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A New Approach for Generalized Partial Derivatives of Non-smooth Functions

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Abstract

In this paper, we define the functional optimization problems corresponding to multi-variable smooth functions, so that their optimal solutions are partial derivatives of these functions. These functional optimization problems are applied for multi-variables nonsmooth functions, so that by solving these problems we obtain generalized partial derivatives. For this purpose, linear programming problems corresponding to the functional optimization problems are obtained, so that their optimal solutions give the approximate generalized partial derivatives. In some illustrative examples, we show the efficiency of our approach by obtaining the generalized partial derivative of some smooth and nonsmooth functions, respectively.

Keywords: Generalized derivative, nonsmooth functions, optimization

Introduction

Nonsmooth functions arise in various branches of science and engineering and in many dynamical system applications; examples include the occurrence of impacting motion in mechanical systems, stick-slip motion in oscillators with friction, switching in electronic circuits, and hybrid dynamics in control systems. Also, many mathematical models used in real world application, which is one of the important concepts in various areas of basic science and engineering, are nonsmooth. For instance, the evolution of DNA sequences or the growth and inheritance of an entire population in biology, the immunology of diseases in medicine, diodes and transistors in electrical circuits, problems in switching systems that arise in air traffic management, scheduling of automated railway systems, and economic market models. These kinds of systems are not differentiable, and so many methods based on differentiability cannot be used. Although the theory of smooth systems still plays an important role in the framework of nonsmooth systems, the vast majority of problems in nonsmooth dynamical systems call for completely new methodologies. As nonsmooth behavior is so important in applications, there is a mature concept known as nonsmooth analysis. Undoubtedly, nonsmooth analysis has developed in many fields of applications, especially optimization, control theory, mechanics, and dynamic systems. Basic tools, called generalized derivative for nonsmooth functions, were developed. The key to solving applied problems, including nonsmooth functions, is the generalized derivative. Note that for solving these problems, we usually need to obtain an approximated derivative for nonsmooth functions; however, there have not been sufficient contributions, or indeed any serious efforts. in this domain up to now.

As is well known, many different generalized derivatives have been introduced, for instance by Clarke [1], Clarke *et al*. [2] and Mordukhovich [3,4]. These generalized derivatives are not practical or applicable for solving problems.

In existing works, for obtaining generalized partial derivative (GPD), there are many conditions and restrictions; for example, in a nonsmooth function $f(.)$, it must be continuous and locally Lipschitz or convex, and nondifferentiable at a given point x of a compact and connected set. Moreover, we see the obtained generalized partial derivative (GPD) of $f(.)$, at $x \in \Omega$ is a set, which may be empty or include several members, and the directional derivative is usually used to introduce GPD. Indeed, the concepts *limsup* and *liminf* are applied to obtain the GPD; both of these could lead to complicated computations.

A new generalized derivative for one-variable and second derivative nonsmooth functions were defined in [5] and [6], respectively. In this paper, we extend this concept of the generalized derivative to the multi-variable nonsmooth functions, and define the GPDs which appear in many applications, especially in optimization problems. Moreover, Erfanian *et al*. [7] used this generalized derivative for solving nonsmooth ordinary differential equations, which are frequently used in many various branches of science and engineering. For instance, in optimal control problems, bang-bang controllers switch discontinuously between maximum and minimum values, control theory which deal with switch control laws.

In our approach, we introduce the especial functional optimization problems for obtaining the GPD of nonsmooth functions. These functional optimization problems approximated with the corresponding linear programming (LP) problems. Our approach for obtaining GPD does not have the above conditions and restrictions of the other generalized derivatives, and is easier to implement, more practical, and gives better results.

The structure of this paper is as follows. Section 2 defines the GPD of nonsmooth functions based on functional optimization. In Section 3, an LP problem to obtain the approximate GPD of nonsmooth functions is introduced. In Section 4, we use our approach for smooth and nonsmooth functions in some examples. The conclusions of this paper is stated in Section 5.

GPD of nonsmooth functions

In this section, we introduce the functional optimization problems where optimal solutions are the partial derivatives of smooth function on $\Omega \subset \mathbb{R}^n$. At first, we state the following Lemma, in which $C(\Omega)$ is the space of continuous function on Ω :

Lemma 1: Let $h(.) \in C(\Omega)$ and $\int_\Omega \eta(x)h(x)dx = 0$ for any $\eta(.) \in C(\Omega)$. Then, $h(x) = 0$ for all $x \in \Omega$.

Proof: Consider $x_0 \in \Omega$, such that $h(x_0) \neq 0$. Without loss of generality, suppose $h(x_0) > 0$, and since $h(.) \in C(\Omega)$, then there is a neighborhood $N(x_0,\delta)$, $\delta > 0$, of x_0, such that $h(x) > 0$ for all $x \in N(x_0,\delta)$. We consider the function $\eta_0(.) \in C(\Omega)$, such that $\eta_0(.)$ is zero on $\Omega \setminus N(x_0,\delta)$ and positive on $N(x_0,\delta)$. Thus, we have;

$$0 < \int_{N(x_0,\delta)} \eta_0(x)h(x)\,dx = \int_\Omega \eta_0(x)h(x)dx \tag{1}$$

so $\int_\Omega \eta_0(x)h(x)dx > 0$, which is a contradiction. Then $h(x) = 0$ for all $x \in \Omega$.□

In this paper, we select an arbitrary (but fixed) index of $i \in \{1,2,\dots,n\}$ and calculate the partial derivative of $f(.) : \Omega \to \mathbb{R}$ with respect to x_i. Without loss of generality, we assume $\Omega = [0,1]^n$, and define $\bar{\Omega} \subset \Omega$ as follows;

$$\bar{\Omega} = \{(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n) : x_j \in [0,1],\ j = 1,2,\dots,n,\ j \neq i\}. \tag{2}$$

Now, we select N as a sufficiently large number, and divide $\overline{\Omega}$ to the similar grids $\overline{\Omega}_j$, $j = 1,2,...,N^{n\text{-}1}$, such that these grids cover the set $\overline{\Omega}$. In the next step, we consider the arbitrary points $s_j \in \overline{\Omega}_j,\ j = 1,2,...,N^{n\text{-}1}$ as $s_j = (s_{j_1},...,s_{j_{i\text{-}1}},s_{j_{i+1}},...,s_{j_n})$. Moreover, we define the following vector for all $s_j \in \overline{\Omega}_j$;

$$r_j(t) = (s_{j_1},...,s_{j_{i\text{-}1}},t,s_{j_{i+1}}...,s_{j_n}), \qquad t \in [0,1]\,, \quad j = 1,2,...,N^{n\text{-}1}. \tag{3}$$

Now we state the following theorem; it is proved by Lemma 1 that $C^1(\Omega)$ is the space of differentiable functions with the continuous derivative on Ω.

Theorem 2: Let $g(.) \in C[0,1]$, $f(.) : \Omega \to \mathbb{R}$ and $h_j(t) = f(r_j(t))$, $t \in [0,1]$ for $j = 1,2,...,N^{n\text{-}1}$. If $\int_0^1 \left(v'(t)h_j(t) + v(t)g(t)\right)dt = 0$ for any $v(.) \in C^1[0,1]$ where $v(0) = v(1) = 0$, Then $h_j(.) \in C^1[0,1]$ and $h'_j(.) = g(.)$.

Proof: We use integration by parts rule and condition $v(0) = v(1) = 0$;

$$\begin{aligned}\int_0^1 v(t)g(t)\,dt &= \left[v(t)G(t)\right]_0^1 - \int_0^1 v'(t)G(t)\,dt \\ &= \left[v(1)G(1) - v(0)G(0)\right] - \int_0^1 v'(t)G(t)\,dt \\ &= -\int_0^1 v'(t)G(t)\,dt.\end{aligned} \tag{4}$$

where $G(t) = \int_0^t g(\tau)\,d\tau$ for $\tau \in [0,1]$. Since $\int_0^1 \left(v'(t)h_j(t) + v(t)g(t)\right)dt = 0$, so by the relation (4), $\int_0^1 v'(t)G(t)\,dt = \int_0^1 v'(t)h_j(t)$ or $\int_0^1 v'(t)\left(G(t) - h_j(t)\right)dt = 0$. Put $\lambda_j(t) = G(t) - h_j(t)$ for each $t \in [0,1]$. By Lemma 1, since $v'(.) \in C[0,1]$ we have $l_j(t) = 0$ for all $t \in [0,1]$; it means that $G(.) = h_j(.)$. Therefore, $h_j(t) = \int_0^t g(\tau)\,d\tau$. Hence, $h_j(.) \in C^1[0,1]$ and $h'_j(.) = g(.)$.□

Let $V = \left\{v_k(.) : v_k(t) = \sin k\pi t : t \in [0,1],\, k = 1,2,...\right\}$. Then, $v_k(0) = v_k(1) = 0$ for $v_k(.) \in V$. We can extend every continuous function $v(.) \in C[0,1]$, which satisfies $v(0) = 0$ as an odd function on $[\text{-}1,1]$. Thus, there is a Sinus expansion for this function on $[\text{-}1,1]$. Now, consider the following theorem.

Theorem 3: Suppose $g(.) \in C[0,1]$, $f(.) : \Omega \to \mathbb{R}$ and $h_j(t) = f(r_j(t))$, $t \in [0,1]$ for $j = 1,2,...,N^{n\text{-}1}$. If $\int_0^1 \left(v'_k(t)h_j(t) + v_k(t)g(t)\right)dx = 0$ for any $v_k(.) \in V$, then $h_j(.) \in C^1[0,1]$ and $h'_j(.) = g(.)$.

Proof: Let $\widehat{V} = \left\{ v(.) \in C^1[0,1] : v(0) = v(1) = 0 \right\}$ and $v(.) \in \widehat{V}$. Since set V is a total set for space $\widehat{V}$, there exist real coefficients $c_1, c_2, \ldots$ such that $v(t) = \sum_{k=1}^{\infty} c_k v_k(t)$ for any $t \in [0,1]$ where $v_k(.) \in V$. Since $\int_0^1 \left(v_k'(t) h_j(t) + v_k(t) g(t) \right) dt = 0$ for any $k = 1, 2, \ldots$ so;

$$\sum_{k=1}^{\infty} c_k \left(\int_0^1 \left(v_k'(t) h_j(t) + v_k(t) g(t) \right) dt \right) = 0 . \tag{5}$$

We know the series $\sum_{k=1}^{\infty} c_k v_k(.)$ is uniformly convergent to the function $v(.)$. So by the relation (5);

$$\int_0^1 \left(\sum_{k=1}^{\infty} c_k \; v_k'(t) \; h_j(t) + \sum_{k=1}^{\infty} c_k v_k(t) \; g(t) \right) dt = 0 . \tag{6}$$

Thus, by (6);

$$\int_0^1 \left(v'(t) h_j(t) + v(t) g(t) \; dt \right) = 0 , \tag{7}$$

where $v'(t) = \sum_{k=1}^{\infty} c_k v_k'(t)$, $t \in [0,1]$. Thus, for any $v(.) \in \widehat{V}$ the relation (7) and conditions of Theorem 2 hold. Then, we have $h_j(.) \in C^1[0,1]$ and $h_j'(.) = g(.)$. □

Here we state the following theorem, and in the next step apply it.

Theorem 4: Let $\varepsilon > 0$ be given a small number, $f(.) : \Omega \to \mathbb{R}$ and $h_j(t) = f(r_j(t))$, $t \in [0,1]$ for $j = 1, 2, \ldots, N^{n-1}$. Then, there exist $\delta > 0$, such that for all $w_p \in (\frac{p-1}{N}, \frac{p}{N})$, $p = 1, 2, \ldots, N$;

$$\int_{w_p - \delta}^{w_p + \delta} \left| h_j(t) - h_j(w_p) - (t - w_p) h_j'(w_p) \right| dt \le \varepsilon \delta^2 \tag{8}$$

Proof: Since $h_j'(w_p) = \lim_{t \to w_p} \dfrac{h_j(t) - h_j(w_p)}{t - w_p}$, $p = 1, 2, \ldots, N$, there exist $\delta_p > 0$, such that for $t \in (w_p - \delta_p, w_p + \delta_p) - \{w_p\}$;

$$\left| \frac{h_j(t) - h_j(w_p)}{t - w_p} - h_j'(w_p) \right| \le \frac{\varepsilon}{2}. \tag{9}$$

Now, suppose that $\delta = \min \left\{\delta_p : p = 1,2,...,N\right\}$. Thus, $(w_p - \delta, w_p + \delta) \subseteq (w_p - \delta_p, w_p + \delta_p)$, and by inequality (9), we obtain the following inequality;

$$\left| h_j(t) - h_j(w_p) - (t - w_p)h_j'(w_p) \right| \leq \frac{\varepsilon}{2}\left| t - w_p\right| \leq \frac{\varepsilon}{2}\delta. \tag{10}$$

Thus by integrating both sides of inequality (10) on $(w_p - \delta, w_p + \delta)$, $p = 1,2,...,N$, we can obtain inequality (8).□

Suppose $g(.) \in C[0,1]$, $f(.) : \Omega \to \mathbb{R}$, $h_j(t) = f(r_j(t))$, $t \in [0,1]$ and $\int_0^1 \left(v_k'(t)\, h_j(t) + v_k(t)g(t)\right) dt = 0$ for any $v_k(.) \in V$. So;

$$\int_0^1 v_k(t)g(t)\ dt = \lambda_{j,k}, \quad v_k(.) \in V, \quad k = 1,2,3,... \tag{11}$$

where

$$\lambda_{j,k} = -\int_0^1 v_k'(t)h_j(t)\ dt\ , k = 1,2,3,.... \tag{12}$$

Now, assume $\varepsilon > 0$ and $\delta > 0$ are two sufficiently small given numbers and $N \in \mathbb{N}$ is a sufficiently large number. For given piecewise continuous function $f(.) : \Omega \to \mathbb{R}$, we define the following functional optimization problem for $j = 1,2,...,N^{n-1}$;

$$(P_j)$$

$$\textit{Minimize} \qquad J_j(g(.)) = \sum_{k=1}^{\infty}\left|\int_0^1 v_k(t)g(t)\ dt - \lambda_{j,k}\right| \tag{13}$$

$$\textit{subject to}$$

$$\begin{aligned} &\int_{w_p-\delta}^{w_p+\delta} \left| h_j(t) - h_j(w_p) - (t - w_p)g(w_p) \right| dx \leq \varepsilon\delta^2, \\ &g(.) \in C[0,1],\ w_p \in (\frac{p-1}{N}, \frac{p}{N}),\ p = 1,2,...,N \end{aligned} \tag{14}$$

where $v_k(.) \in V$ for all $k = 1,2,3,\ldots$.

Theorem 5: Let $f : \Omega \to \mathbb{R}$ and $g_j^*(.) \in C[0,1]$ be the optimal solution of the functional optimization problem P_j, for $j = 1,2,...,N^{n-1}$, defined by (13) - (14). Then;

$$\frac{\partial f}{\partial x_i}(s_{j_1},...,s_{j_{i-1}},t,s_{j_{i+1}},...,s_{j_n}) = g_j^{\ *}(t), \quad t \in [0,1], \tag{15}$$

where $s_j = (s_{j_1},...,s_{j_{i-1}},s_{j_{i+1}},...,s_{j_n}) \in \overline{\Omega}_j,\ j = 1,2,...,N^{n-1}$.

Proof: Assume $s_j = (s_{j_1},...,s_{j_{i-1}},s_{j_{i+1}},...,s_{j_n}) \in \bar{\Omega}_j,\ j = 1,2,...,N^{n-1}$ and $h'_j(t) = \frac{\partial f}{\partial x_i}(r_j(t))$, $t \in [0,1]$. It is trivial $\sum_{k=1}^{\infty}\left|\int_0^1 v_k(t)g(t)\ dt - \lambda_{j,k}\right| \geq 0$ for all $g(.) \in C[0,1]$. From Theorem 3, we have $\left|\int_0^1 v_k(t)h'_j(t)\ dt - \lambda_{j,k}\right| = 0$, $v_k(.) \in V$. Hence, for all $g(.) \in C[0,1]$
$0 = \sum_{k=1}^{\infty}\left|\int_0^1 v_k(t)h'_j(t)\ dt - \lambda_{j_i,k}\right| \leq \sum_{k=1}^{\infty}\left|\int_0^1 v(t)g(t)\ dt - \lambda_{j,k}\right|$. On the other hand, $h'_j(.) \in C[0,1]$. Thus, from Theorem 4, function $h'_j(.)$ satisfies the constraints of the problem (10)-(11). Thus, $h'_j(.)$ is the optimal solution of functional optimization (10) - (11). Hence, if function $g_j^*(.) \in C[0,1]$ is the optimal solution of problem (13) - (14), then $\frac{\partial f}{\partial x_i}(r_j(t)) = g_j^*(t),\ t \in [0,1]$. □

Now, the GPD of nonsmooth function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ will be defined as follows:

Definition 6: Let $i \in \{1,2,...,n\}$ be a fixed and arbitrary index and $s_j = (s_{j_1},...,s_{j_{i-1}},s_{j_{i+1}},...,s_{j_n}) \in \bar{\Omega}_j$, $j = 1,2,...,N^{n-1}$. Moreover, let function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a piecewise continuous nonsmooth function and $g_j^*(.)$, $j = 1,2,...,N^{n-1}$ be the optimal solution of the functional optimization problem (10)-(11). We denote the GPD of $f(.)$ with respect to variable x_i by $\partial_i f$, and define it as $\partial_i f(s_{j_1},...,s_{j_{i-1}},t,s_{j_{i+1}},...,s_{j_n}) = g_j^*(t),\ t \in [0,1]$.

Remark 7: Note that if $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is a smooth function, then for fixed index $i \in \{1,2,...,n\}$ and;

$$s_j = (s_{j_1},...,s_{j_{i-1}},s_{j_{i+1}},...,s_{j_n}) \in \bar{\Omega}_j\ ,\ j = 1,2,...,N^{n-1} \tag{16}$$

we have;

$$\partial_i f(s_{j_1},...,s_{j_{i-1}},t,s_{j_{i+1}},...,s_{j_n}) = \frac{\partial f}{\partial x_i}(s_{j_1},...,s_{j_{i-1}},t,s_{j_{i+1}},...,s_{j_n}), \tag{17}$$

for $t \in [0,1]$.

Further, if $f(.)$, is a nonsmooth function, then the GPD of $f(.)$, with respect to variable x_i is an approximation for the first derivative of function $f(.)$, with respect to variable x_i.

However, the functional optimization problem (13) - (14) is an infinite dimensional problem. In the next section, we convert this problem to the corresponding finite dimensional problem. Finally, for finding GPD, we propose the LP problems.

Transformation of problem to LP problem

It is obvious that we can extend any function $g(.) \in C[0,1]$ on interval [-1, 1] as a Fourier series;

$$g(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k \cos(k\pi x) + b_k \sin(k\pi x)\right) \tag{18}$$

where coefficients $a_0, a_1, ...$ and $b_1, b_2, ...$ satisfy the following relations;

$$a_k = \int_{-1}^{1} \cos(k\pi x) g(x) dx \,,\; b_k = \int_{-1}^{1} \sin(k\pi x) g(x) dx,\;\; k = 1, 2, ... \tag{19}$$

On the other hand, we have $\lim_{k\to\infty} a_k = 0$ and $\lim_{k\to\infty} b_k = 0$. Then, there exists $M \in \mathbb{N}$, such that for all $k \geq M+1$, we have $a_k \approx 0$ and $b_k \approx 0$.
Hence, from the above point and from definition of functions $v_k(.)$ $k = 1, 2, ...$ the problems (P_j), $j = 1, 2, ..., N^{n-1}$ corresponding to (13) - (14) are approximated as the following finite dimensional problems (FP);

$$(FP_j)$$

$$Minimize \qquad J_j(g(.)) = \sum_{k=1}^{M}\left|\int_0^1 v_k(t) g(t)\, dt - \lambda_{j,k}\right| \tag{20}$$

$$\int_{w_p-\delta}^{w_p+\delta} \left| h_j(t) - h_j(w_p) - (t - w_p) g(w_p) \right| dx \leq \varepsilon\delta^2, \tag{21}$$

$$g(.) \in C[0,1],\; w_p \in \left(\frac{p-1}{N}, \frac{p}{N}\right),\; p = 1, 2, ..., N$$

where M is a given large number. We assume that $v_{kp} = v_k(w_p),\; g_p = g(w_p),\; h_j^{1,p} = h_j(w_p - \delta)$, $h_j^{2,p} = h_j(w_p + \delta)$ and $h_j^p = h_j(w_p)$ for all $p = 1, 2, ..., N$. In addition, we choose the arbitrary points $w_p \in (\frac{p-1}{N}, \frac{p}{N}),\; p = 1, 2, ..., N$. By trapezoidal and midpoint integration rules, problem (20) - (21) can be approximated as the following problem, where $g_{j1}, g_{j2}, ..., g_{jm}$ are unknown variables;

$$(FP_j)$$

$$Minimize \quad \sum_{k=1}^{M}\left|\; \delta \sum_{p=1}^{N} v_{kp} g_{jp} - \lambda_{j,k} \right| \tag{22}$$

$$\left| h_j^{1,p} - h_j^p + \delta g_{jp} \right| + \left| h_j^{2,p} - h_j^p - \delta g_{jp} \right| \leq \varepsilon\delta,$$

$$p = 1, 2, ..., M \tag{23}$$

Now, problem (22) - (23) can be converted to the following equivalent finite linear programming (FLP) problem (see [8,9]), where $g_{jp},\ p = 1,...,m$, $\mu_k,\ k = 1,2,...,M$ and y_p, z_p, u_p, q_p for $p = 1,2,...,N$ are unknown variables of the problem;

$$
\begin{aligned}
&(FLP_j) \qquad Minimize \qquad \sum_{k=1}^{M} \mu_k \\
&subject\ to \\
&-\mu_k + \delta \sum_{p=1}^{N} v_{kp} g_{jp} \le \lambda_{j,k}, \quad k = 1,2,...,N \\
&-\mu_k - \delta \sum_{p=1}^{N} v_{kp} g_{jp} \le -\lambda_{j_i,k}, \quad k = 1,2,...,N \\
&(u_p + q_p) + (y_p + z_p) \le \varepsilon\delta, \quad p = 1,2,...,N \\
&u_p - q_p - \delta g_{jp} = h_j^{1,p} - h_j^p, \quad p = 1,2,...,N \\
&y_p - z_p + \delta g_{jp} = h_j^{2,p} - h_j^p, \quad p = 1,2,...,N \\
&y_p, z_p, u_p, q_p, \mu_k \ge 0, \quad k = 1,2,...,M; \ p = 1,2,...,N.
\end{aligned} \tag{24}
$$

where λ_{jk} for $j = 1,2,\ldots,N^{n-1}$, $k = 1,2,\ldots,M$ satisfy the relation (12).

Remark 8: Note that the chosen numbers for ε and δ must be sufficiently small, and points $w_p \in (\frac{p-1}{N}, \frac{p}{N}),\ p = 1,2,\ldots,N$ can be chosen as arbitrary numbers.

Remark 9: Note that if g^*_{jp}, p=1,...,N are optimal solutions of problem FLP_j corresponding to (24), then $\partial_i f(s_{j_1},...,s_{j_{i-1}}, w_p, s_{j_{i+1}},...,s_{j_n}) \approx g^*_{jp}$ for $p = 1,\ldots,N$.

In the next section, we obtain the GPD of some smooth and nonsmooth functions using our approach.

Numerical examples

In this section, we find the GPD of smooth and nonsmooth two variable functions with respect to x_1 (means $\partial_1 f$) in three examples, using problem (24). Here we assume $\varepsilon = \delta = 0.01$, $n = 2, M = 20, N = 100$ and $w_p = 0.01p$ for all $p = 1,...,99$. Moreover, we divide $\bar{\Omega} = [0,1]$ to the similar grids $\bar{\Omega}_j,\ j = 1,2,...,100$, and select arbitrary points $s_j \in \bar{\Omega}_j$ as $s_j = 0.005 + (j-1)0.01 \quad j = 1,2,...,100$. Indeed, we set $r_j(t) = (t, s_j)$ for $j = 1,2,...,100$ and $t \in [0,1]$. The problems (24) for $j = 1,2,...,100$ are solved for functions in these examples using the simplex method (see [8]) using MATLAB software.

Example 1: Consider function $f(x_1,x_2) = x_1^3 + x_2^2$ on $\Omega = [0,1]\times[0,1]$ which is a differentiable function. The graph of this function is illustrated in **Figure 1**. We obtain the GPD of function $f(.)$ with respect to x_1 by solving problem (24), which is shown in **Figure 2**.

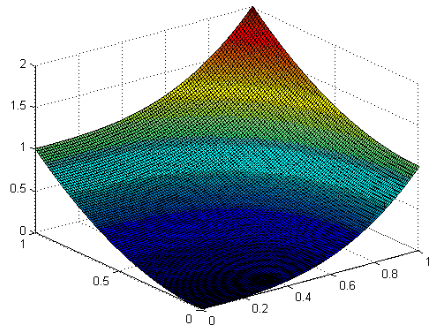

Figure 1 The graph of function $f(.)$ for Ex. 1.

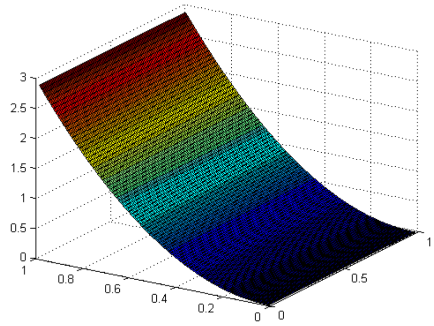

Figure 2 The graph of GPD of $f(.)$ for Ex. 1.

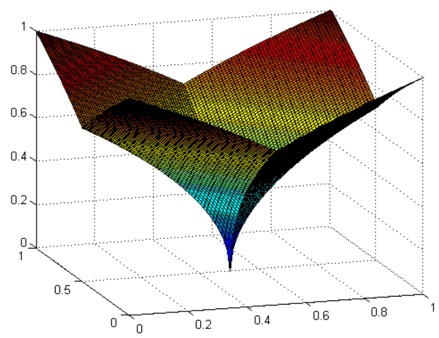

Figure 3 The graph of function $f(.)$ for Ex. 2.

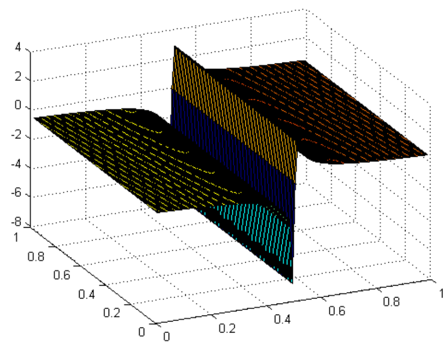

Figure 4 The graph of GPD of $f(.)$ for Ex. 2.

Example 2: Consider the function $f(x_1,x_2) = \sqrt{|x_1 - 0.5| + |x_2 - 0.5|}$ on $\Omega = [0,1]\times[0,1]$, which is a nondifferentiable function in all points of the set $\{(x_1,x_2) \in [0,1]^2 : x_1 = 0.5 \; or \; x_2 = 0.5\}$. This function is illustrated in **Figure 3**. Using problem (24), we obtain the GPD of this function with respect to x_1, which is shown in **Figure 4**.

Example 3: Consider the function $f(x_1,x_2) = |\sin(2\pi x_1)| - |\cos(2\pi x_2)|$ on $\Omega = [0,1]\times[0,1]$, which is a nondifferentiable function in all points of the set $\{(x_1,x_2) \in [0,1]^2 : \sin(2\pi x_1) = 0 \;\; or \;\; \cos(2\pi x_2) = 0\}$. We show this function in **Figure 5**. Using problem (24), we obtain the GPD of this function with respect to x_1, which is shown in **Figure 6**.

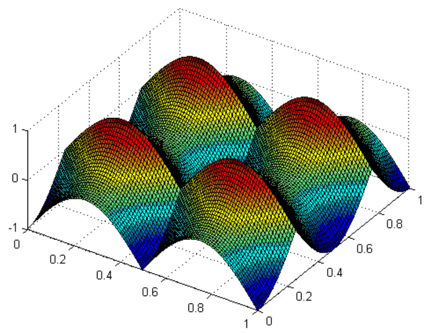

Figure 5 The graph of function $f(.)$ for Ex. 3.

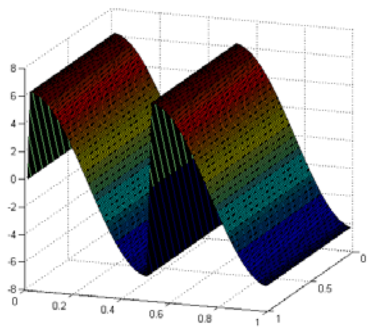

Figure 6 The graph of GPD of $f(.)$ for Ex. 3.

Conclusions

In this paper, we defined a new GPD for nonsmooth functions as an optimal solution of a functional optimization on the $\Omega = [0,1]^n$. We approximated this functional optimization problem by using LP problems. The definition of GPD in this paper has the following properties and advantages:

1) Here, the GPD for smooth functions is the partial derivative of these functions.

2) In our approach, the derivative of piecewise continuous functions is defined, but the other approaches usually define it for special functions, such as continuous and Lipschiptz or convex functions.

3) It is easy to implement and is practical in respect to the other existing approaches.

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