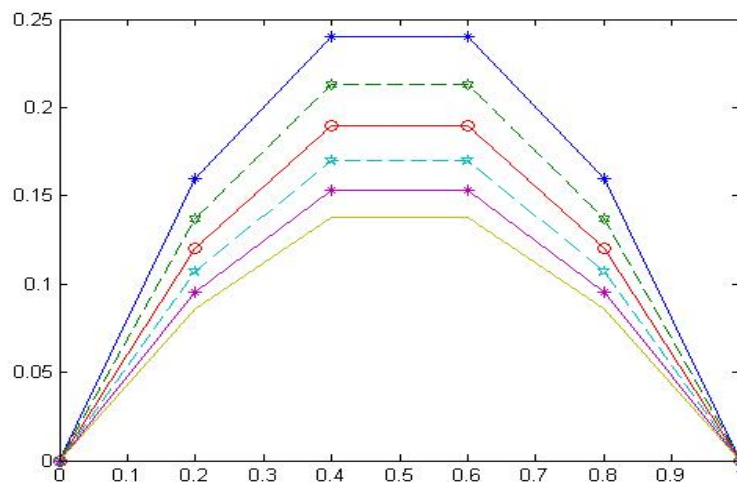


Figure 2 Numerical solution of $u(x,t)$ at different time steps when $\alpha = 0.9$ and $\beta = 1.8$.

Conclusions

This method is very useful to find the numerical solution of semilinear fractional partial differential equations. Stability as well as convergence of the implicit finite difference method is developed by using the matrix method. The theoretical results are demonstrated with the help of numerical problems.

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