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Variational Approach Applied to Study the Lifting of a Non-Newtonian Fluid on a Vertically Moving Belt

Ali Ahmad FAROOQ[1,4,*], Abdul Majeed SIDDIQUI[2], Tahira HAROON[3] and Muhammad Afzal RANA[4]

[1]*COMSATS Institute of Information Technology, Abbottabad, Pakistan*
[2]*Department of Mathematics, York Campus, Pennsylvania State University, United States*
[3]*COMSATS Institute of Information Technology, Islamabad, Pakistan*
[4]*Department of Basic Sciences, Riphah International University, Islamabad, Pakistan*

([*]Corresponding author's e-mail: aliahmedfarooq@yahoo.com)

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Abstract

This paper provides an investigation regarding the modeling and analysis of the thin film flow of a non-Newtonian Williamson fluid on a vertically moving belt. The governing nonlinear differential equation is first integrated analytically and then solved by using the Variational Iteration Method (VIM). The results of the present method are also compared with those obtained by the Adomian Decomposition Method (ADM) and a very good agreement is observed. This comparison reveals that VIM may be considered as an efficient alternative method for solving nonlinear problems arising in the area of fluid mechanics. Expressions for some important physical quantities such as volume flux, average velocity, the belt speed for the lifting of non-Newtonian fluid are also derived.

Keywords: Thin film flow, moving belt, Williamson fluid model, nonlinear problem, VIM

Introduction

Thin film flows of non-Newtonian fluids have a variety of practical applications in nonlinear sciences and engineering industries. Very little effort has been devoted to the analytical study of thin film flow problems arising in the lifting and drainage of non-Newtonian fluids utilizing various geometries; see for instance [1-7]. The governing differential equations of such problems are in general highly nonlinear and complicated. It is well known that most of these types of nonlinear equations do not have an exact solution. Therefore, these problems should be treated by using some numerical or analytical methods. Analytical study of such nonlinear problems is important not only because of their technological significance but also due to the interesting mathematical features presented by the governing differential equations of the flow.

Apart from numerical methods, several analytical techniques such as Homotopy Analysis Method (HAM), Homotopy Perturbation Method (HPM), Adomian Decomposition Method (ADM) and Variational Iteration Method (VIM) are proposed to find approximate analytical, and if possible in closed form, solutions of such nonlinear equations, see [8-17], and the references therein. In this paper, we aim to apply the VIM and the ADM in finding the approximate analytical solutions of a highly nonlinear differential equation arising in the thin film flow problems of a non-Newtonian Williamson fluid. The VIM is a powerful analytical technique was first introduced by Ji-Huan He and has been used by many mathematicians to solve various nonlinear equations [11-17]. This method gives rapidly convergent successive approximations of the exact solutions if such a solution exists. As in this problem, the governing equation is highly nonlinear whose exact solution is very difficult, a few approximations can be used for numerical purposes with a high degree of accuracy. For comparison, the same problem is also solved by ADM. The results show that the difference between the two solutions is negligible. This comparison is benched-marked against a numerical solution.

These methods have already been successfully used for the solutions of various linear and nonlinear problems see [11-17]. An important advantage of these methods over the numerical methods is that they provide series solutions in the form of functions of a single variable. These forms of solutions can be used to evaluate analytical expressions of various flow parameters of physical relevance. The convergence of these methods is systematically discussed in [18,19].

Mathematical model

We consider a container filled with a Williamson fluid. A wide belt passes through this container, which moves vertically upward with a constant speed V_0. The moving belt picks up a thin film fluid of uniform thickness δ and gravity tries to make the fluid film drain down the belt. We assume that the flow is steady, laminar and uniform and the pressure is atmospheric pressure. We choose the $x-axis$ normal to the belt and the $y-axis$ is taken along the belt which is in an upward direction.

The appropriate boundary conditions for the problem are;

$$v = V_0 \quad \text{at } x = 0 \text{ (no slip condition)}, \tag{1}$$

$$S_{xy} = 0 \quad \text{at } x = \delta \text{ (free surface)}, \tag{2}$$

where S_{xy} is the shear stress component of the Williamson fluid.

The balance of momentum in terms of extra stresses has the form;

$$\rho \frac{d\mathbf{V}}{dt} = -\nabla p + \rho \mathbf{f} + \nabla.\mathbf{S}, \tag{3}$$

where $\mathbf{V} = (u, v, w)$ is the velocity vector, ρ is the constant density, $\mathbf{f} = (\mathrm{f}_1, \mathrm{f}_2, \mathrm{f}_3)$ is the body force per unit mass, $\frac{d}{dt}$ is the material time derivative, p is the dynamic pressure. The fluid is assumed to be incompressible, therefore;

$$\nabla.\mathbf{V} = \mathbf{0} \tag{4}$$

The extra stress tensor $\mathbf{S}$ for the Williamson fluid is defined as [20];

$$\mathbf{S} = \left[\mu_\infty + \left(\mu_0 + \mu_\infty \right) \left(1 - \Gamma \dot{\boldsymbol{\gamma}} \right)^{-1} \right] \dot{\boldsymbol{\gamma}}, \tag{5}$$

where μ_∞ is the infinite shear rate viscosity, μ_0 is the zero shear rate viscosity, Γ is the time constant and $\dot{\boldsymbol{\gamma}}$ is defined as;

$$\dot{\boldsymbol{\gamma}} = \sqrt{\frac{1}{2}\sum_{i}\sum_{j}\gamma_{ij}\gamma_{ji}} = \sqrt{\frac{1}{2}\boldsymbol{\Pi}}, \tag{6}$$

where $\boldsymbol{\Pi}$ is the second invariant strain tensor. We consider the constitutive Eq. (5), the case for which $\mu_\infty = 0$ and $\Gamma\dot{\boldsymbol{\gamma}} < 1$. The component of extra stress tensor therefore, can be written as;

$$\mathbf{S} = \mu_0\left[\left(1-\Gamma\dot{\boldsymbol{\gamma}}\right)^{-1}\right]\dot{\boldsymbol{\gamma}} = \mu_0\left[\left(1+\Gamma\dot{\boldsymbol{\gamma}}\right)\right]\dot{\boldsymbol{\gamma}} \tag{7}$$

We seek the velocity field and the extra stress tensor of the form;

$$\mathbf{V} = \left[0, v(x), 0\right], \qquad \mathbf{S} = \mathbf{S}(x) \tag{8}$$

when (8) is used in Eqs. (3) and (4), the continuity equation is identically satisfied and the momentum equation without dynamic pressure is reduced to;

$$0 = \frac{dS_{xx}}{dx} + \rho \mathrm{f}_1, \tag{9}$$

$$0 = \frac{dS_{xy}}{dx} + \rho \mathrm{f}_2, \tag{10}$$

where f_1 and f_2 are the components of body force in x and y directions, respectively. Since the $y-axis$ is taken to be in an upward direction and the force due to gravity is in a downward direction, so the above equations take the form;

$$0 = \frac{dS_{xx}}{dx}, \tag{11}$$

$$0 = \frac{dS_{xy}}{dx} - \rho \mathrm{g}. \tag{12}$$

Integrating once Eq. (12) with respect to x and using the boundary condition given in Eq. (2), we obtain;

$$S_{xy} = \rho g(x-\delta) \tag{13}$$

using Eqs. (7) and (8), one can write the Eq. (13) describing the flow in the form;

$$\mu_0\frac{dv}{dx} + \mu_0\Gamma\left(\frac{dv}{dx}\right)^2 = \rho g(x-\delta) \tag{14}$$

Introducing the following dimensionless parameters;

$$f = \frac{v}{V_0},\ \eta = \frac{x}{\delta},\ We = \frac{\Gamma V_0}{\delta},\ K = \frac{\rho g \delta^2}{\mu_0 V_0}$$

Therefore, the governing differential equation for the problem with the boundary condition (1) becomes;

$$\frac{df}{d\eta} + We\left(\frac{df}{d\eta}\right)^2 = K(\eta - 1), \tag{15}$$

$$f = 1 \text{ at } \eta = 0. \tag{16}$$

It is noted that Eq. (15) along with one boundary condition (16) is a nonlinear first order ordinary differential equation. Here, we are interested in finding the approximate analytical solutions by using VIM and ADM.

Solution by VIM

To apply the variational iteration method (VIM), we write Eq. (15) in the form;

$$Lf(\eta) + Nf(\eta) = g(\eta), \tag{17}$$

where $L(f) = \frac{df}{d\eta}$ is the linear term, $N(f) = We\left(\frac{df}{d\eta}\right)^2$ is a nonlinear term and $g(\eta) = K(\eta - 1)$ is the forcing term. We can construct a correction functional according to the variational iteration method [9,13] as;

$$f_{n+1} = f_n + \int_0^{\eta} \lambda(x)\left(Lf_n(x) + N\tilde{f}_n(x) - g(x)\right)dx, \tag{18}$$

where $\lambda(x)$ is a general Lagrange multiplier which can be identified optimally via variational iteration theory, the subscript n denotes the nth approximation, and $\tilde{f}_n(x)$ is considered as a restricted variation, that is $\delta \tilde{f}_n(x) = 0$.

After some calculations, we obtain the following stationary conditions;

$$\lambda'(\eta) = 0,\ \ 1 + \lambda(\eta) = 0 \tag{19}$$

The Lagrange multiplier can therefore be identified as $\lambda = -1$. As a result, we obtain the following iteration formula;

$$f_{n+1}(\eta) = f_n(\eta) - \int_0^{\eta}\left(\frac{df_n}{dx} + We\left(\frac{df_n}{dx}\right)^2 - K(x-1)\right)dx,\ n \geq 0 \tag{20}$$

We start with initial approximation $f_0(\eta)=1$. The next iterates $f_1(\eta)$, $f_2(\eta)$, $f_3(\eta)$,..., are given below respectively;

$$f_1(\eta)=1+\frac{K}{2}\left((\eta-1)^2-1\right) \tag{21}$$

$$f_2(\eta)=1+\frac{K}{2}\left((\eta-1)^2-1\right)-\frac{WeK^2}{3}\left((\eta-1)^3+1\right) \tag{22}$$

$$\begin{aligned}f_3(\eta)=1&+\frac{K}{2}\left((\eta-1)^2-1\right)-\frac{2WeK^2}{3}\left((\eta-1)^3+1\right)+\frac{We^2K^3}{2}\left((\eta-1)^4-1\right)\\&-\frac{We^3K^4}{5}\left((\eta-1)^5+1\right)\end{aligned}, \tag{23}$$

$$\begin{aligned}f_4(\eta)=1&+\frac{K}{2}\left((\eta-1)^2-1\right)-WeK^2\left((\eta-1)^3+1\right)+\frac{3We^2K^3}{2}\left((\eta-1)^4-1\right)\\&-\frac{7We^3K^4}{5}\left((\eta-1)^5+1\right)+\frac{5We^4K^5}{3}\left((\eta-1)^6-1\right)-\frac{8We^5K^6}{7}\left((\eta-1)^7+1\right)-\frac{We^7K^8}{9}\left((\eta-1)^9+1\right)\end{aligned} \tag{24}$$

Hence, the series solution in general gives;

$$\begin{aligned}f(\eta)=1&+\frac{K}{2}\left((\eta-1)^2-1\right)-WeK^2\left((\eta-1)^3+1\right)+\frac{3We^2K^3}{2}\left((\eta-1)^4-1\right)\\&-\frac{7We^3K^4}{5}\left((\eta-1)^5+1\right)+\frac{5We^4K^5}{3}\left((\eta-1)^6-1\right)-\frac{8We^5K^6}{7}\left((\eta-1)^7+1\right)-\frac{We^7K^8}{9}\left((\eta-1)^9+1\right)\end{aligned} \tag{25}$$

Here it should be noted that for $We=0$, we get a solution for the Newtonian fluid [1]. By back substitution of values of dimensionless parameters, we obtain the solution (25) in dimensionless form as;

$$\begin{aligned} v(x) &= V_0 + \frac{\rho g}{\mu_0}\left(x^2 - 2\delta x\right) - \left(\frac{\Gamma}{\mu_0}\right)\left(\frac{\rho g}{\mu_0}\right)^2\left((x-\delta)^3 + \delta^3\right) \\ &+ \frac{3}{2}\left(\frac{\Gamma}{\mu_0}\right)^2\left(\frac{\rho g}{\mu_0}\right)^3\left((x-\delta)^4 - \delta^4\right) - \frac{7}{5}\left(\frac{\Gamma}{\mu_0}\right)^3\left(\frac{\rho g}{\mu_0}\right)^4\left((x-\delta)^5 + \delta^5\right) \\ &+ \frac{5}{3}\left(\frac{\Gamma}{\mu_0}\right)^4\left(\frac{\rho g}{\mu_0}\right)^5\left((x-\delta)^6 - \delta^6\right) - \frac{8}{7}\left(\frac{\Gamma}{\mu_0}\right)^5\left(\frac{\rho g}{\mu_0}\right)^6\left((x-\delta)^7 + \delta^7\right) \\ &- \frac{1}{9}\left(\frac{\Gamma}{\mu_0}\right)^7\left(\frac{\rho g}{\mu_0}\right)^8\left((x-\delta)^9 + \delta^9\right) + \ldots\ldots. \end{aligned} \tag{26}$$

Flow rate and average film velocity

The flow rate per unit width is given by the formula;

$$Q = \int_0^\delta v(x)dx \tag{27}$$

Substituting (26) in (27) and then integrating, we obtain the flow rate for a Williamson fluid as;

$$\begin{aligned} Q &= V_0\delta - \frac{2\rho g}{3\mu}\delta^3 - \frac{3}{4}\left(\frac{\Gamma}{\mu_0}\right)\left(\frac{\rho g}{\mu_0}\right)^2\delta^4 - \frac{9}{5}\left(\frac{\Gamma}{\mu_0}\right)^2\left(\frac{\rho g}{\mu_0}\right)^3\delta^5 - \frac{7}{6}\left(\frac{\Gamma}{\mu_0}\right)^3\left(\frac{\rho g}{\mu_0}\right)^4\delta^6 \\ &- \frac{10}{7}\left(\frac{\Gamma}{\mu_0}\right)^4\left(\frac{\rho g}{\mu_0}\right)^5\delta^7 - \left(\frac{\Gamma}{\mu_0}\right)^5\left(\frac{\rho g}{\mu_0}\right)^6\delta^8 - \frac{1}{10}\left(\frac{\Gamma}{\mu_0}\right)^7\left(\frac{\rho g}{\mu_0}\right)^8\delta^{10} \end{aligned} \tag{28}$$

The average velocity $\bar{V}$ is given by;

$$\begin{aligned} \bar{V} &= \frac{Q}{\delta} = V_0 - \frac{2\rho g}{3\mu}\delta^2 - \frac{3}{4}\left(\frac{\Gamma}{\mu_0}\right)\left(\frac{\rho g}{\mu_0}\right)^2\delta^3 - \frac{9}{5}\left(\frac{\Gamma}{\mu_0}\right)^2\left(\frac{\rho g}{\mu_0}\right)^3\delta^4 - \frac{7}{6}\left(\frac{\Gamma}{\mu_0}\right)^3\left(\frac{\rho g}{\mu_0}\right)^4\delta^5 \\ &- \frac{10}{7}\left(\frac{\Gamma}{\mu_0}\right)^4\left(\frac{\rho g}{\mu_0}\right)^5\delta^6 - \left(\frac{\Gamma}{\mu_0}\right)^5\left(\frac{\rho g}{\mu_0}\right)^6\delta^7 - \frac{1}{10}\left(\frac{\Gamma}{\mu_0}\right)^7\left(\frac{\rho g}{\mu_0}\right)^8\delta^9 \end{aligned} \tag{29}$$

It is observed from Eq. (29) that there will be a net upward flow of liquid if $\bar{V} > 0$ which implies that;

$$V_0 > \frac{2\rho g}{3\mu}\delta^2 + \frac{3}{4}\left(\frac{\Gamma}{\mu_0}\right)\left(\frac{\rho g}{\mu_0}\right)^2\delta^3 + \frac{9}{5}\left(\frac{\Gamma}{\mu_0}\right)^2\left(\frac{\rho g}{\mu_0}\right)^3\delta^4 + \frac{7}{6}\left(\frac{\Gamma}{\mu_0}\right)^3\left(\frac{\rho g}{\mu_0}\right)^4\delta^5$$
$$+\frac{10}{7}\left(\frac{\Gamma}{\mu_0}\right)^4\left(\frac{\rho g}{\mu_0}\right)^5\delta^6 + \left(\frac{\Gamma}{\mu_0}\right)^5\left(\frac{\rho g}{\mu_0}\right)^6\delta^7 + \frac{1}{10}\left(\frac{\Gamma}{\mu_0}\right)^7\left(\frac{\rho g}{\mu_0}\right)^8\delta^9 \tag{30}$$

Inequality (30) provides a reasonable estimation for the belt speed to lift the Williamson fluid. It shows that a large belt speed is needed to lift a fluid of small viscosity. As a special case, when $We = 0$, the inequality (30) becomes;

$$V_0 > \frac{\rho g}{3\mu}\delta^2, \tag{31}$$

which is true for a Newtonian fluid [1].

Solution by ADM

To apply ADM to our nonlinear equation, first we rewrite it in the following operator form [13-15];

$$Lf(\eta) = -We\left(Nf(\eta)\right) + K(\eta - 1), \tag{32}$$

where $L = \dfrac{d}{d\eta}$ is a linear invertible operator and $Nf(\eta) = \left(\dfrac{df}{d\eta}\right)^2$. Applying the inverse operator L^{-1} on both sides of Eq. (32), we get;

$$L^{-1}\left(Lf(\eta)\right) = -WeL^{-1}\left(\left(Nf(\eta)\right)\right) + L^{-1}\left(K(\eta - 1)\right) \tag{33}$$

so that;

$$f(\eta) = f(0) + L^{-1}\left(K(\eta - 1)\right) - WeL^{-1}\left(\left(Nf(\eta)\right)\right) \tag{34}$$

and using the boundary condition given in Eq. (14) we obtain;

$$f(\eta) = 1 + K\left(\frac{\eta^2}{2} - \eta\right) - WeL^{-1}\left(\left(N_2 f(\eta)\right)\right) \tag{35}$$

We decompose $f(\eta)$ and the nonlinear terms $Nf(\eta) = \left(\dfrac{df}{d\eta}\right)^2$, respectively, as follows;

$$f(\eta) = \sum_{n=0}^{\infty} f_n(\eta)$$
$$Nf(\eta) = \sum_{n=0}^{\infty} A_n \tag{36}$$

The first few terms of the adomian polynomial A_n are given by;

$$A_0 = \left(\frac{df_0}{d\eta}\right)^2,$$
$$A_1 = 2\left(\frac{df_0}{d\eta}\right)\frac{df_1}{d\eta},$$
$$A_2 = \left(\frac{df_1}{d\eta}\right)^2 + 2\left(\frac{df_0}{d\eta}\right)\frac{df_2}{d\eta},$$
$$A_3 = 2\frac{df_1}{d\eta}\frac{df_2}{d\eta} + 2\left(\frac{df_0}{d\eta}\right)\frac{df_3}{d\eta}, \tag{37}$$
.
.
.

We identify the zeroth component $f_0(\eta)$ by;

$$f_0(\eta) = 1 + \frac{K}{2}\left((\eta-1)^2 - 1\right), \tag{38}$$

and the remaining components $f_{n+1}(\eta)$ by the recurrence relation;

$$f_{n+1}(\eta) = -WeL^{-1}\sum_{n=0}^{\infty} A_n \quad n \geq 0 \tag{39}$$

Using the same algorithms as are used in [13-15], we obtain the following components;

$$f_1(\eta) = -\frac{WeK^2}{3}\left((\eta-1)^3 + 1\right), \tag{40}$$

$$f_2(\eta) = \frac{We^2K^3}{2}\left((\eta-1)^4 - 1\right), \tag{41}$$

$$f_3(\eta) = -We^3K^4\left((\eta-1)^5 + 1\right), \tag{42}$$

$$f_4(\eta) = \frac{7We^4K^5}{3}\left((\eta-1)^6 - 1\right) \tag{43}$$

and so on. In this manner, the rest of the terms in the decomposition series can be calculated.

Summing up, we write the solution in the decomposition series form;

$$f(\eta) = f_0 + f_1 + f_2 + f_3 + f_4 + \ldots\ldots$$

This, after inserting the values of f_0, f_1, f_2, f_3 and f_4 from (38) - (43), becomes;

$$\begin{aligned} f(\eta) &= 1+\frac{K}{2}\left((\eta-1)^2-1\right)-\frac{WeK^2}{3}\left((\eta-1)^3+1\right)+\frac{We^2K^3}{2}\left((\eta-1)^4-1\right) \\ &\quad -We^3K^4\left((\eta-1)^5+1\right)+\frac{7We^4K^5}{3}\left((\eta-1)^6-1\right)+.... \end{aligned} \tag{44}$$

The solution of the corresponding problem for the Newtonian fluid can be obtained by setting $We = 0$.

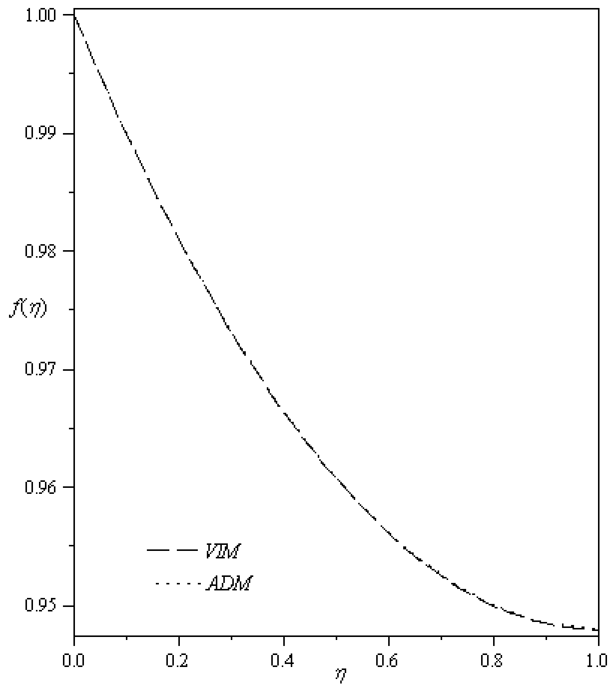

Figure 1 Comparison of VIM solution and ADM solution for $We = 0.1, K = 0.5$.

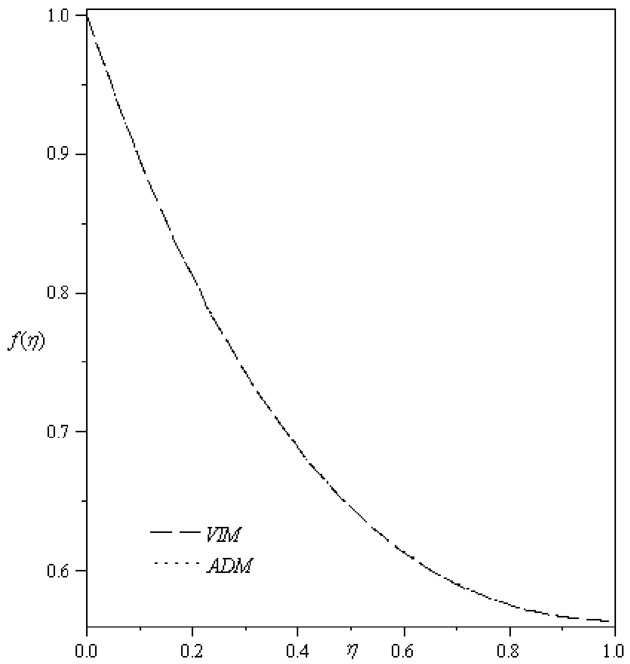

Figure 2 Comparison of VIM solution and ADM solution for $We = 0.3, K = 0.1$.

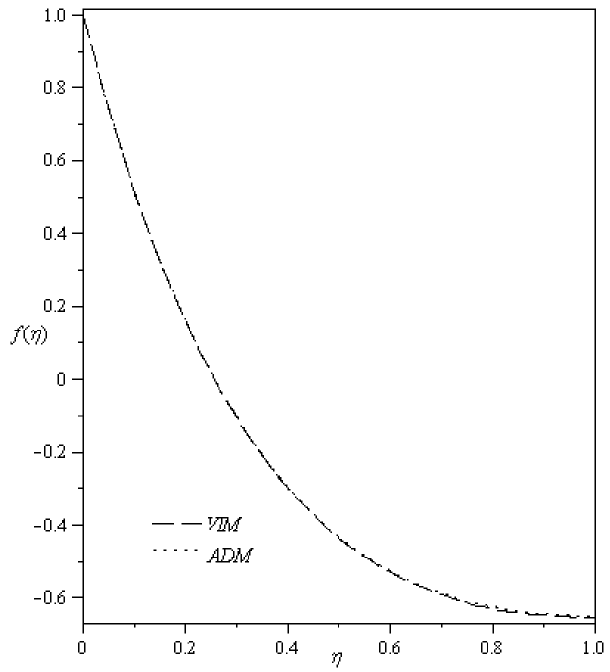

Figure 3 Comparison of VIM solution and ADM solution for $We = 0.5, K = 0.3$.

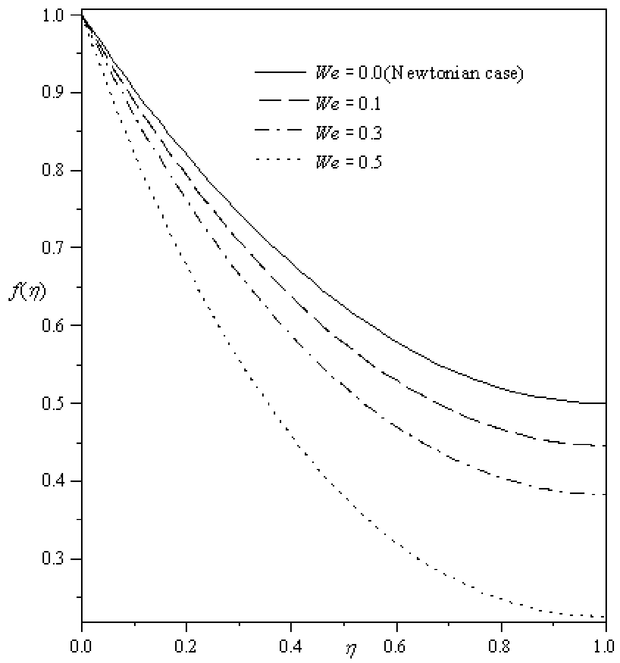

Figure 4 Effect of We on the velocity profile when $K = 0.5$.

Figures 1 - 3 are drawn to show a numerical comparison between the 4 iteration solution obtained by VIM and the 4 terms of the ADM solution for different values of We and K and no visible difference is observed. The quantitative effects of the material parameter We on the velocity profile obtained by VIM are observed physically in **Figure 4**. It can be seen that for the Williamson fluid, when the material parameter We increases for fixed values of K, the fluid velocity decreases.

Conclusions

The thin film flow of a Williamson fluid on a vertical moving belt is discussed. The governing nonlinear equation is solved by using VIM. In order to verify the efficiency of this method, the same problem is also solved by ADM. A very good agreement between the results of VIM and those obtained by ADM is observed, which confirm the validity of VIM. In comparison with the results of ADM, one can see that the 4 terms an approximation of the VIM is more effective than the four terms solution of ADM. Moreover, the series solution obtained by VIM converges faster than that by ADM and there is less computation work needed in comparison with the Adomian decomposition method. An estimation of the belt speed required to lift the fluid is also recorded. This estimation can be used for experimental verification.

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