

Determination of the Position Vectors of Curves from Intrinsic Equations in G_3

Handan ÖZTEKIN* and Serpil TATLIPINAR

Department of Mathematics, Firat University, Elazig, Turkey

(*Corresponding author's e-mail: handanoztekin@gmail.com)

Received: 17 May 2012, Revised: 28 April 2014, Accepted: 2 May 2014

Abstract

In this paper, we investigate the position vectors of the curves and the general helices in Galilean space G_3 . We find the differential equations for the position vectors of such curves.

Keywords: Galilean space, position vector, helix

Introduction

The helix is an important concept in differential geometry. The helices provide various conditions and results, so they contribute to comprehension of this geometry. Firstly, in 1802 Lancret presented the helices and in 1845 de Saint Venant proved the first condition for these curves. This well known condition is "A necessary and sufficient condition that a curve be a general helix that the function $f = \frac{\tau}{\kappa}$ is constant along the curve, where κ and τ denote the curvature and the torsion, respectively".

In Euclidean 3-space E^3 , a general helix is a curve in which the tangent makes a constant angle with a fixed straight line called the axis of the general helix. The first 2 coordinates of a helix provide circular motion while the third coordinate lifts the curve out of the plane. In addition, a curve is defined uniquely by its curvature and torsion as a function of natural parameters. The curvature and torsion of this curve parameterized with s can be defined by $\kappa = \kappa(s), \tau = \tau(s)$, respectively, which are called the natural or intrinsic equations of the curve. Scientists studying the helix have found some characterizations in Euclidean space [1-6].

The basic concepts of Euclidean plane geometry are points and straight lines, also the best known theorem is Pisagor theorem. But in nature, every surface is not a plane and every line is not a straight line like Euclidean geometry. The parallel postulate which is the 5th postulate in Euclid's Elements, is a distinctive axiom. The non-Euclidean geometry is a geometry which does not provide the parallel postulate. Furthermore, the absolute geometry (or neutral geometry) is a geometry which is independent of this 5th postulate (only assumes the first 4 postulates). In the 19th century Gauss and Bessel entertained the idea of non-Euclidean geometry. Moreover, Bolyai and Lobachevsky showed hyperbolic geometry and again Bernhard investigated the bases of elliptic geometry. These are very special types of Riemannian geometry of constant positive curvature and constant negative curvature, respectively.

Galilean geometry is a non-Euclidean geometry. Firstly, Galilean space was investigated by Keli in 1869. The geometry of Galilean space G_3 was widely developed by Röchel [7]. The ruled surfaces were described by Kamenarovic and also studied by Divjak and Milin-Sipus [8,9]. Furthermore Ergut, Bektas and Ogrenmis found the characterizations for the curves and the helices in Galilean space [10-13].

The literature survey indicated that, there are no position vectors of curves in Galilean 3-space. Thus the study is proposed to serve such a need. In this paper, we prove that the position vector of every space curve in G_3 satisfies a vector differential equation of fourth order. We obtain position vectors of a general

helix in the parametric form. Also, we give some examples to illustrate how to find the position vector from the intrinsic equations of general helices.

Basic notions and properties

The Galilean space G_3 is a 3 dimensional complex projective space P_3 , in which the absolute figure $\{w, f, I_1, I_2\}$ consists of a real plane w (the absolute plane), a real line $f \subset w$ (the absolute line) and two complex conjugate points $I_1, I_2 \in f$ (the absolute points).

We shall take, as a real model of the space G_3 , a real projective space P_3 , with the absolute $\{w, f\}$ consisting of a real plane $w \subset G_3$, and a real line $f \subset w$, on which an elliptic involution has been defined.

Let be in homogeneous coordinates;

$$\begin{aligned} w \dots x_0 &= 0, \\ f \dots x_0 &= x_1 = 0, \\ \varepsilon : (0:0:x_2:x_3) &\rightarrow (0:0:x_3:-x_2). \end{aligned} \quad (1)$$

In the nonhomogeneous coordinates, the similarity group H_8 has the form;

$$\begin{aligned} x' &= a_{11} + a_{12}x \\ y' &= a_{21} + a_{22}x + a_{23} \cos \alpha y + a_{23} \sin \alpha z \\ z' &= a_{31} + a_{32}x - a_{23} \sin \alpha y + a_{23} \cos \alpha z \end{aligned} \quad (2)$$

where a_{ij} and α are real numbers.

For $a_{12} = a_{23} = 1$, we have the subgroup B_6 , the group of Galilean motions;

$$\begin{aligned} x' &= a + x \\ B_{6\dots} y' &= b + cx + y \cos \alpha + z \sin \alpha \\ z' &= d + ex - y \sin \alpha + z \cos \alpha \end{aligned} \quad (3)$$

In G_3 there are 4 classes of lines:

- (proper) non-isotropic lines- they do not meet the absolute line f .
- (proper) isotropic lines- lines that do not belong to the plane w but meet the absolute line f .
- unproper non-isotropic lines- all lines of w but f .
- the absolute line f .

Planes $x = \text{const.}$ are Euclidean and so is the plane w . Other planes are isotropic. In what follows, for $a_{12} = a_{23} = 1$, defines the group $B_6 \subset H_8$ of isometrics of the Galilean space G_3 .

For a curve $c: I \rightarrow G_3, I \subseteq R$ parameterized by the invariant parameter $s = x$, is given in the coordinate form;

$$c(x) = (x, y(x), z(x)), \quad (4)$$

the curvature $\kappa(x)$ and the torsion $\tau(x)$ are defined by;

$$\begin{aligned}\kappa(x) &= \sqrt{y''^2(x) + z''^2(x)}, \\ \tau(x) &= \frac{\det(c'(x), c''(x), c'''(x))}{\kappa^2(x)}.\end{aligned}\quad (5)$$

The associated moving trihedron is given by;

$$\begin{aligned}T &= c'(x) = (1, y'(x), z'(x)), \\ N &= \frac{1}{\kappa(x)} c''(x) = \frac{1}{\kappa(x)} (0, y''(x), z''(x)), \\ B &= \frac{1}{\kappa(x)} (0, -z''(x), y''(x)).\end{aligned}\quad (6)$$

The vectors T , N and B are called the vectors of the tangent, principal normal and binormal line, respectively.

For their derivatives the following Frenet's formulae hold [10];

$$\begin{aligned}T' &= \kappa N, \\ N' &= \tau B, \\ B' &= -\tau N,\end{aligned}\quad (7)$$

Definition 1 Let α be a curve in 3-dimensional Galilean space G_3 , and $\{T, N, B\}$ be the Frenet frame in 3-dimensional Galilean space G_3 along α . If κ and τ are positive constants along α , then α is called a circular helix with respect to the Frenet frame, [11].

Definition 2 Let α be a curve in 3-dimensional Galilean space G_3 , and $\{T, N, B\}$ be the Frenet frame in 3-dimensional Galilean space G_3 along α . A curve α such that;

$$\frac{\kappa}{\tau} = \text{const.} \quad (8)$$

is called a general helix with respect to Frenet frame, [10].

Positions vectors of curves and helices in G_3

Theorem 1 Let $\varphi = \varphi(s)$ be a unit speed curve in Galilean space G_3 . Then position φ satisfies the following vector differential;

$$\frac{d}{ds} \left[\frac{1}{\tau} \frac{d}{ds} \left(\frac{1}{\kappa} \frac{d^2 \varphi}{ds^2} \right) \right] + \frac{\tau}{\kappa} \frac{d^2 \varphi}{ds^2} = 0. \quad (9)$$

Proof. Let $\varphi = \varphi(s)$ be a unit speed curve in G_3 . If we consider Eq. (7), we have;

$$B = \frac{1}{\tau} \frac{d}{ds} \left(\frac{1}{\kappa} \frac{dT}{ds} \right) \quad (10)$$

and we can write the last equation of (7) as follows;

$$\frac{d}{ds} \left[\frac{1}{\tau} \frac{d}{ds} \left(\frac{1}{\kappa} \frac{dT}{ds} \right) \right] + \frac{\tau}{\kappa} \frac{dT}{ds} = 0, \quad (11)$$

where $\frac{d\varphi}{ds} = T$, so we have a vector differential equation of 4th order (9).
 From Eq. (11), we can write the following equation;

$$\frac{d}{d\theta} \left(f \frac{d^2 T}{d\theta^2} \right) + \frac{1}{f} \frac{dT}{d\theta} = 0, \quad (12)$$

where $f = f(\theta) = \frac{\kappa(\theta)}{\tau(\theta)}$ and $\theta = \int \kappa(s) ds$. This means the position vector of an arbitrary space curve can be determined by the solution of the above equation.

Theorem 2 Let $\varphi = \varphi(s)$ be a unit speed curve in Galilean space G_3 . Then the principal normal vector N satisfies the following vector differential;

$$f(\theta) N''(\theta) + f'(\theta) N'(\theta) + \frac{1}{f(\theta)} N(\theta) = 0, \quad (13)$$

where $f(\theta) = \frac{\kappa(\theta)}{\tau(\theta)}$ and $\theta = \int \kappa(s) ds$.

Proof. Let $\varphi = \varphi(s)$ be a unit speed curve in G_3 . If we consider this curve as $\varphi = \varphi(\theta)$ where $\theta = \int \kappa(s) ds$, we hold;

$$\begin{bmatrix} T'(\theta) \\ N'(\theta) \\ B'(\theta) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{1}{f(\theta)} \\ 0 & -\frac{1}{f(\theta)} & 0 \end{bmatrix} \begin{bmatrix} T(\theta) \\ N(\theta) \\ B(\theta) \end{bmatrix} \quad (14)$$

where $f(\theta) = \frac{\kappa(\theta)}{\tau(\theta)}$. From the new Frenet Eq. (14), after some calculations we have;

$$B(\theta) = f(\theta) N'(\theta) \quad (15)$$

and differentiating the last equation, we have the vector differential (13).

If we investigate and solve Eq. (13), we obtain the space curve $\varphi = \varphi(s)$ as follows;

$$\varphi(s) = \int \left(\int \kappa(s) N(s) ds \right) ds + C, \quad (16)$$

and in parametric representation;

$$\varphi(\theta) = \int \frac{1}{\kappa(\theta)} \left(\int N(\theta) d\theta \right) d\theta + C, \quad (17)$$

where $\theta = \int \kappa(s) ds.$

Theorem 3 The position vector of a general helix in the natural parameter form;

$$\varphi(s) = \int \left(1, \cos[\cot[\alpha] \int \kappa(s) ds] + \sin[\cot[\alpha] \int \kappa(s) ds] \cos[\alpha] \right) ds + C. \quad (18)$$

or in the parametric form;

$$\varphi(\phi) = \frac{\tan[\alpha]}{\kappa(\phi)} \int (1, \cos[\phi] + \sin[\phi] \cos[\alpha]) d\phi + C, \phi = \cot[\alpha] \int \kappa(s) ds. \quad (19)$$

Proof. If φ is a general helix in Galilean space G_3 , we can write $f(\theta) = \tan[\alpha]$. So, the Eq. (19) becomes;

$$\frac{d^3 T}{d\theta^3} + \cot^2[\alpha] \frac{dT}{d\theta} = 0 \quad (20)$$

or

$$\frac{d^3 T}{d\phi^3} + \frac{dT}{d\phi} = 0, \phi = \cot[\alpha] \int \kappa(s) ds. \quad (21)$$

If we write the tangent vector $T = (T_1, T_2, T_3)$ the general solution of (21) takes the form;

$$T(\phi) = T_i(\phi) e_i = (a_i \cos[\phi] + b_i \sin[\phi] + c_i) e_i, i = 1, 2, 3 \quad (22)$$

where $a_i, b_i, c_i \in R$ for $i = 1, 2, 3$.

Since the curve φ is a general helix, the tangent vector of φ makes a constant angle α with the constant vector field which is called the axis of the helix. Therefore, without loss of generality, we take the axis of helix is parallel to e_3 . Then $T_3 = \langle T, e_3 \rangle = \cos[\alpha]$, $a_3 = b_3 = 0$ and $c_3 = \cos[\alpha]$. Also, since the tangent vector T is a unit vector, we have;

$$T_1 = 1 \quad (23)$$

and

$$a_1 \cos[\phi] + b_1 \sin[\phi] + c_1 = 1. \quad (24)$$

So, we have;

$$T(\phi) = (1, a_2 \cos[\phi] + b_2 \sin[\phi] + c_2, \cos[\alpha]). \quad (25)$$

From Eqs. (23), (24) and (25), without loss of generality, we can write;

$$T(\phi) = (1, \cos[\phi] + \sin[\phi], \cos[\alpha]) \quad (26)$$

where $a_2 = b_2 = 1, c_2 = 0$. If we integrate Eq. (26) with respect to s such that $\phi = \cot[\alpha] \int \kappa(s) ds$, we get Eqs. (18) and (19). This completes the proof.

Examples

Example 1 If we consider a curve with $\kappa = \frac{\sin[\alpha]}{a}, \tau = \frac{\cos[\alpha]}{a}$, the position vectors take the following form;

$$\varphi(s) = \int \left(1, \cos \left[\cot[\alpha] \int \kappa(s) ds \right] + \sin \left[\cot[\alpha] \int \kappa(s) ds \right], \cos[\alpha] \right) ds + C. \quad (27)$$

If we integrate Eq. (27) and put $s = \frac{a}{\cos[\alpha]} \phi$, we get;

$$\varphi(\phi) = \frac{a}{\cos[\alpha]} (\phi, \sin[\phi] - \cos[\phi], \cos[\alpha]\phi) + C, \quad (28)$$

which is the parametric representation of the curve.

Example 2 If we consider a curve with $\kappa = \frac{\sin[\alpha]}{as}, \tau = \frac{\cos[\alpha]}{as}$, the position vectors take the following form;

$$\varphi(\phi) = \frac{a}{\cos[\alpha]} \int \exp\left(-\frac{a}{\cos[\alpha]} \phi\right) (1, \cos[\phi] + \sin[\phi], \cos[\alpha]) d\phi + C. \quad (29)$$

If we integrate Eq. (29), we get;

$$\begin{aligned} \varphi(\phi) = & \frac{a \cos[\alpha]}{a^2 + \cos[\alpha]^2} \exp\left(\frac{a}{\cos[\alpha]} \phi\right) \left(\frac{a^2 + \cos[\alpha]^2}{a \cos[\alpha]}, \sin[\phi] - \cos[\phi] \right. \\ & \left. + \frac{a}{\cos[\alpha]} (\sin[\phi] + \cos[\phi]), \frac{a^2 + \cos[\alpha]^2}{a} \right) + C, \end{aligned} \quad (30)$$

which is the parametric representation of the curve.

Example 3 If we consider a curve with $\kappa = \frac{1}{as}, \tau = \frac{b}{as}$, the position vectors take the following form;

$$\varphi(\phi) = a \tan[\alpha] \int \exp(a \tan[\alpha] \phi) (1, \cos[\phi] + \sin[\phi], \cos[\alpha]) d\phi + C. \quad (31)$$

Then, if we integrate Eq. (31), we get;

$$\begin{aligned} \varphi(\phi) = & \frac{a \tan[\alpha]}{1 + a^2 \tan^2[\alpha]} \exp(a \tan[\alpha] \phi) \left(\frac{1 + a^2 \tan^2[\alpha]}{a \tan[\alpha]}, \sin[\phi] - \cos[\phi] \right. \\ & \left. + a \tan[\alpha] (\sin[\phi] + \cos[\phi]), \frac{1 + a^2 \tan^2[\alpha]}{a} \sin[\alpha] \right) + C, \end{aligned} \quad (32)$$

which is the parametric representation of the curve.

Example 4 If we consider a curve with $\kappa = \frac{1}{as+c}, \tau = \frac{b}{as+c}$, the position vectors are similar to Example 3.

Conclusions

In this study, we give the position vectors of the curves and the general helices in Galilean 3-space. We prove that the position vector of every space curve in Galilean 3-space satisfies a vector differential equation of 4th order. Furthermore we get the position vector of a general helix in the parametric form.

References

- [1] AT Ali and M Turgut. Position vector of a time-like slant helix in Minkowski 3-space. *J. Math. Anal. Appl.* 2010; **365**, 559-69.
- [2] M Külahcı and M Ergüt. Bertrand curves of AW(k)-type in Lorentzian space. *Nonlinear Anal.* 2009; **70**, 1725-31.
- [3] DA Brannan, MF Esplen and JJ Gray. *Geometry*. Cambridge University Press, England, 1999.
- [4] C Camcı, K İlarıslan, L Kula and HH Hacısalıhoğlu. Harmonic curvatures and generalized helices in E^n . *Chaos Soliton. Fract.* 2009; **40**, 2590-6.
- [5] K İlarıslan and Ö Boyacıoğlu. Position vectors of a spacelike W-curve in Minkowski space E_1^3 . *Bull. Korean Math. Soc.* 2007; **44**, 429-38.
- [6] M Turgut and S Yılmaz. Contributions to classical differential geometry of the curves in E^3 . *Scientia Magna* 2008; **4**, 5-9.
- [7] O Röchel. *Die Geometrie des Galileischen Raumes*. Arbeitsbericht, Leoben, 1984.

- [8] I Kamenarovic. Existence theorems for ruled surfaces in the Galilean space G_3 . *Rad. Hazu. Math.* 1991; **456**, 183-96.
- [9] BJ Pavkovic and I Kamenarovic. The equiform differential geometry of curves in the Galilean space G_3 . *Glasnik Matematicki* 1987; **22**, 449-57.
- [10] M Ergüt and AO Öğrenmiş. Some characterizations of spherical curves in Galilean space G_3 . *J. Adv. Res. Pure Math.* 2009; **1**, 18-26.
- [11] AO Öğrenmiş, M Bektaş and M Ergüt. The characterizations for helices in the Galilean space G_3 . *Int. J. Pure Appl. Math.* 2004; **16**, 31-6.
- [12] AO Öğrenmiş, M Ergüt and M Bektaş. On the helices in the Galilean space G_3 . *Iranian J. Sci. Tech. A* 2007; **31**, 177-81.
- [13] M Bektaş. The characterizations of general helices in the 3-dimensional Pseudo-Galilean space. *Soochow J. Math.* 2005; **31**, 441-7.