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Analytical Investigation on 2-D Unsteady MHD Viscoelastic Flow between Moving Parallel Plates Using RVIM and HPM

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Abstract

In this paper the velocity fields associated with the two-dimensional unsteady magnetohydrodynamic (MHD) flow of a viscous fluid between moving parallel plates have been investigated. The governing Navier-Stokes equations for the flow are reduced to a fourth order nonlinear ordinary differential equation. The Homotopy Perturbation Method (HPM) and Reconstruction of Variational Iteration Method (RVIM) have been used to achieve analytical solutions. The obtained approximate results have been compared with numerical ones and results from pervious works in some cases. It has been shown that the current study is accurate and validated and can be used for other nonlinear cases.

Keywords: Approximate solution, MHD viscoelastic flow, moving parallel plates, Homotopy Perturbation Method, Reconstruction of Variational Iteration Method

Introduction

The study of flow of an electrically conducting fluid has applications in many engineering problems such as magnetohydrodynamic (MHD) power generators, MHD pumps, accelerators, plasma studies, geothermal energy extractions, the boundary layer control, aerodynamic heating, electrostatic precipitation, etc. The subject of MHD is largely perceived to have been initiated by Swedish electrical engineer Hannes Alfven [1] in 1942. The problem of unsteady flow of a MHD viscous fluid between 2 parallel plates in motion normal to their own surfaces independent of each other and arbitrary with respect to time is a type of unsteady flow which is met frequently in many hydrodynamical machines and apparatuses. Such a flow problem lends itself to applications in liquid metal lubrications, for instance [2]. The theoretical and experimental studies of flow between 2 parallel plates have been conducted by many researchers [3-6].

MHD viscoelastic flow was the main interest of many pervious researches [7-9]. Most fluid mechanical problems have non-linear behavior inherently. There are few phenomena in different fields of science occurring linearly. A lot of scientific phenomena like heat and mass transfer ones function nonlinearly. These nonlinear equations cannot be solved using ordinary methods. To overcome the shortcomings, many new techniques have appeared in the literature, for example, the Homotopy Perturbation Method (HPM) [10], Variational Iteration Method (VIM) [11], Differential Transformation Method (DTM) [12], Homotopy Analysis Method (HAM) [13,14] and Adomian Decomposition Method (ADM) [15]. In this paper we apply the HPM and VIM to a nonlinear ordinary differential equation derived from the unsteady MHD flow of a viscous fluid between moving parallel plates. The obtained approximate result is compared to the numerical solution in numerical cases.

Materials and methods

The non-dimensional equation of 2-D unsteady MHD viscoelastic flow between moving parallel plates can be written in the following form [5];

$$f^{(iv)}(\eta) - M^2 f''(\eta) = R\left(f(\eta)f'''(\eta) - f'(\eta)f''(\eta) - \eta f'''(\eta) - \left(2 + \frac{1}{\rho}\right) f''(\eta)\right) \tag{1}$$

where $R = \frac{\alpha v}{\rho}$ is the parameter stating the movement of the plates ($R > 0$ similar to the plate moving apart, while $R < 0$ similar to the plates moving together) and M^2 is the magnetic parameter. Taking $\rho = 1$ gives a governing steady-state equation. We may permit the density to keep a parameter in the model, and so we deliberately affect the density of the fluid on the achieved solutions.

Moreover, as our similarity variable g holds both the spatial variable y as well as the temporal variable η, there is no steady-state assumption in such a case. This allows us to pay more attention to both cases in which the plates move apart and also the case in which the plates move together. As a matter of fact, in the case of a squeezing flow, a steady-state assumption will be meaningless, since the plates may only move toward one another in finite time. Furthermore, Eq. (10) is held subject to the boundary conditions which follow from the Eq. (1) boundary conditions;

$$f(0) = 0, \quad f(1) = 1, \quad f'(1) = 0, \quad f'(0) = 0 \tag{2}$$

In order to solve Eq. (1) by using HPM, we consider the following nonlinear differential equation;

$$A(u) - g(r) = 0, \quad r \in \Omega \tag{3}$$

with the boundary conditions of;

$$B(u, \partial u/\partial n) = 0, \quad r \in \Gamma \tag{4}$$

where A, B , $g(r)$ and Γ are a general differential operator, a boundary operator, a known analytical function, and the boundary of domain Ω .

Generally speaking the operator A can be divided into a linear part L and a nonlinear part $N(u)$. Eq. (3) can therefore, be rewritten as;

$$L(u) + N(u) - g(r) = 0 \tag{5}$$

We construct a homotopy of Eq. (4) $f(r, p): \Omega \times [0,1] \to R$, which satisfies;

$$H(f, p) = (1 - p)[L(f) - L(u_0)] + p[A(f) - g(r)] = 0, \quad p \in [0,1], \quad r \in \Omega \tag{6}$$

or,

$$H(f, p) = L(f) - L(u_0) + pL(u_0) + p[N(f) - g(r)] = 0 \tag{7}$$

So, we can construct a homotopy of the system as follows;

$$H(f, p) = (1 - p)\left(f^{(iv)}\right) + p\left(-M^2 f''(\eta) - R\left(f(\eta)f'''(\eta) - f'(\eta)f''(\eta) - \eta f'''(\eta) - \left(2 + \frac{1}{\rho}\right)\right)\right) \tag{8}$$

where $p \in [0,1]$ is an embedding parameter, while f_0 is an initial approximation of Eq. (4) which satisfies the boundary condition. We consider f as follows;

$$f = f_0 + pf_1 + p^2 f_2 + p^3 f_3 + \cdots \tag{9}$$

Setting $p = 1$ yields an approximate solution of the equation to;

$$f = \lim_{p \to 1} f = f_0 + f_1 + f_2 + \cdots \tag{10}$$

Assuming $f''' = 0$ and substituting f from Eq. (10) into Eq. (8) and rearranging based on powers of p-terms, we have;

$$\begin{aligned} & p^0 : f_0^{(iv)} = 0 \\ & f_0(0) = 0,\ \ f_0'(0) = 0,\ \ f_0(1) = 1,\ \ f_0'(1) = 0 \end{aligned} \tag{11}$$

$$\begin{aligned} & p^1 : f_1^{(iv)} - R\left(f_0 f_0''' - f_0' f_0'' - t f_0''' - \left(2 + \frac{1}{\rho}\right) f_0'' \right) - M^2 f_0'' = 0 \\ & f_1(0) = 0,\ \ f_1'(0) = 0,\ \ f_1(1) = 0,\ \ f_1'(1) = 0 \end{aligned} \tag{12}$$

$$\begin{aligned} & p^2 : f_2^{(iv)} - R\left(f_0 f_1''' + f_1 f_0''' - f_0' f_1'' - f_1' f_0'' - t f_1''' - \left(2 + \frac{1}{\rho}\right) f_1'' \right) - M^2 f_1'' = 0 \\ & f_2(0) = 0,\ \ f_2'(0) = 0,\ \ f_2(1) = 0,\ \ f_2'(1) = 0 \end{aligned} \tag{13}$$

Solving the above equations results in the following answers;

$$f_0(t) = -2t^3 + 3t^2 \tag{14}$$

$$\begin{aligned} f_1(t) = & -\frac{2R}{35}t^7 + \frac{R}{5}t^6 + \left(\frac{R}{10\rho} - \frac{M^2}{10}\right)t^5 + \left(\frac{M^2}{4} - \frac{R}{4\rho} - \frac{R}{2}\right)t^4 + \left(\frac{17R}{35} + \frac{R}{5\rho} - \frac{M^2}{5}\right)t^3 \\ & + \left(\frac{M^2}{20} - \frac{R}{20\rho} - \frac{9R}{70}\right)t^2 \end{aligned} \tag{15}$$

$$\begin{aligned}
f_2(t) &= \frac{4R^2}{5775}t^{11} - \frac{2R^2}{525}t^{10} + \left(\frac{19R^2}{1260} - \frac{RM^2}{420} + \frac{R^2}{140\rho}\right)t^9 + \left(\frac{3RM^2}{280} - \frac{11R^2}{280} - \frac{3R^2}{280\rho}\right)t^8 \\
&+ \left(\frac{R^2}{150\rho} - \frac{RM^2}{150} + \frac{103R^2}{2450} - \frac{M^4}{420} - \frac{R^2}{420\rho^2} + \frac{RM^2}{210\rho}\right)t^7 + \left(\frac{2R^2}{75\rho} + \frac{R^2}{105} + \frac{M^4}{120} + \frac{R^2}{120\rho^2} - \frac{RM^2}{60\rho}\right)t^6 \\
&+ \left(\frac{31RM^2}{700} - \frac{R^2}{100\rho^2} + \frac{RM^2}{50\rho} - \frac{33R^2}{700} - \frac{31R^2}{700\rho} - \frac{M^4}{100}\right)t^5 \\
&+ \left(\frac{R^2}{240\rho^2} - \frac{2RM^2}{105} - \frac{RM^2}{120\rho} + \frac{3R^2}{140} + \frac{M^4}{240} + \frac{2R^2}{105\rho} + \frac{31RM^2}{700} - \frac{M^4}{100}\right)t^4 \\
&+ \left(\frac{2267R^2}{485100} + \frac{R^2}{420\rho} - \frac{RM^2}{420} - \frac{RM^2}{2100\rho} + \frac{R^2}{4200\rho^2} + \frac{M^4}{4200}\right)t^3 \\
&+ \left(\frac{3RM^2}{1400} + \frac{RM^2}{1400\rho} - \frac{R^2}{2800\rho^2} - \frac{69R^2}{21560} - \frac{3R^2}{1400\rho} - \frac{M^4}{2800}\right)t^2
\end{aligned} \tag{16}$$

We avoid listing the other components. However according to Eq. (10) we can obtain f as follows;

$$f \approx f_0 + f_1 + f_2 + \cdots \tag{17}$$

In order to illustrate the basic concepts of Reconstruction of Variational Iteration Method (RVIM), the following nonlinear partial differential equations can be considered;

$$Lu(x,t) + Ru(x,t) + Nu(x,t) = g(x,t) \tag{18}$$

where R is a linear operator which has partial derivatives with respect to x, L is the linear time derivative operator, $Nu(x,t)$ is a nonlinear term and $g(x,t)$ is an inhomogeneous term. According to RVIM, the following iteration formula can be constructed.

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(Lu_n + Ru_n + Nu_n - g)d\tau \tag{19}$$

where λ is the general Lagrange multiplier which can be identified optimally via the Laplace transformation. We consider the linear part of the differential equation, then we perform a Laplace transformation to that (with the boundary condition equal zero), and put the answer equal to $-1^n\left(n = order \quad of \quad derivation\right)$. Performing an inverse Laplace and $t \to \tau - t$ we can easily obtain the Lagrangian multiplyer. For this problem, we consider the linear part of Eq. (10), then we perform a Laplace transformation to that (with the boundary condition equal zero), and putting it equal to -1^4;

$$L\left(f^{(4)} - M^2 f'' + R\left(2 + \frac{1}{\rho}\right)f''\right) = s^4F(s) - M^2s^2F(s) + 2Rs^2F(s) + \frac{Rs^2F(s)}{\rho} = 1 \tag{20}$$

Then;

$$F(s) = -\frac{\rho}{s^2\left(-s^2\rho + M^2\rho - 2R\rho - R\right)} \tag{21}$$

with performing an inverse Laplace and $t \to \tau - t$ we will obtain the Lagrangian multiplayer;

$$\lambda = -\frac{\rho\left(-\rho K^2(t-\tau) - \rho K\sinh\left(K(t-\tau)\right)\right)}{\left(M^2\rho - 2R\rho - R\right)^2} \tag{22}$$

where $K^2 = \dfrac{M^2\rho - 2R\rho - R}{\rho}$

According to Eqs. (27) and (28), the correction functional of Eq. (10) can be settled in the following form;

$$f_{n+1}(\eta) = f_n(\eta) + \int_0^\eta \lambda\left(f_n^{(4)}(\tau) - M^2 f_n''(\tau) - R\left(f_n(\tau)f_n'''(\tau) - f_n'(\tau)f_n''(\tau) - \tau f_n'''(\tau) - \left(2 + \frac{1}{\rho}f_n''(\tau)\right)\right)\right)d\tau \tag{23}$$

with the initial function;

$$f_0(\eta) = \frac{\left(-\rho + \cosh(K) + \cosh(K\eta) - \cosh\left(K(\eta-1) - \rho K\eta\sinh(K)\right)\right)}{\left(-2\rho + 2\cosh(K) - \rho K\eta\sinh(K)\right)} \tag{24}$$

Using the above variational formula in Eq. (32) and substituting the initial function in Eq. (33) the solution will be obtained.

Results and discussion

In this section, we will investigate the analytical approximate results achieved by the HPM and RVIM. In **Table 1**, a comparison between the approximate solutions achieved by HPM and RVIM and numerical ones is presented.

Table 1 HPM and RVIM in comparison with the Runge Kutta forth order method when $R = 1$, $M = 2$, $\rho = 1$.

T	$\mathbf{f_{NM}}$	$\mathbf{f_{HPM}}$	$\mathbf{f_{RVIM}}$
0	0.00000	0.00000	0.00000
0.2	0.10425	0.10425	0.10414
0.4	0.35216	0.35215	0.35217
0.6	0.64783	0.64782	0.64783
0.8	0.89574	0.89574	0.8974
1	1.0000	1.0000	1.0000

Figure 1 shows that the obtained results are valid. In **Figures 2a - 2b** the effect of the R parameter on the y direction velocity profile is presented. **Figure 2a** shows that an increase in the R parameter value (when the plates are moving apart) leads to an increase in the velocity peak point (maximum velocity) in the y direction. **Figure 2b** shows that a decreasing R value, in case the plates are moving together (the so-called squeezing flow) causes a decrease in the velocity peak point in the y direction.

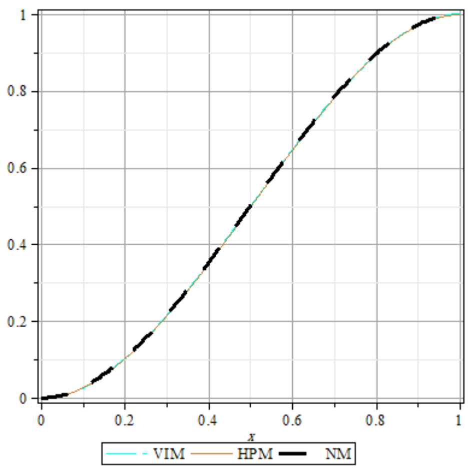

Figure 1 Velocity profile in the x-direction for $R = 1, M = 2, \rho = 1$.

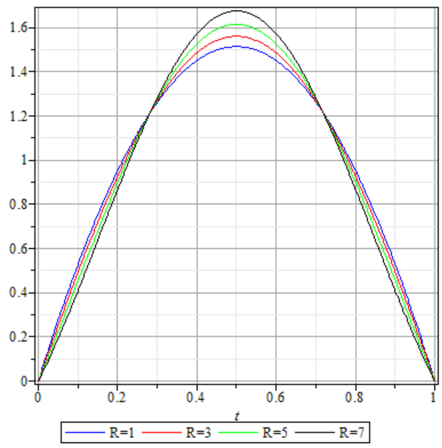

Figure 2a Profiles for the velocity in the y-direction, f(g), for, M = 2 and variable $R > 0$.

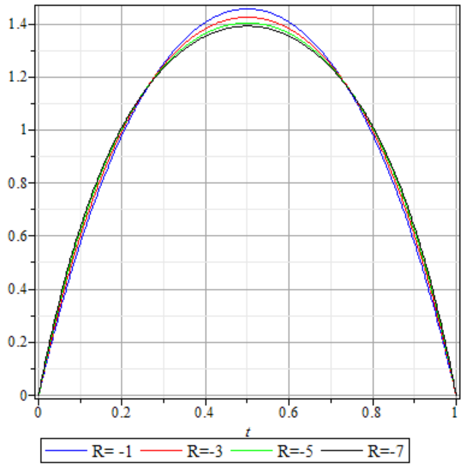

Figure 2b Profiles for the velocity in the *y*-direction, (f '), for, M = 2 and variable $R < 0$.

In **Figure 3** we can see the effect of the magnetic field strength on the velocity in the *y* direction. As can be seen by the increasing M^2 the maximum velocity decreases. Therefore, increasing the electrical conductivity of the fluid or increasing the magnitude of the magnetic field results in a non-uniform decrease in the *y*-direction velocity. **Figure 4** shows similar results when the plates are allowed to move together.

A comparison between RVIM solution and HAM solution [16] for velocity in the *y*-direction when $R = 1,\ M = 1,\ \rho = 1$ is listed in **Table 2**.

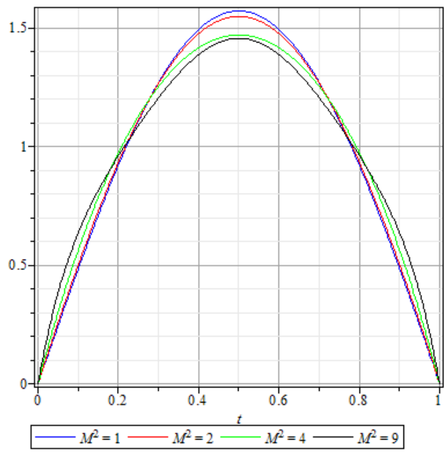

Figure 3 Profiles for the velocity in the *y*-direction, $f'(\eta)$, for R = 3, and variable M^2.

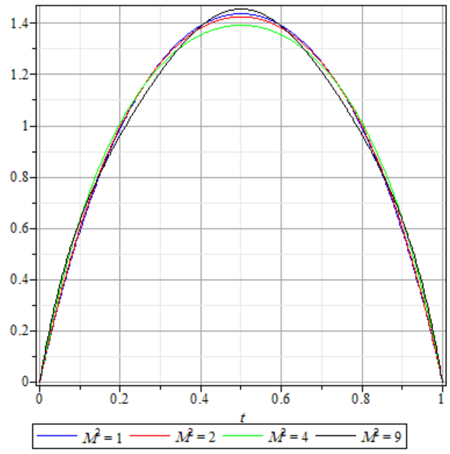

Figure 4 Profiles for the velocity in the *y*-direction, $f'(\eta)$, for R = –3, and variable M^2.

Table 2 Comparison between RVIM solution and HAM solution [16] when $R = 1, M = 1, \rho = 1$.

T	f'_{RVIM}	f'_{HAM} [16]	Error (%)
0	0.00000	0.00000	0.00000
0.1	0.52641	0.52407	0.234
0.2	0.95082	0.93324	1.758
0.3	1.26266	1.23891	2.375
0.4	1.45344	1.43303	2.041
0.5	1.51767	1.49926	1.841

Conclusions

In this paper we studied the 2-D unsteady MHD viscoelastic flow between moving parallel plates by using 2 powerful analytical methods: the HPM and RVIM. Velocity profiles in both the *x* and *y* directions have been investigated in some numerical cases. The high accuracy and validity of the methods shows that this study can be used for other nonlinear problems.

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