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Determination of Approximate Periods of *Duffing-harmonic* Oscillator

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Abstract

We introduced an analytical technique based on harmonic balance method (HBM) to determine approximate periods of a nonlinear *Duffing-harmonic* oscillator. Generally, a set of nonlinear algebraic equations are appeared when HBM is formulated. Investing analytically of such kinds of algebraic equations are a tremendously difficult task and cumbersome. In the present study, the offered technique gives desired results and to avoid numerical complexity. It is remarkable important that a third-order approximate period gives excellent agreement compared with numerical solution. The method is mainly illustrated by strongly nonlinear *Duffing-harmonic* oscillator but it is also useful for many other nonlinear oscillating systems arising in nonlinear sciences and engineering.

Keywords: Approximate periods, harmonic balance method, Duffing-harmonic oscillator, Power series solutions, Perturbation Method

Introduction

Many complex problems in nature are due to nonlinear phenomena. Nowadays, nonlinear processes are one of the biggest challenges in finding solutions and are not easy to control, because the nonlinear characteristic of the system abruptly changes due to small changes of valid parameters, including time. Thus, the issue becomes more complicated and, hence, needs an ultimate solution. Therefore, the study of approximate solutions of nonlinear differential equations (NDEs) plays a crucial role in understanding the internal mechanisms of nonlinear phenomena. Advanced nonlinear techniques are significant in solving inherent nonlinear problems, particularly those involving differential equations, dynamical systems, and related areas. In recent years, mathematicians, engineers, and physicists have made significant improvements in finding new mathematical tools related to NDEs and dynamical systems, whose understanding will rely not only on analytical techniques, but also on numerical and asymptotic methods. These professionals have established many effective and powerful methods to handle the NDEs.

The study of given nonlinear problems is of crucial importance, not only in all areas of physics, but also in engineering and other disciplines, since most phenomena in our world are essentially nonlinear and are described by NDEs. Moreover, obtaining exact solutions for nonlinear oscillatory problems has many difficulties. It is very difficult to solve nonlinear problems and, in general, it is often more difficult to get an analytical approximation solution than a numerical one for a given nonlinear problem. There are many analytical approaches to solve NDEs. One of the popular methods is Perturbation Method [1-3], which is the most versatile tools available in nonlinear analysis of engineering problems and they are constantly being developed and applied to ever more complex problems. However, they are known to be almost useless in the strongly nonlinear oscillatory systems. As a result, due to conquer this weak-point, in recent year, a number of researchers have devoted their time and effort to find potent approaches for investigating to the strongly nonlinear phenomena. As the earliest effort, they developed a large variety of approximate methods commonly used for nonlinear oscillatory systems especially for solving strongly nonlinear oscillators including He's Homotopy Perturbation Method [4], Differential Transforms Method [5,6], Max-Min Approach Method [7,8], Algebraic Method [9], Parameter Expansion Method and Variational Iteration Method [10-12], Amplitude Frequency Formulation Method [13], Iteration Method [14,15], Energy Balance Method [16-18], He's Energy Balance Method [19], Rational Energy Balance Method [20], Rational Harmonic Balance Method [21], Residue Harmonic Balance Method [22-24] and so on. The harmonic balance method (HBM) is another technique for solving strongly nonlinear systems. Borges *et. al.* [25] and Bobylev *et. al.* [26] was first provided overviews of HBM. Mickens [27-29] was first applied HBM in truly nonlinear oscillators. Due to his contribution he is known as father of HBM. Afterwards, Belendez *et al.* [30] and others researchers [31-36] has significantly improved the HBM. The HBM provides a general technique for calculating approximations to the periodic solutions of linear as well as NDEs. The significance of the method is that it may be applied to differential equations for which the nonlinear terms are not small. In this paper, the higher order approximate periods (mainly third approximation periods) have been obtained for a *Duffing-harmonic* oscillator.

The method

Let us consider a strongly nonlinear differential equation;

$$\ddot{x} + \omega_0^2 x = -\varepsilon\, f(x, \dot{x}), \quad [x(0) = a_0,\, \dot{x}(0) = 0]\,, \tag{1}$$

where $f(x,\dot{x})$ is a nonlinear function, such that $f(-x,-\dot{x}) = -f(x,\dot{x})$, $\omega_0 \geq 0$, and ε is a constant.

Consider a periodic solution of Eq. (1) in the form;

$$x = a_0(\rho\cos(\omega t) + u\cos(3\omega t) + v\cos(5\omega t) + w\cos(7\omega t) + z\cos(9\omega t)\cdots)\,, \tag{2}$$

where a_0, ρ, and ω are constants. If $\rho = 1 - u - v - \cdots$ and the initial phase $\varphi_0 = 0$, solution Eq. (2) readily satisfies the initial conditions $[x(0) = a_0,\, \dot{x}(0) = 0]$.

Substituting Eq. (2) into Eq. (1), and expanding $f(x,\dot{x})$ in a Fourier series, displays that it takes the following algebraic identity;

$$\begin{aligned} a_0[\rho(\omega_0^2 - \omega^2)\cos(\omega t) + u(\omega_0^2 - 9\omega^2)\cos(3\omega t) + \cdots] = -\varepsilon[&F_1(a_0, u, \cdots)\cos(\omega t) \\ &+ F_3(a_0, u, \cdots)\cos(3\omega t) + \cdots] \end{aligned} \tag{3}$$

By comparing the coefficients of equal harmonics of Eq. (3), the following nonlinear algebraic equations are found;

$$\rho(\omega_0^2 - \omega^2) = -\varepsilon F_1, \quad u(\omega_0^2 - 9\omega^2) = -\varepsilon F_3, \quad v(\omega_0^2 - 25\omega^2) = -\varepsilon F_5, \cdots \tag{4}$$

with the help of the first equation, ω^2 is eliminated from all the rest of Eq. (4). Thus, Eq. (4) takes the following form;

$$\rho\omega^2 = \rho\omega_0^2 + \varepsilon F_1, \quad 8\omega_0^2 u\rho = \varepsilon(\rho F_3 - 9uF_1), \quad 24\omega_0^2 v\rho = \varepsilon(\rho F_5 - 25vF_1), \cdots \tag{5}$$

Substitution $\rho = 1 - u - v - \cdots$, and simplification, the second-, third-equations of Eq. (5) take the following form;

$$u = G_1(\omega_0, \varepsilon, a_0, u, v, \cdots, \lambda_0), \quad v = G_2(\omega_0, \varepsilon, a_0, u, v, \cdots, \lambda_0), \cdots, \tag{6}$$

where $G_1, G_2, \cdots$ exclude respectively the linear terms of $u, v, \cdots$.

Whatever the values of ω_0 and a_0, there exists a parameter $\mu_0(\omega_0, \varepsilon, a_0) << 1$, such that $u, v, \cdots$ are expandable in the following power series in terms of λ_0 as;

$$u = U_1\lambda_0 + U_2\lambda_0^2 + \cdots, \quad v = V_1\lambda_0 + V_2\lambda_0^2 + \cdots, \quad \cdots \tag{7}$$

where $U_1, U_2, \cdots, V_1, V_2, \cdots$ are constants.

Finally, by substituting the values of $u, v, \cdots$ from Eq. (7) into the first equation of Eq. (5), ω is determined. This completes the determination of all related functions for the proposed periodic solution as given in Eq. (2), and using $T = \dfrac{2\pi}{\omega}$, the approximate periods have been calculated.

Example

Let us consider the following *Duffing-harmonic* oscillator;

$$\ddot{x} + x^3/(1+x^2) = 0 \tag{8}$$

Eq. (8) is written in another form as;

$$\ddot{x} + x^3 - x^5 + x^7 - \cdots = 0\,. \tag{9}$$

Herein, we have determined second- and third-order approximations of the period for the *Duffing-harmonic* oscillator.

Let us consider a two-term solution, *i.e.*, $x = a_0(\rho\cos(\omega_2 t) + u\cos(3\omega_2 t))$ for Eq. (9). Substituting this solution along with $\rho = 1-u$ into Eq. (9), Eq. (3) becomes;

$$\begin{aligned}(1-u)\,\omega_2^2\cos(\omega_2 t) + 9u\omega_2^2\cos(3\omega_2 t) &= 3a_0^2/4 - 5a_0^4/8 - 3a_0^2u/2 + 9a_0^2u^2/4 + \cdots)\cos(\omega_2 t) \\ &\quad + (a_0^2/4 - 5a_0^4/16 + 3a_0^2u/4 - 9a_0^2u^2/4 + \cdots)\cos(3\omega_2 t)]/4 + HOH,\end{aligned} \tag{10}$$

where *HOH* represents higher order harmonics.

Now, comparing the coefficients of equal harmonics, the following equations are obtained;

$$\begin{aligned}(1-u)\,\omega_2^2 &= 3a_0^2/4 - 5a_0^4/8 - 3a_0^2u/2 + 25a_0^4u/16 + 9a_0^2u^2/4 - 15a_0^4u^2/4 + \cdots \\ 9u\omega_2^2 &= a_0^2/4 - 5a_0^4/16 + 3a_0^2u/4 - 5a_0^4u/16 - 9a_0^2u^2/4 + 5a_0^4u^2/2 - 63a_0^6u^2/32 + \cdots\end{aligned} \tag{11}$$

From the first equation of Eq. (11), it becomes;

$$\omega_2^2 = (3a_0^2/4 - 5a_0^4/8 - 3a_0^2u/2 + 25a_0^4u/16 + 9a_0^2u^2/4 - 15a_0^4u^2/4 + \cdots)/(1-u) \tag{12}$$

By elimination of ω_2^2 from the second equations of Eq. (11), with the help of Eq. (12) and simplification, the following nonlinear algebraic equation of u is found;

$$\begin{aligned}
&a_0^2/4-5a_0^4/16+21a_0^6/64-25a_0^2u/4+45a_0^4u/8-21a_0^6u/4+21a_0^2u^2/2-45a_0^4u^2/4\\
&+189a_0^6u^2/16-16a_0^2u^3+25a_0^4u^3-1995a_0^6u^3/64+23a_0^2u^4/2-675a_0^4u^4/16\\
&+2275a_0^6u^4/32+355a_0^4u^5/8-2009a_0^6u^5/16-85a_0^4u^6/4+2499a_0^6u^6/16\\
&-3844a_0^6u^7/32+1365a_0^6u^8/32=0
\end{aligned} \tag{13}$$

For the *Duffing-harmonic* oscillator, the series of u presented in Eq. (13) is invalid. Herein, u is substituted by $u_0+u_2a_0^2+u_4a_0^4+\cdots$ into Eq. (13); equating the coefficients of $a_0^2,\ a_0^4,\cdots$ yields;

$$\begin{aligned}
&1-25u_0+42u_0^2-64u_0^3+46u_0^4=0,\\
&-5/4+45u_0/2-45u_0^2+100u_0^3-675u_0^4/4+355u_0^5/2-85u_0^6-25u_2+84u_0u_2\\
&-192u_0^2u_2+184u_0^3u_2=0,\\
&21/16-21u_0+189u_0^2/4-1995u_0^3/16+2275u_0^4/8-2009u_0^5/4+2499u_0^6/4\\
&-3843u_0^7/8+1365u_0^8/8+45u_2/2-90u_0u_2+300u_0^2u_2-675u_0^3u_2+1775u_0^4u_2/2\\
&-510u_0^5u_2+42u_2^2-192u_0u_2^2+276u_0^2u_2^2-25u_4+84u_0u_4-192u_0^2u_4+184u_0^3u_4=0,\\
&\cdots
\end{aligned} \tag{14}$$

The coefficients of u_0, u_2, u_4, respectively in the 3 equations of Eq. (14) are 25. In Eq. (14), the equations of u_0, u_2, u_4, can be written as;

$$\begin{aligned}
&u_0=\lambda(1+42u_0^2-64u_0^3+46u_0^4),\\
&u_2=\lambda(-5/4+45u_0/2-45u_0^2+100u_0^3-675u_0^4/4+355u_0^5/2-85u_0^6+84u_0u_2\\
&-192u_0^2u_2+184u_0^3u_2,\\
&u_4=\lambda(21/16-21u_0+189u_0^2/4-1995u_0^3/16+2275u_0^4/8-2009u_0^5/4+2499u_0^6/4\\
&-3843u_0^7/8+1365u_0^8/8+45u_2/2-90u_0u_2+300u_0^2u_2-675u_0^3u_2+1775u_0^4u_2/2\\
&-510u_0^5u_2+42u_2^2-192u_0u_2^2+276u_0^2u_2^2+84u_0u_4-192u_0^2u_4+184u_0^3u_4=0,
\end{aligned} \tag{15}$$

where $\lambda=1/25$.

Therefore, the power series solutions of these equations in terms of λ are obtained as;

$$\begin{aligned}
&u_0=\lambda+42\lambda^3-64\lambda^4+3574\lambda^5-13440\lambda^6+394320\lambda^7-2391424\lambda^8+\cdots,\\
&u_2=-\frac{5}{4}\lambda+\frac{45}{2}\lambda^2-150\lambda^3+3175\lambda^4-\frac{107795}{4}\lambda^5+\frac{1009705}{2}\lambda^6-5099980\lambda^7+\cdots,\\
&u_4=\frac{21}{16}\lambda-\frac{393}{8}\lambda^2+\frac{6735}{8}\lambda^3-\frac{221163}{16}\lambda^4+\frac{1888951}{8}\lambda^5-\frac{28175843}{8}\lambda^6+\cdots.
\end{aligned} \tag{16}$$

Now, substituting the value of $u=u_0+u_2a^2+u_4a^4$, where u_0, u_2, u_4 are calculated by Eq. (16), into Eq. (12), the period of oscillation is calculated as;

$$T_2=7.401780/a_0+2.975753a_0+\cdots. \tag{17}$$

In a similar way, the method can be used to determine higher order approximations. In this article, a third approximate solution is obtained;

$$x = a_0 \cos(\omega_3 t) + a_0 u(\cos(3\omega_3 t) - \cos(\omega_3 t)) + a_0 v(\cos(5\omega_3 t) - \cos(\omega_3 t)) , \tag{18}$$

Substituting Eq. (18) into Eq. (9) and equating the coefficients of $\cos(\omega_3 t)$, $\cos(3\omega_3 t)$, and $\cos(5\omega_3 t)$, the following equations are obtained;

$$\begin{aligned}
&(1-u-v)\omega_3^2 = 3a_0^2/4 - 5a_0^4/8 + 35a_0^6/64 - 3a_0^2u/2 + 25a_0^4u/16 - 49a_0^6u/32 \\
&+9a_0^2u^2/4 - 15a_0^4u^2/4 + 147a_0^6u^2/32 - 3a_0^2u^3/2 + 25a_0^4u^3/4 - 175a_0^6u^3/16 \\
&-25a_0^4u^4/4 - 9a_0^2v/4 + 45a_0^4v/16 - 49a_0^6v/16 + 9a_0^2uv/2 - 10a_0^4uv + 231a_0^6uv/16 \\
&-3a_0^2u^2v + 75a_0^4u^2v/4 + \cdots \\
&9u\omega_3^2 = a_0^2/4 - 5a_0^4/16 + 21a_0^6/64 + 3a_0^2u/4 - 5a_0^4u/16 - 9a_0^2u^2/4 + 5a_0^4u^2/2 \\
&-63a_0^6u^2/32 + 2a_0^2u^3 - 25a_0^4u^3/4 + 525a_0^6u^3/64 + 125a_0^4u^4/16 + 5a_0^4v/16 \\
&-21a_0^6v/32 - 3a_0^2uv/2 + 5a_0^4uv/2 - 63a_0^6uv/32 + 3a_0^2u^2v/2 - 75a_0^4u^2v/8 + \cdots \\
&25v\omega_3^2 = -a_0^4/16 + 7a_0^6/64 + 3a_0^2u/4 - 15a_0^4u/16 + 7a_0^6u/8 - 3a_0^2u^2/4 \\
&+5a_0^4u^2/2 - 231a_0^6u^2/64 - 25a_0^4u^3/8 + 525a_0^6u^3/64 + 25a_0^4u^4/16 + 3a_0^2v/2 \\
&-25a_0^4v/16 + 91a_0^6v/64 - 9a_0^2uv/2 + 35a_0^4uv/4 - 189a_0^6uv/16 + 15a_0^2u^2v/4 \\
&-75a_0^4u^2v/4 + 315a_0^6u^2v/8 + \cdots
\end{aligned} \tag{19}$$

From the first equation of Eq. (19), it yields;

$$\begin{aligned}
&\omega_3^2 = (3a_0^2/4 - 5a_0^4/8 + 35a_0^6/64 - 3a_0^2u/2 + 25a_0^4u/16 - 49a_0^6u/32 \\
&+9a_0^2u^2/4 - 15a_0^4u^2/4 + 147a_0^6u^2/32 - 3a_0^2u^3/2 + 25a_0^4u^3/4 - 175a_0^6u^3/16 \\
&-25a_0^4u^4/4 - 9a_0^2v/4 + 45a_0^4v/16 - 49a_0^6v/16 + 9a_0^2uv/2 - 10a_0^4uv + 231a_0^6uv/16 \\
&-3a_0^2u^2v + 75a_0^4u^2v/4 + \cdots)/(1-u-v)
\end{aligned} \tag{20}$$

By eliminating ω_3^2 from the second and third equations of Eq. (19) with the help of Eq. (20) and simplification, the following nonlinear algebraic equations of u and v are found;

$$\begin{aligned}
&-a_0^2/4 + 5a_0^4/16 - 21a_0^6/64 + 25a_0^2u/4 - 45a_0^4u/8 + 21a_0^6u/4 - 21a_0^2u^2/2 \\
&+45a_0^4u^2/4 - 189a_0^6u^2/16 + 16a_0^2u^3 - 25a_0^4u^3 + 1995a_0^6u^3/64 - 23a_0^2u^4/2 \\
&+675a_0^4u^4/16 + a_0^2v/4 - 5a_0^4v/8 + 63a_0^6v/64 - 18a_0^2uv + 365a_0^4uv/16 \\
&-105a_0^6uv/4 + 141a_0^2u^2v/4 - 605a_0^4u^2v/8 + 3507a_0^6u^2v/32 - 47a_0^2u^3v/2 \\
&+\cdots = 0, \\
&a_0^4/16 - 7a_0^6/64 - 3a_0^2u/4 + 7a_0^4u/8 - 49a_0^6u/64 + 3a_0^2u^2/2 - 55a_0^4u^2/16 \\
&+287a_0^6u^2/64 - 3a_0^2u^3/4 + 45a_0^4u^3/8 - 189a_0^6u^3/16 - 75a_0^4u^4/16 + 315a_0^6u^4/16 \\
&+69a_0^2v/4 - 113a_0^4v/8 + 791a_0^6v/64 - 123a_0^2uv/4 + 445a_0^4uv/16 - 1547a_0^6uv/64 \\
&+189a_0^2u^2v/4 - 255a_0^4u^2v/4 + 3843a_0^6u^2v/64 - 135a_0^2u^3v/4 + \cdots = 0.
\end{aligned} \tag{21}$$

Again we observe that, for the *Duffing-harmonic* oscillator, the series of u and v presented in Eq. (21) are invalid. Herein, u is substituted by $u_0 + u_2 a_0^2 + u_4 a_0^4 + \cdots$, and v is substituted by $v_0 + v_2 a_0^2 + v_4 a_0^4 + \cdots$ into Eq. (21), and then, by equating the coefficients of $a_0^2,\ a_0^4, \cdots$, yields;

$$\begin{aligned}
&1 - 25u_0 + 42u_0^2 - 64u_0^3 + 46u_0^4 - v_0 + 72u_0v_0 - 141u_0^2v_0 + 94u_0^3v_0 - 3v_0^2 - 117u_0v_0^2 \\
&+ 147u_0^2v_0^2 + 5v_0^3 + 70u_0v_0^3 - 2v_0^4 = 0, \\
&3u_0 - 6u_0^2 + 3u_0^3 - 69v_0 + 123u_0v_0 - 189u_0^2v_0 + 135u_0^3v_0 + 207v_0^2 - 405u_0v_0^2 \\
&+ 270u_0^2v_0^2 - 354v_0^3 + 426u_0v_0^3 + 216v_0^4 = 0,
\end{aligned} \tag{22}$$

$$\begin{aligned}
&5/16 - 45u_0/8 + 45u_0^2/4 - 25u_0^3 + 675u_0^4/16 - 355u_0^5/8 + 85u_0^6/4 + 25u_2/4 - 21u_0u_2 \\
&+ 48u_0^2u_2 - 46u_0^3u_2 - 5v_0/8 + 365u_0v_0/16 - 605u_0^2v_0/8 + 1125u_0^3v_0/8 - 285u_0^4v_0/2 \\
&+ 965u_0^5v_0/16 - 18u_2v_0 + 141u_0u_2v_0/2 - 141u_0^2u_2v_0/2 - 5v_0^2/16 - 265u_0v_0^2/4 \\
&+ 855u_0^2v_0^2/4 - 290u_0^3v_0^2 + 2465u_0^4v_0^2/16 + 117u_2v_0^2/4 - 147u_0u_2v_0^2/2 + 25v_0^3/8 \\
&+ 465u_0v_0^3/4 - 1125u_0^2v_0^3/4 + 1545u_0^3v_0^3/8 - 35u_2v_0^3/2 - 95v_0^4/16 - 895u_0v_0^4/8 \\
&+ 1175u_0^2v_0^4/8 + 5v_0^5 + 715u_0v_0^5/16 - 25v_0^6/16 + v_2/4 - 18u_0v_2 + 141u_0^2v_2/4 \\
&- 47u_0^3v_2/2 + 3v_0v_2/2 + 117u_0v_0v_2/2 - 147u_0^2v_0v_2/2 - 15v_0^2v_2/4 - 105u_0v_0^2v_2/2 \\
&+ 2v_0^3v_2 = 0,
\end{aligned} \tag{23}$$

$$\begin{aligned}
&1/16 + 7u_0/8 - 55u_0^2/16 + 45u_0^3/8 - 75u_0^4/16 + 3u_0^5/2 + u_0^6/16 - 3u_2/4 + 3u_0u_2 \\
&- 9u_0^2u_2/4 - 113v_0/8 + 445u_0v_0/16 - 255u_0^2v_0/4 + 915u_0^3v_0/8 - 1005u_0^4v_0/8 \\
&+ 981u_0^5v_0/16 - 123u_2v_0/4 + 189u_0u_2v_0/2 - 405u_0^2u_2v_0/4 + 495v_0^2/8 - 1665u_0v_0^2/8 \\
&+ 3105u_0^2v_0^2/8 - 1585u_0^3v_0^2/4 + 1355u_0^4v_0^2/8 + 405u_2v_0^2/4 - 135u_0u_2v_0^2 - 395v_0^3/2 \\
&+ 625u_0v_0^3 - 1675u_0^2v_0^3/2 + 3525u_0^3v_0^3/8 - 213u_2v_0^3/2 + 5785v_0^4/16 - 835u_0v_0^4 \\
&+ 9015u_0^2v_0^4/16 - 2833v_0^5/8 + 6971u_0v_0^5/16 + 569v_0^6/4 + 69v_2/4 - 113u_0v_2/4 \\
&+ 189u_0^2v_2/4 - 135u_0^3v_2/4 - 207v_0v_2/2 + 405u_0v_0v_2/2 - 135u_0^2v_0v_2 + 531v_0^2v_2/2 \\
&- 639u_0v_0^2v_2/2 - 216v_0^3v_2 = 0, \\
&\cdots
\end{aligned} \tag{24}$$

In Eqs. (22) - (24) the equations of u_0, v_0, u_2, v_2 can be written as;

$$u_0 = \lambda(1 + 42u_0^2 - 64u_0^3 + 46u_0^4 - v_0 + 72u_0v_0 - 141u_0^2v_0 + 94u_0^3v_0 + \cdots) \tag{25}$$

$$v_0 = \mu(u_0 - 2u_0^2 + u_0^3 + 41u_0v_0 - 63u_0^2v_0 + 45u_0^3v_0 + 69v_0^2 - 135u_0v_0^2 + \cdots) \tag{26}$$

$$\begin{aligned}
u_2 = \lambda(&-5/4 + 45u_0/2 - 45u_0^2 + 100u_0^3 - 675u_0^4/4 + 355u_0^5/2 - 85u_0^6 + 84u_0u_2 \\
&- 192u_0^2u_2 + 184u_0^3u_2 + 5v_0/2 - 365u_0v_0/4 + 605u_0^2v_0/2 + \cdots)
\end{aligned} \tag{27}$$

$$\begin{aligned}
v_2 = \mu(&-1/12 - 7u_0/6 + 55u_0^2/12 - 15u_0^3/2 + 25u_0^4/4 - 2u_0^5 - u_0^6/12 + u_2 - 4u_0u_2 \\
&+ 3u_0^2u_2 + 113v_0/6 - 445u_0v_0/12 + 85u_0^2v_0/2 + \cdots)
\end{aligned} \tag{28}$$

where λ is defined in Eq. (15) and $\mu = 1/23$. The algebraic relation between λ and μ are;

$$\mu = 25\lambda/23 \tag{29}$$

Now, solving Eqs. (25) and (26) and then Eqs. (27) and (28) simultaneously in terms of λ;

$$\begin{aligned}
u_0 &= \lambda + \frac{941}{23}\lambda^3 + \frac{378}{23}\lambda^4 + \frac{1626871}{529}\lambda^5 + \cdots \\
v_0 &= \frac{25}{23}\lambda^2 - \frac{50}{23}\lambda^3 + \frac{49725}{529}\lambda^4 - \frac{128400}{529}\lambda^5 + \cdots \\
u_2 &= -\frac{5}{4}\lambda + \frac{6236}{276}\lambda^2 - \frac{41725}{276}\lambda^3 + \frac{17309425}{6348}\lambda^4 - \frac{110254165}{6348}\lambda^5 + \cdots \\
v_2 &= -\frac{25}{276}\lambda - \frac{725}{276}\lambda^2 + \frac{337625}{6348}\lambda^3 - \frac{954025}{1587}\lambda^4 + \frac{355128650}{36501}\lambda^5 + \cdots
\end{aligned} \tag{30}$$

Substituting the values of $u = u_0 + u_2 a^2 + \cdots$ and $v = v_0 + v_2 a^2 + \cdots$ where u_0, u_2, and v_0, v_2, are calculated by Eq. (30) into Eq. (20), the third-order approximate period of oscillation is calculated as;

$$T_3 = 7.415647/a_0 + 2.935536 a_0 + \cdots \tag{31}$$

Results and discussions

We illustrate the accuracy of a new analytical technique by comparing the approximate periods previously obtained with the exact period T_{ex}. For this nonlinear problem, the exact period is;

$$T_{ex} = 7.4163\cdots/a_0 + 2.93048\cdots a_0 + \cdots. \tag{32}$$

which is stated in Belendez *et al.* [4].

The second- and third-order approximate periods, obtained in this study by applying analytical technique to the aforementioned *Duffing-harmonic* oscillator, are the following;

$$T_2 = 7.40158/a_0 + 2.97549 a_0 + \cdots, \tag{33}$$

$$T_3 = 7.415647/a_0 + 2.935536 a_0 + \cdots. \tag{34}$$

Belendez *et al.* [4] investigated the approximate periods for the nonlinear *Duffing-harmonic* oscillator by using He's homotopy perturbation method. He obtained the following first-order approximate periods in 2 different forms as;

$$T_a = 7.2552/a_0 + 2.7207 a_0 + \cdots \tag{35}$$

$$T_b = 7.2552/a_0 + 3.0230 a_0 + \cdots \tag{36}$$

Comparing all the approximate periods, the accuracy of the results obtained in this paper using an analytical technique is better than those obtained previously existing results. It is noted that, the third-order approximate period gives almost same fashion with exact periods. It has been mentioned that, the solution procedure of Belendez *et al.* [4] is laborious, especially for obtaining the higher approximations.

Therefore, second- and third-order approximate periods have not been calculated. On the other hand, the technique offered in this article is simple, easy, and highly efficient. Comparing the results obtained in this article with those previously obtained by several authors, it is shown that the proposed method is simpler than several existing procedures. The advantages of this method include its simplicity, its computational efficiency, and its ability to objectively find better agreement in third-order approximate periods.

Conclusions

An analytical technique has been established based on HBM to find approximate periods for strongly nonlinear *Duffing-harmonic* oscillators. The approximate periods show good agreement comparing with corresponding numerical solutions. We can see in third-order approximate period, the percentage errors of first and second terms are 0.0088 % and −0.1725 %, respectively. In comparison with previously published methods, determination of the solutions is straightforward, quite easy and simple. To sum up, we can say that the method presented in this article to determine approximate periods for a *Duffing-harmonic* oscillator can be considered as an efficient alternative to the previously proposed methods.

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