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An Analytic Study on Time-Fractional Fisher Equation using Homotopy Perturbation Method

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Abstract

In this paper, the homotopy perturbation method (HPM) is effectively applied to obtain the approximate analytic solutions of the time-fractional Fisher equation (TFFE) with initial conditions. The fractional derivatives are described in the Caputo sense. The initial approximation can be determined by imposing the initial conditions. Some examples are given. Numerical results show that the HPM is easy to implement and is accurate when applied to TFFE.

Keywords: Time-fractional Fisher's equation, homotopy perturbation method, Caputo fractional derivative

Introduction

Fractional calculus is considered to be the generalization of the classical (or integer order) calculus, with a history of at least 300 years. It can be dated back to Leibniz's letter to L'Hospital, in which the meaning of the one-half order derivative was first discussed [1]. Although it has such a long history, research in the field remains in the realm of theory, due to the lack of proper mathematical analysis methods and real applications. However, the use of Fractional differential equations (FDEs) in mathematical models has become increasingly popular in recent years. The main reason for this is that the theory of derivatives of fractional (noninteger) stimulates considerable interest in the areas of mathematics, physics, engineering and other sciences. The number of scientific and engineering problems involving fractional calculus [2] is already large and is still growing, and perhaps fractional calculus will be the calculus of this century.

In the present paper we use the HPM to construct an approximate solution to the Fisher equation with time-fractional derivative of the form;

$$\begin{cases} D_t^{\alpha}u(x,y,t) = D_{xx}u(x,t) + \lambda u(x,t)(1-u(x,t)), \quad (x,t)\in[0,1]\times[0,1], \\ u(x,0) = u_0(x), \end{cases} \tag{1}$$

where $1<\alpha\leq 1$, λ is real parameter, D_t^{α} denotes the Caputo fractional derivative in time and $u_0(x)$ is the given function. $D_{xx} = \frac{\partial^2 u(x,t)}{\partial x^2} = u_{xx}$ is the linear differential operator.

The time-fractional Fisher equation (TFFE), which is a mathematical model for a wide range of important physical phenomena, is a partial differential equation obtained from the classical Fisher equation by replacing the time derivative with a fractional derivative of order α, $1<\alpha\leq 1$. Eq. (1) is known as Fisher's equation [3], which describe the propagation of a virile mutant in an infinitely long habitat. It also represents a model equation for the evolution of a neutron population in a nuclear reactor and a prototype model for a spreading flame. Eq. (1) becomes one of the most important classes of nonlinear equations because of its occurrence in many biological and chemical processes.

Many researchers have studied Fisher's equation. Olmos and Shizgal [4] presented a pseudo-spectral approach to obtain the numerical solutions. The finite difference algorithms have been reported by different authors, such as Parekh and Puri [5], Twizell *et al*. [6], Mickens [7,8], and Rizwan-Uddin [9]. The finite element method and Galerkin finite element method are used by some authors, Tang and Weber [10], Carey and Shen [11] and Roessler and Hüssner [12]. Also, some other methods have been conducted to derive the solutions for Fisher's equation. For more details about these investigations, the reader is advised to see Refs. [13-23] and the references therein.

We can see from the above description that there are many works about Fisher's equation. However, there are few articles about the numerical methods for TFFE. In this paper, we aim to effectively employ the HPM to establish the numerical solutions for Eq. (1). Using the HPM, the numerical results of Eq. (1) can be obtained within a few iterations. The HPM has been successfully applied to solve many types of linear and nonlinear problems in science and engineering by many authors [24-29] and also been used to solve fractional differential equations, all in the Caputo sense [30-39].

This paper is organized as follows: In Section 2, some basic definitions and properties of fractional calculus theory are given. In Section 3, the basic idea of the HPM for TFFE is given. In Section 4, we obtain the numerical solution of time-fractional Fisher equations with initial conditions.

Preliminaries

In this section, we give some basic definitions and properties of fractional calculus theory which are further used in this article.

Definition 2.1 [40,41] A real function $u(t)$, $t > 0$ is said to be in space $C_\theta(\theta \in R)$ if there exists a real number $p > \theta$, such that $u(t) = t^p u_1(t)$, where $u_1(t) \in C(0,\infty)$, and it is said to be in the space C_θ^n if and only if $u^n \in C_\theta$, $n \in N$.

Definition 2.2 [40,41] The Riemann-Liouville fractional integral operator of order α≥0 of a function $u(t) \in C_\theta, \theta \geq -1$, is defined as;

$$\begin{cases} J^\alpha u(t) = \dfrac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau) d\tau, \ \alpha > 0, \tau > 0, \\ J^0 u(t) = u(t). \end{cases} \tag{2}$$

For $u(t) \in C_\theta, \theta \geq -1, \alpha, \beta \geq 0$ and $\gamma \geq -1$, some properties of the operator J^α, which are needed here, are as follows;

$$i).\ J^\alpha J^\beta u(t) = J^{\alpha+\beta} u(t); \quad ii).\ J^\alpha J^\beta u(t) = J^\beta J^\alpha u(t); \quad iii).\ J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}.$$

Definition 2.3 [40,41] The fractional derivative in the Caputo sense of $u(t) \in C_{-1}^m, m \in N, t > 0$ is defined as;

$$D_t^{\alpha}u(t)=\begin{cases} J^{\alpha-m}\dfrac{d^m}{dt^m}u(t)=\dfrac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}u^m(\tau)d\tau,\ m-1<\alpha<m, \\ \dfrac{d^m}{dt^m}u(t), \alpha=m. \end{cases} \tag{3}$$

Lemma 2.4 [40,41] If $m<\alpha+1\le m+1, m\in N, f\in C_{\theta}^{m}, \theta\ge -1$, then the following 2 properties hold;

$$i). D_t^{\alpha}[J^{\alpha}u(t)]=u(t);\quad ii). J^{\alpha}[D_t^{\alpha}u(t)]=u(t)-\sum_{k=0}^{m-1} u^{(k)}(0^{+})\frac{t^k}{k!}. \tag{4}$$

The HPM for time-fractional Fisher's equation

We apply the HPM [24,25] to the time-fractional differential equations with initial conditions. We consider the following equation;

$$\begin{cases} D_t^{\alpha}u(x,y,t)=D_{xx}u(x,t)+\lambda u(x,t)(1-u(x,t)), (x,t)\in[0,1]\times[0,1], \\ u(x,0)=u_0(x), \end{cases} \tag{5}$$

where $u(x,t)$ is an unknown function, and x and t denote spatial and temporal independent variables. By means of the HPM, one first constructs the homotopy which satisfies the relation;

$$D_t^{\alpha}u(x,t)=p[D_{xx}u(x,t)+\lambda u(x,t)(1-u(x,t))], \tag{6}$$

where $p\in[0,1]$ is the embedding parameter.

The HPM [24,25] consists of decomposing the solution $u(x,t)$ in Taylor series about the embedding parameter p into a sum of components given by the infinite series expanded;

$$u(x,t)=\sum_{m=0}^{\infty} u_m(x,t)p^m, \tag{7}$$

where

$$u_m(x,t)=\frac{1}{m!}\frac{\partial^m u(x,t)}{\partial p^m}|_{p=0}. \tag{8}$$

As $p=1$, we can get that;

$$u(x,t)=\sum_{m=0}^{\infty} u_m(x,t). \tag{9}$$

We substitute (7) into (6), then differentiate the result m-times with respect to p and divide them by $m!$. After that, let $p=0$ and apply the inverse operator J^{α} to the both sides of them, then by (4) we can get;

$$u_m(x,t) = J^{\alpha}[D_{xx}u_{m-1}(x,t) + \lambda u_{m-1}(x,t) - \lambda \sum_{i=0}^{m-1} u_i(x,t)u_{m-i-1}(x,t)], \tag{10}$$

for $m = 1, 2, \cdots$.

The method suggests that the zeroth component u_0 is usually defined as terms arising from initial conditions. Furthermore, we equate selected components of (10), and obtain recursively;

$$\begin{cases} u_0 = u_0(x), \\ u_m(x,t) = J^{\alpha}[D_{xx}u_{m-1}(x,t) + \lambda u_{m-1} - \lambda \sum_{i=0}^{m-1} u_i u_{m-i-1}], m \geq 1. \end{cases} \tag{11}$$

The convergence of the decomposition method has been discussed in [41] and all components will be easily determined; hence, the decomposition method provides a reliable technique that requires less work if compared with traditional techniques.

Applications

We will apply the HPM to the following fractional differential equations.

Example 1 We consider the following time-fractional problem;

$$\begin{cases} D_t^{\alpha}u(x,t) = D_{xx}u(x,t) + \lambda u(x,t)(1-u(x,t)), \quad (x,t) \in [0,1]\times[0,1], \\ u(x,0) = u_0(x), \end{cases} \tag{12}$$

with $\lambda = 1$ and $u(x,0) = \beta$, where β is a constant. The corresponding integer order problem, $\alpha = 1$ in the limit sense, has the exact solution;

$$u_{exact,1}(x,t) = \frac{\beta e^t}{1-\beta+\beta e^t}. \tag{13}$$

By the analysis in Section 3, we construct the homotopy which satisfies the relation

$$\begin{cases} D_t^{\alpha}u(x,t) = p[D_{xx}u(x,t) + u(x,t) - u^2(x,t)], \\ u(x,0) = \beta. \end{cases} \tag{14}$$

We assume the solution of Eq. (12) to be in the form;

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots. \tag{15}$$

Substituting (15) into (14) and equating the coefficients of like powers of p, we get the following set of differential equations;

$$p^0: \ D_t^{\alpha} u_0 = 0, u_0(x) = \beta, \qquad p^1: \ D_t^{\alpha} u_1 = u_0 - u_0^2,$$
$$p^2: \ D_t^{\alpha} u_2 = u_1 - 2u_0 u_1, \qquad p^3: \ D_t^{\alpha} u_3 = u_2 - 2u_0 u_2 - u_0^1,$$
$$p^4: \ D_t^{\alpha} u_4 = u_3 - 2u_0 u_3 - 2u_1 u_2, \ \cdots$$
$$\Downarrow$$
$$u_0(x,t) = \beta; \ u_1(x,t) = (\beta - \beta^2)\frac{t^{\alpha}}{\Gamma(1+\alpha)}, \ u_2(x,t) = (\beta - 3\beta^2 + 2\beta^3)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)},$$
$$u_3(x,t) = (\beta - 5\beta^2 + 8\beta^3 - 4\beta^4)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} - (\beta^2 - 2\beta^3 + \beta^4)\frac{\Gamma(1+2\alpha)t^{3\alpha}}{\Gamma^2(1+\alpha)\Gamma(1+3\alpha)},$$
$$u_4(x,t) = (1-2\beta)[(\beta - 5\beta^2 + 8\beta^3 - 4\beta^4) - (\beta^2 - 2\beta^3 + \beta^4)\frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}]\frac{t^{4\alpha}}{\Gamma(1+4\alpha)}$$
$$-2(\beta - \beta^2)(\beta - 3\beta^2 + 2\beta^3)\frac{\Gamma(1+3\alpha)t^{4\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)\Gamma(1+4\alpha)}, \tag{16}$$
$$\cdots,$$

the rest of components can be obtained using Maple or Mathematica in the same manner. It should be pointed out that when $\alpha = 1$, the results obtained by (16) become;

$$\begin{aligned} u_0(x,t) &= \beta, \\ u_1(x,t) &= \beta(1-\beta)t, \\ u_2(x,t) &= \beta(1-\beta)(1-2\beta)\frac{t^2}{2!}, \\ u_3(x,t) &= \beta(1-\beta)(1-6\beta+6\beta^2)\frac{t^3}{3!}, \\ u_4(x,t) &= \beta(1-\beta)(1-2\beta)(1-12\beta+12\beta^2)\frac{t^4}{4!}, \\ &\cdots, \end{aligned} \tag{17}$$

moreover, we get the solution in a series form as;

$$\begin{aligned} u(x,t) = \beta + \beta(1-\beta)t + +\beta(1-\beta)(1-2\beta)\frac{t^2}{2!} + \beta(1-\beta)(1-6\beta+6\beta^2)\frac{t^3}{3!} \\ +\beta(1-\beta)(1-2\beta)(1-12\beta+12\beta^2)\frac{t^4}{4!} + \cdots, \end{aligned} \tag{18}$$

and, after some algebra, the solution in a closed form is given by;

$$u(x,t) = \frac{\beta e^t}{1-\beta+\beta e^t} \tag{19}$$

which is in full agreement with the results in [21]. Then, the 4 th-order approximation solution can be obtained as $u(x,t) \approx \Phi_4(x,t) = u_0 + u_1 + u_2 + u_3 + u_4$.

Figure 1 shows the exact solution of the corresponding integer order problem $u_{exact,1}(x,t)$ along with the HPM 4th-order approximation solution of Eq. (12) with $\beta = 2/3$ for different values of α. For $\alpha = 1$ and $\beta = 2/3$, in **Figure 2**, we draw the absolution error functions $error(x,t) = |u_{exact,1}(x,t) - \Phi_4(x,t)|$, where $u_{exact,1}(x,t)$ is the form of (13).

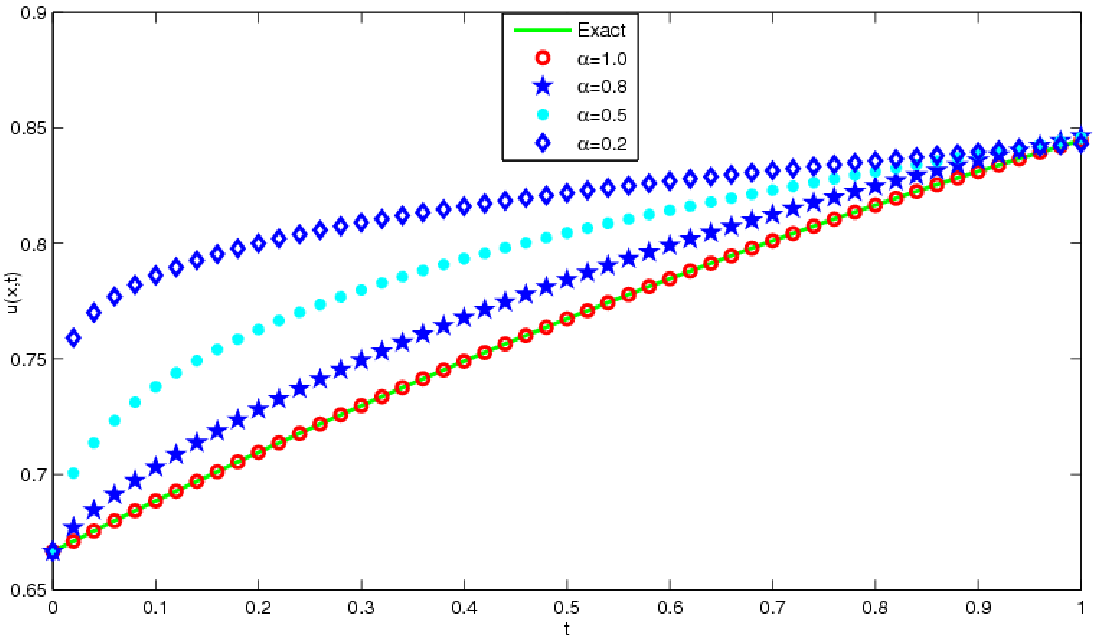

Figure 1 The 4th-order approximation solutions of $u(x,t)$ of example 1 for different values of α.

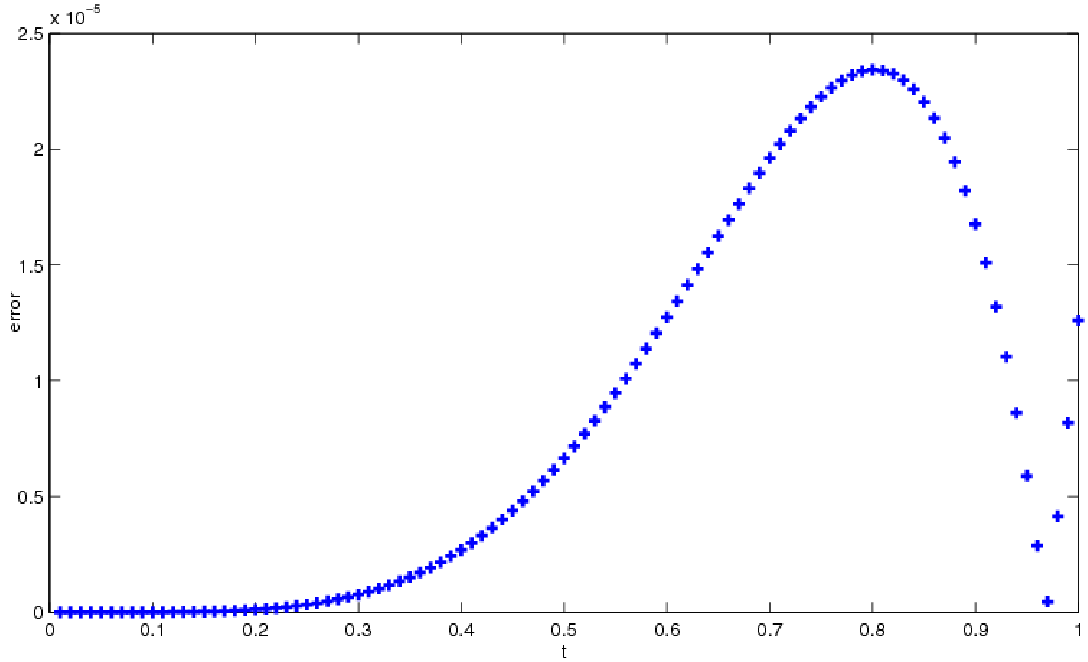

Figure 2 Absolute error obtained by 4th-order approximation for $\alpha = 1$.

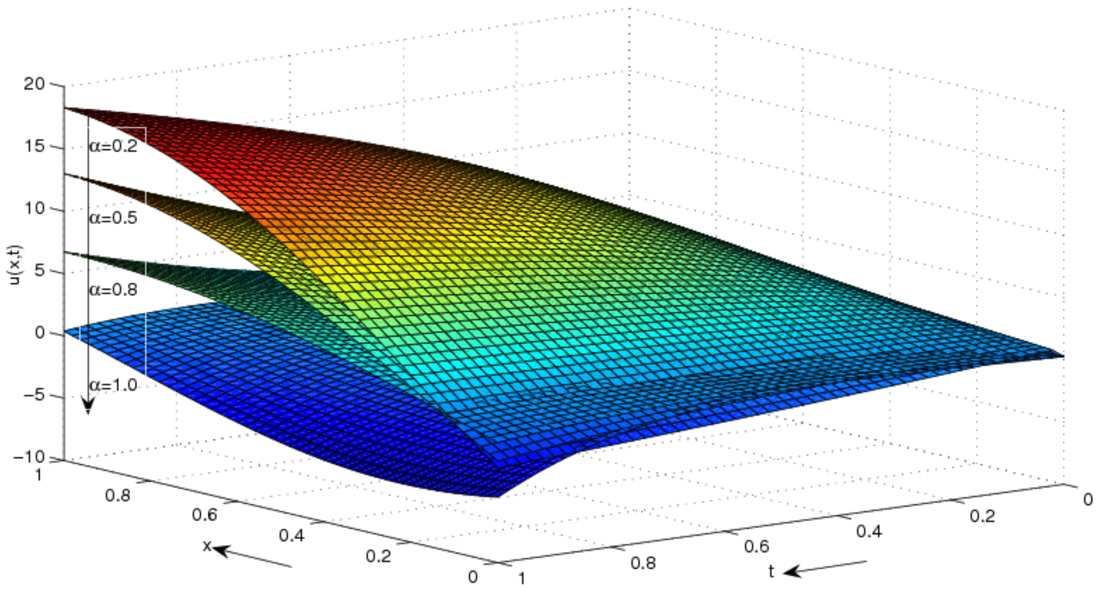

Figure 3 The surfaces show the solution $u(x,t)$ of example 2, obtained by 3rd-order approximation for different values of α.

Example 2 We consider the following time-fractional Fisher equation;

$$\begin{cases} D_t^{\alpha} u(x,t) = D_{xx} u(x,t) + \lambda u(x,t)(1-u(x,t)), \quad (x,t) \in [0,1]\times[0,1], \\ u(x,0) = u_0(x), \end{cases} \tag{20}$$

with $\lambda = 6$ and $u(x,0) = \frac{1}{(1+e^x)^2}$. The corresponding integer order problem, $\alpha = 1$ in the limit sense, has the exact solution;

$$u_{exact,2}(x,t) = \frac{1}{(1+e^{x-5t})^2}. \tag{21}$$

By the analysis in Section 3, we construct the homotopy which satisfies the relation;

$$\begin{cases} D_t^{\alpha} u(x,t) = p[D_{xx} u(x,t) + 6u(x,t) - 6u^2(x,t)], \\ u(x,0) = \frac{1}{(1+e^x)^2}, \end{cases} \tag{22}$$

As per the analysis in Example 1, we can get following set of differential equations;

$$\begin{aligned}
&p^0: \ D_t^{\alpha} u_0 = 0, u_0(x) = \frac{1}{(1+e^x)^2}, \\
&p^1: \ D_t^{\alpha} u_1 = D_{xx} u_0 + 6u_0 - 6u_0^2, \\
&p^2: \ D_t^{\alpha} u_2 = D_{xx} u_1 + 6u_1 - 12u_0 u_1, \\
&p^3: \ D_t^{\alpha} u_3 = D_{xx} u_2 + 6u_2 - 12u_0 u_2 - 6u_1^2, \\
&\quad \cdots \\
&\quad \Downarrow \\
&u_0(x,t) = \frac{1}{(1+e^x)^2}, \\
&u_1(x,t) = 10 \frac{e^x}{(1+e^x)^3} \frac{t^{\alpha}}{\Gamma(1+\alpha)}, \\
&u_2(x,t) = 50 \frac{e^x(-1+2e^x)}{(1+e^x)^4} \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, \\
&u_3(x,t) = 50e^x(5 - 6e^x - 15e^{2x} + 20e^{3x} - 12e^x \frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}) \frac{t^{3\alpha}}{(1+e^x)^6 \Gamma(1+3\alpha)}, \\
&\quad \cdots,
\end{aligned} \tag{23}$$

the rest of components can be obtained using Maple or Mathematica in the same manner, and when $p = 1$ the solution is thus obtained as;

$$u = u_0 + u_1 + u_2 + u_3 + u_4 \cdots. \tag{24}$$

It should be pointed out that when $\alpha = 1$, the results obtained by (23) become;

$$\begin{aligned}
u_0(x,t) &= \frac{1}{(1+e^x)^2}, \\
u_1(x,t) &= 10\frac{e^x}{(1+e^x)^3}t, \\
u_2(x,t) &= 25\frac{e^x(-1+2e^x)}{(1+e^x)^4}t^2, \\
u_3(x,t) &= -\frac{125}{3}\frac{e^x(-1+7e^x-4e^{2x})}{(1+e^x)^5}t^3, \\
&\cdots,
\end{aligned} \tag{25}$$

moreover, we get the solution in a series form as;

$$u(x,t) = \frac{1}{(1+e^x)^2} + 10\frac{e^x}{(1+e^x)^3}t + 50\frac{e^x(-1+2e^x)}{(1+e^x)^4}\frac{t^2}{2!} - 250\frac{e^x(-1+7e^x-4e^{2x})}{(1+e^x)^5}\frac{t^3}{3!} + \cdots, \tag{26}$$

and after some algebra, the solution in a closed form is given by;

$$u(x,t) = \frac{1}{(1+e^{x-5t})^2}, \tag{27}$$

which is in full agreement with the results in [22].

The 3rd-order approximation solution can be obtained as $u(x,t) \approx \Psi_3(x,t) = u_0 + u_1 + u_2 + u_3$. **Figure 3** shows the solution $u(x,t)$ of Example 2, obtained by 3rd-order approximation for different values of α. It is seen from **Figure 3** that $u(x,t)$ increases with the increase in x for given t but decreases with the increase in α.

When $\alpha = 1$, it is worth noting that by applying the scheme proposed above for the Fisher equation;

$$D_t^{\alpha} u(x,t) = D_{xx} u(x,t) + \lambda u(x,t)(1 - u(x,t)) \tag{28}$$

with the initial condition $u(x,0) = 1/(1+e^{\sqrt{\frac{\lambda}{6}}x})^2$, we can get the solution;

$$u(x,t) = \frac{1}{(1+e^{\sqrt{\frac{\lambda}{6}}x - \frac{5}{6}\lambda t})^2}, \tag{29}$$

which is the same as the results in [20].

Conclusions

In this paper, the HPM was used for obtaining approximate solutions of the TFFEs with initial conditions. The examples are presented to illustrate the accuracy of the present scheme of HPM. The approximate solutions were almost identical to analytic solutions of the considered equations for $\alpha = 1$. It may be concluded that this methodology is a very powerful and efficient technique in finding approximate solutions for a wide classes of problems. Also, it is to be noted that the accuracy can be improved by computing more terms of approximated solutions. This shows that the HPM is a very useful method to get high-precision numerical solutions for many problems.

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