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Analytic and Approximate Solutions of Space-Time Fractional Telegraph Equations via Laplace Transform

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Abstract

In this paper, we consider a fractional model of telegraph equation in terms of voltage and current. The fractional derivatives are taken in the Caputo sense. The numerical algorithm based on the homotopy perturbation transform method (HPTM) is applied to obtain analytic and approximate solutions of the space-time fractional telegraph equations. The HPTM is combined in the form of Laplace transform and homotopy perturbation method. The results obtained by the HPTM show that the approach is easy to implement and computationally very attractive.

Keywords: Laplace transform method, homotopy perturbation method, space-time fractional telegraph equations, transmission line

Introduction

It would be difficult to imagine a world without communication systems. A plethora of guided fixed line telephones as well as a multitude of unguided systems to serve cellular phones are evident in our surrounding world. High frequency communication systems continue to benefit from significant industrial attention, triggered by a host of radio frequency (RF) and microwave communication (MW) systems. These systems use the transmission media for transferring the information carrying signal from one point to another point. This transmission media can be divided into 2 groups, namely, guided and unguided transmission lines. In guided medium the signal is transferred through the coaxial cable or transmission line. In guided transmission media, specifically cable transmission medium is investigated to address the problem of efficient telegraphic transmission. A cable transmission medium can be classified as a guided transmission medium and represents a physical system that directly propagates the information between 2 or more locations. In order to optimize the guided communication system it is necessary to determine or project power and signal losses in the system, because all the systems have such losses. To determine these losses and eventually ensure a maximum output, it is necessary to model some kind of equation with which to calculate these losses. In this paper a fractional model for the telegraph equation in terms of voltage and current for a section of a transmission line has been developed. The most important advantage of using fractional derivatives in these and other applications is their non-local property. It is well known that the integer order differential operator is a local operator but the fractional order differential operator is non-local. This means that the next state of a system depends not only upon its current state but also upon all of its historical states. This is more realistic and it is one reason why fractional calculus has become more and more popular [1-11]. In general, it is difficult to obtain an exact solution for a fractional differential equation. So numerical methods attracted the interest of researchers, the perturbation method is one of these. But the perturbation methods have some limitations e.g., the approximate solution involves series of small parameters which poses difficulty since majority of nonlinear problems have no small parameters at all. Although appropriate choices of small parameters some time leads to ideal solution but in most of the cases unsuitable choices lead to serious effects in the solutions. The space- and time-fractional telegraph equations have been studied by many researchers notably Momani [12] and Yildirim [13] by using Adomian's decomposition method (ADM) and homotopy perturbation method (HPM) respectively. The HPM was first introduced by He [14]. Recently, the HPM has been studied by many authors to handle linear and nonlinear equations arising in physics and engineering [15-19]. The HPTM is a combination of Laplace transform method, homotopy perturbation method (HPM) and He's polynomials. In recent years, many researchers have obtained the solutions of partial differential equations by using various methods combined with the Laplace transform method. Among these are Laplace decomposition method (LDM) [20-22] and HPTM [23-25].

The objective of the present paper is to extend the application of the HPTM to obtain analytic and approximate solutions to the space-time fractional telegraph equations. The advantage of this technique is its capability of combining two powerful methods for obtaining exact and approximate analytical solutions for nonlinear equations. The fact that the HPTM solves nonlinear problems without using Adomian's polynomials is a clear advantage of this technique over the decomposition method.

Basic definitions of fractional calculus

In this section, we mention the following basic definitions of fractional calculus.

Definition 1. The Riemann-Liouville fractional integral operator of order $\alpha > 0,$ of a function $f(t) \in C_\mu, \mu \geq -1$ is defined as [5];

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad (\alpha > 0), \tag{1}$$

$$J^0 f(t) = f(t). \tag{2}$$

For the Riemann-Liouville fractional integral we have;

$$J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\alpha+\gamma}. \tag{3}$$

Definition 2. The fractional derivative of $f(t)$ in the Caputo sense is defined as [8];

$$D^\alpha f(t) = J^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \tag{4}$$

for $n-1 < \alpha \leq n, \; n \in N, \quad t > 0.$

From Definition 2, the following result can be easily obtained;

$$D^\alpha t^\beta = \begin{cases} \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, n-1 < \alpha \leq n, \beta > n-1, \beta \in R, \\ 0, \qquad n-1 < \alpha \leq n, \beta \leq n-1. \end{cases} \tag{5}$$

Definition 3. The Laplace transform of the Caputo derivative is given by Caputo [8]; see also Kilbas *et al*. [11] of the form;

$$L\,[D^{\alpha} f(t)] = s^{\alpha}\, L[\,f(t)] - \sum_{r=0}^{n-1} s^{\alpha-r-1}\, f^{(r)}(0+),\ (n-1<\alpha\le n). \tag{6}$$

Fractional model for telegraph equation

We consider an infinitesimal piece of telegraph cable wire as an electrical circuit presented in **Figure 1** and assume that the cable is imperfectly insulated so that there are both capacitance and current leakage to ground.

Let;

$x =$ distance from sending end of the cable,
$u(x,t) =$ voltage at any point and any time, on the cable,
$i(x,t) =$ current at any point and any time, on the cable,
$R =$ resistance of the cable,
$C =$ capacitance to the ground,
$L =$ inductance of the cable and
$G =$ conductance to the ground.

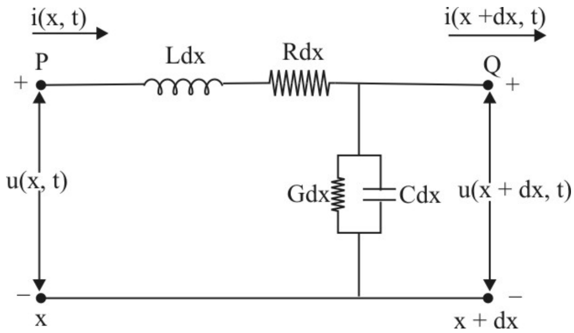


Figure 1 Schematic diagram of telegraphic transmission line with leakage.

According to Ohm’s law the voltage across the register is given as;

$$u = iR \tag{7}$$

and the voltage drop across the inductor is given by;

$$u = L\frac{di}{dt} \tag{8}$$

and the voltage drop across the capacitor is expressed as;

$$u = \frac{1}{C}\int i dt. \tag{9}$$

The voltage at terminal Q is equal to the voltage at terminal P, minus the drop in voltage along the element PQ, so if Eqs. (7), (8) and (9) are combined, it leads to the following result;

$$u(x+dx,t)-u(x,t)=-[Rdx]i-[Ldx]\frac{di}{dt}. \tag{10}$$

Taking $dx \to 0,$ and differentiating (10) with respect to $x,$ we arrive at the following result;

$$\frac{\partial u}{\partial x}=-Ri-L\frac{di}{dt}. \tag{11}$$

Likewise, the current at terminal Q is equal to the current at terminal P, minus the current through leakage to the ground, so we have;

$$i(x+dx,t)=i(x,t)-[Gdx]u-i_C\,dx. \tag{12}$$

The equation for current through the capacitor is given as;

$$i_C=C\frac{\partial u}{\partial t}. \tag{13}$$

Now, differentiating (10) with respect to t and (13) with respect to x and eliminating the derivatives of $u,$ we arrive at the following result [26];

$$c^2\frac{\partial^2 i}{\partial x^2}=\frac{\partial^2 i}{\partial t^2}+(\xi+\eta)\frac{\partial i}{\partial t}+\xi\eta i. \tag{14}$$

Similarly, we have;

$$c^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}+(\xi+\eta)\frac{\partial u}{\partial t}+\xi\eta u, \tag{15}$$

where $\xi=\frac{G}{C},\eta=\frac{R}{L}$ and $c^2=\frac{1}{LC}.$
Eqs. (14) and (15) are called one-dimensional hyperbolic second-order telegraph equations.
If the fractional derivative model is used to present the space and time derivatives, the Eqs. (14) and (15) assumes the forms;

$$c^2\frac{\partial^\alpha i}{\partial x^\alpha}=\frac{\partial^\beta i}{\partial t^\beta}+(\xi+\eta)\frac{\partial^\gamma i}{\partial t^\gamma}+\xi\eta i \tag{16}$$

and

$$c^2\frac{\partial^\alpha u}{\partial x^\alpha}=\frac{\partial^\beta u}{\partial t^\beta}+(\xi+\eta)\frac{\partial^\gamma u}{\partial t^\gamma}+\xi\eta u,\ \ 1<\alpha,\ \beta\le 2,\ \ 0<\gamma\le 1. \tag{17}$$

HPTM for space-time fractional telegraph equation

Consider the following general space-time fractional telegraph equation;

$$D_t^{\alpha} u(x,t) = a_1 D_t^{\beta} u(x,t) + a_2 D_t^{\gamma} u(x,t) + a_3 u(x,t) + f(x,t), \tag{18}$$

where $1 < \alpha, \beta \le 2, 0 < \gamma \le 1, x, t \ge 0, u(0,t) = h(t),\ u_x(0,t) = g(t)$, and a_1, a_2, a_3 are constants. Applying the Laplace transform (denoted in this paper by L) on both sides of Eq. (18), we get;

$$L[D_t^{\alpha} u(x,t)] = L[a_1 D_t^{\beta} u(x,t) + a_2 D_t^{\gamma} u(x,t) + a_3 u(x,t)] + L[f(x,t)]. \tag{19}$$

Using the property of the Laplace transform, we have;

$$L[u(x,t)] = \frac{h(t)}{s} + \frac{g(t)}{s^2} + \frac{1}{s^{\alpha}} L[f(x,t)] + \frac{1}{s^{\alpha}} L[a_1 D_t^{\beta} u(x,t) + a_2 D_t^{\gamma} u(x,t) + a_3 u(x,t)]. \tag{20}$$

Operating with the Laplace inverse on both sides of Eq. (20) gives;

$$u(x,t) = G(x,t) + L^{-1}\left[\frac{1}{s^{\alpha}} L[a_1 D_t^{\beta} u(x,t) + a_2 D_t^{\gamma} u(x,t) + a_3 u(x,t)]\right], \tag{21}$$

where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Now, we apply the HPM;

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \tag{22}$$

Substituting Eq. (22) in Eq. (21), we get;

$$\begin{aligned}\sum_{n=0}^{\infty} p^n u_n(x,t) = G(x,t) + p\Bigg(L^{-1}\Bigg[\frac{1}{s^{\alpha}} L\Bigg[a_1 D_t^{\beta}\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right) \\ + a_2 D_t^{\gamma}\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right) + a_3\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)\Bigg]\Bigg]\Bigg),\end{aligned} \tag{23}$$

which is the coupling of the Laplace transform method and the HPM. Comparing the coefficients of like powers of p, we have;

$$p^0 : u_0(x,t) = G(x,t),$$

$$p^1 : u_1(x,t) = L^{-1}\left[\frac{1}{s^{\alpha}} L\left[a_1 D_t^{\beta} u_0(x,t) + a_2 D_t^{\gamma} u_0(x,t) + a_3 u_0(x,t)\right]\right],$$

$$p^2 : u_2(x,t) = L^{-1}\left[\frac{1}{s^{\alpha}} L\left[a_1 D_t^{\beta} u_1(x,t) + a_2 D_t^{\gamma} u_1(x,t) + a_3 u_1(x,t)\right]\right], \tag{24}$$

$$p^3 : u_3(x,t) = L^{-1}\left[\frac{1}{s^{\alpha}} L\left[a_1 D_t^{\beta} u_2(x,t) + a_2 D_t^{\gamma} u_2(x,t) + a_3 u_2(x,t)\right]\right],$$

$$\vdots$$

Numerical examples and error estimation

In this section, we discuss the implementation of our numerical method and investigate its accuracy and stability by applying it to numerical examples on space-time fractional telegraph equations.

Example 1. We consider the following time-fractional telegraph equation;

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} + 2\frac{\partial^{\alpha} u}{\partial t^{\alpha}} + u, \qquad x,t \ge 0,\ 1 < \alpha \le 1, \tag{25}$$

subject to the initial conditions;

$$u(x,0) = e^x,\ u_t(x,0) = -2e^x. \tag{26}$$

Applying the Laplace transform on the both sides of Eq. (25), subject to the initial conditions (26), we have;

$$L[u(x,t)] = \frac{e^x}{s} - \frac{2e^x}{s^2} + \frac{1}{s^{2\alpha}} L\left[\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^{\alpha} u}{\partial t^{\alpha}} - u\right]. \tag{27}$$

The inverse Laplace transform implies that;

$$u(x,t) = e^x - 2te^x + L^{-1}\left[\frac{1}{s^{2\alpha}} L\left[\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^{\alpha} u}{\partial t^{\alpha}} - u\right]\right]. \tag{28}$$

Now applying the HPM, we get;

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = (1-2t)e^x + p\left(L^{-1}\left[\frac{1}{s^{2\alpha}} L\left[\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right) - 2\frac{\partial^{\alpha}}{\partial t^{\alpha}}\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right) - \left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)\right]\right]\right). \tag{29}$$

Comparing the coefficients of like powers of p, we have;

$$p^0 : u_0(x,t) = (1-2t)e^x,$$

$$p^1 : u_1(x,t) = L^{-1}\left[\frac{1}{s^{2\alpha}} L\left[\frac{\partial^2}{\partial x^2}(u_0) - 2\frac{\partial^\alpha}{\partial t^\alpha}(u_0) - u_0\right]\right] = 4e^x \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}, \tag{30}$$

$$p^2 : u_2(x,t) = L^{-1}\left[\frac{1}{s^{2\alpha}} L\left[\frac{\partial^2}{\partial x^2}(u_1) - 2\frac{\partial^\alpha}{\partial t^\alpha}(u_1) - u_1\right]\right] = -8e^x \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)},$$

$$\vdots$$

Therefore, the HPTM series solution is;

$$u(x,t) = e^x\left(1 - 2t + \frac{4t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{8t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \cdots\right). \tag{31}$$

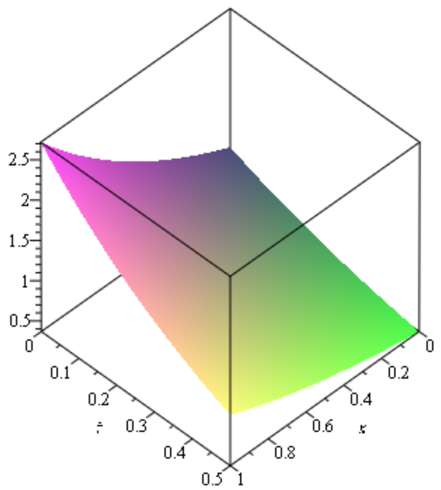
Setting α = 1 in (31), we reproduce the solution of the problem as follows;

$$u(x,t) = e^x\left(1 - 2t + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + \cdots\right). \tag{32}$$

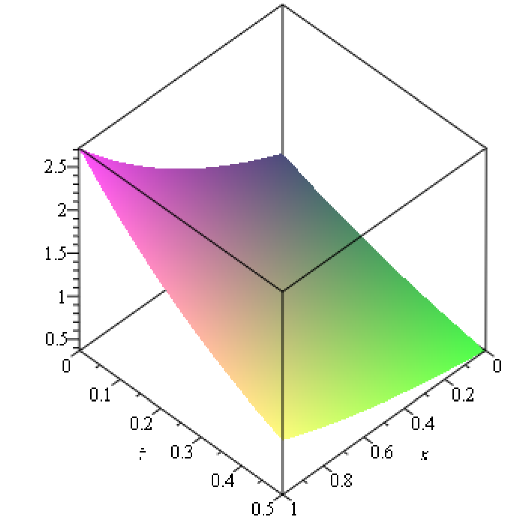
This solution is equivalent to the exact solution in a closed form;

$$u(x,t) = e^{x-2t}. \tag{33}$$

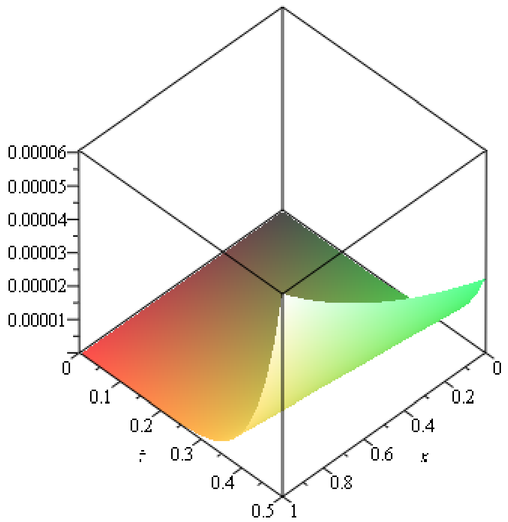
It is clear that no linearization or perturbation was used and a closed form solution is obtainable by adding more terms to the HPTM series. The numerical results for the exact solution (33) and the approximate solution (31) obtained by HPTM, for the special case α = 1, are shown in **Figure 2**. It can be seen from the **Figure 2** that the solution obtained by the present method is nearly identical with the exact solution. It is to be noted that only the seventh order term of the HPTM is used in evaluating the approximate solutions for **Figure 2**. It is evident that the efficiency of this approach can be dramatically enhanced by computing further terms of $u(x,t)$ when the HPTM is used.



(a)



(b)



(c)

Figure 2 The surface shows the solution $u(x,t)$ for Eqs. (25) - (26) when α = 1: (a) exact solution (b) approximate solution (c) $|u_{ex} - u_{app}|$.

Example 2. Next, we consider the following space-fractional telegraph equation;

$$\frac{\partial^\alpha u}{\partial x^\alpha} = \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u, \qquad x, t \geq 0, \;\; 1 < \alpha \leq 2, \tag{34}$$

subject to the initial conditions;

$$u(0,t) = e^{-t},\ u_x(0,t) = e^{-t}. \tag{35}$$

Applying the Laplace transform on the both sides of Eq. (34), we have;

$$L[u(x,t)] = \frac{e^{-t}}{s} + \frac{e^{-t}}{s^2} + \frac{1}{s^{\alpha}} L\left[\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u\right]. \tag{36}$$

The inverse Laplace transform implies that;

$$u(x,t) = e^{-t} + xe^{-t} + L^{-1}\left[\frac{1}{s^{\alpha}} L\left[\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u\right]\right]. \tag{37}$$

Now applying the HPM, we get;

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = e^{-t} + xe^{-t} + p\left(L^{-1}\left[\frac{1}{s^{\alpha}} L\left[\frac{\partial^2}{\partial t^2}\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)\right.\right.\right.$$
$$\left.\left.\left. + \frac{\partial}{\partial t}\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right) + \left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)\right]\right]\right). \tag{38}$$

Comparing the coefficients of like powers of p, we have;

$$\begin{aligned}
p^0 &: u_0(x,t) = e^{-t}(1+x),\\
p^1 &: u_1(x,t) = L^{-1}\left[\frac{1}{s^{\alpha}} L\left[\frac{\partial^2}{\partial t^2}(u_0) + \frac{\partial}{\partial t}(u_0) + u_0\right]\right]\\
&\qquad = e^{-t}\left[\frac{x^{\alpha}}{\Gamma(\alpha+1)} + \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}\right],\\
p^2 &: u_2(x,t) = L^{-1}\left[\frac{1}{s^{\alpha}} L\left[\frac{\partial^2}{\partial t^2}(u_1) + \frac{\partial}{\partial t}(u_1) + u_1\right]\right]\\
&\qquad = e^{-t}\left[\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right],\\
&\vdots
\end{aligned} \tag{39}$$

Therefore, the HPTM series solution is;

$$u(x,t) = e^{-t}\left(1 + x + \frac{x^{\alpha}}{\Gamma(\alpha+1)} + \frac{x^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)} + \cdots\right). \tag{40}$$

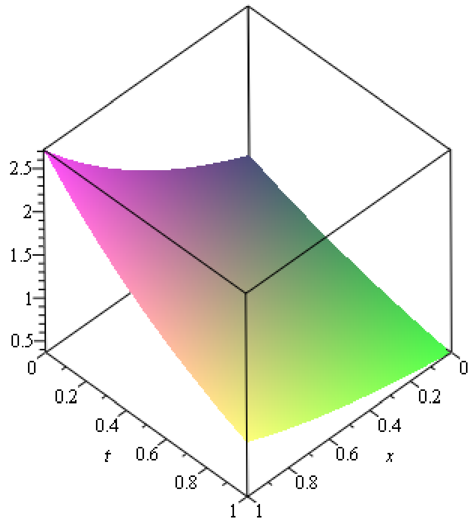
Setting α = 2 in (40), we reproduce the solution of the problem as follows;

$$u(x,t) = e^{-t}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right). \tag{41}$$

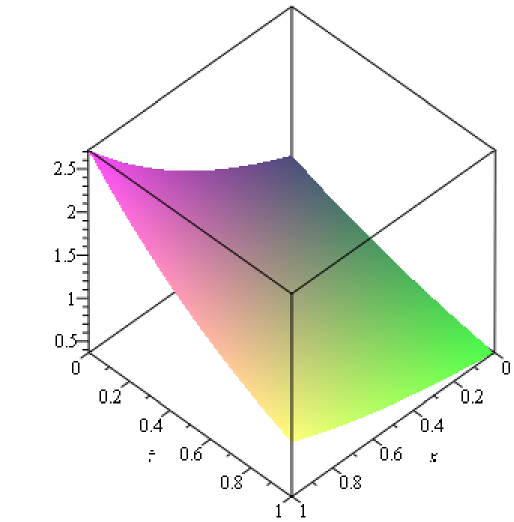
This solution is equivalent to the exact solution in a closed form;

$$u(x,t) = e^{x-t}. \tag{42}$$

The numerical results for the exact solution (42) and the approximate solution (40) obtained by HPTM, for the special case α = 2, are shown in **Figure 3**. It can be seen from the **Figure 3** that the solution obtained by the present method is nearly identical with the exact solution. The approximate solutions when α = 1.25 and α = 1.75 are shown by **Figures 4a** and **4b** respectively. It is to be noted that only the third order term of the HPTM is used in evaluating the approximate solutions for **Figures 3** and **4**.



(a)



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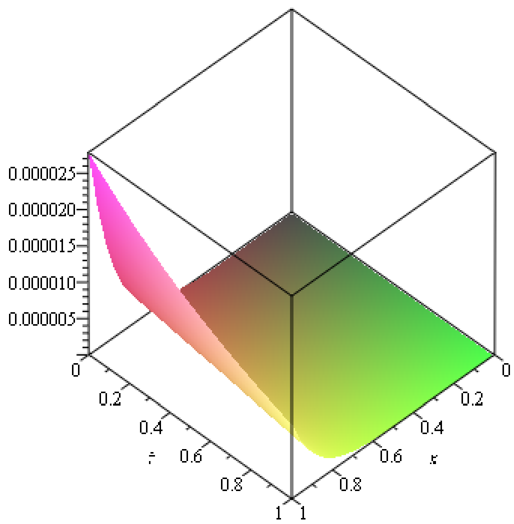
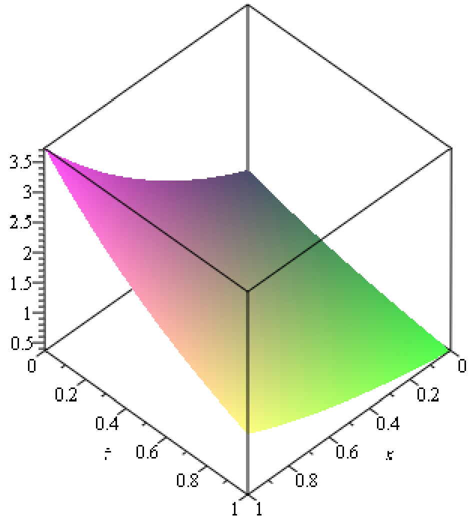
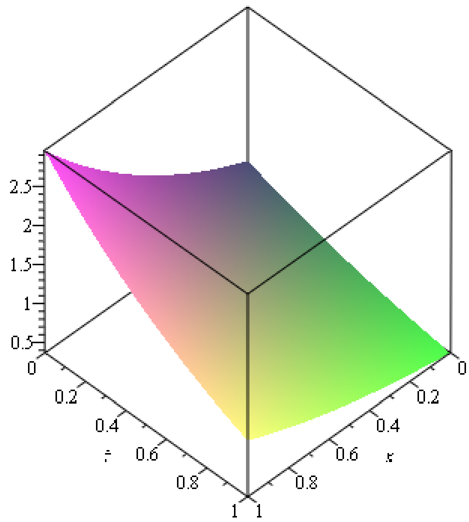


(c)

Figure 3 The surface shows the solution $u(x,t)$ for Eqs. (34) - (35) when α = 2: (a) exact solution (b) approximate solution (c) $|u_{ex} - u_{app}|$.



(a)



(b)

Figure 4 The surface shows the solution $u(x,t)$ for Eqs. (34) - (35): (a) α = 1.25, (b) α = 1.75.

Example 3. Finally, we consider the following nonhomogeneous space-time fractional telegraph equation;

$$\frac{\partial^{\alpha} u}{\partial x^{\alpha}} = \frac{\partial^2 u}{\partial t^2} + \frac{\partial^{\beta} u}{\partial t^{\beta}} + u - x^2 - t + 1, \qquad x, t \geq 0, \;\; 1 < \alpha \leq 2, 0 < \beta \leq 1, \tag{43}$$

subject to the initial conditions;

$$u(0,t)=t,\ u_x(0,t)=0. \tag{44}$$

Operating with the Laplace transform on the both sides of Eq. (43), we have;

$$L[u(x,t)]=\frac{t}{s}-\frac{2}{s^{\alpha+3}}-\frac{t}{s^{\alpha+1}}+\frac{1}{s^{\alpha+1}}+\frac{1}{s^{\alpha}}L\left[\frac{\partial^2 u}{\partial t^2}+\frac{\partial^{\beta}u}{\partial t^{\beta}}+u\right]. \tag{45}$$

The inverse Laplace transform implies that;

$$u(x,t)=t-\frac{2x^{\alpha+2}}{\Gamma(\alpha+3)}-\frac{tx^{\alpha}}{\Gamma(\alpha+1)}+\frac{x^{\alpha}}{\Gamma(\alpha+1)}+L^{-1}\left[\frac{1}{s^{\alpha}}L\left[\frac{\partial^2 u}{\partial t^2}+\frac{\partial^{\beta}u}{\partial t^{\beta}}+u\right]\right]. \tag{46}$$

Now applying the HPM, we get;

$$\sum_{n=0}^{\infty} p^n u_n(x,t)=t-\frac{2x^{\alpha+2}}{\Gamma(\alpha+3)}-\frac{tx^{\alpha}}{\Gamma(\alpha+1)}+\frac{x^{\alpha}}{\Gamma(\alpha+1)}+p\left(L^{-1}\left[\frac{1}{s^{\alpha}}L\left[\frac{\partial^2}{\partial t^2}\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)\right.\right.\right.$$
$$\left.\left.\left.+\frac{\partial^{\beta}}{\partial t^{\beta}}\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)+\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)\right]\right]\right). \tag{47}$$

Comparing the coefficients of like powers of p, we have;

$$p^0: u_0(x,t)=t-\frac{2x^{\alpha+2}}{\Gamma(\alpha+3)}-\frac{tx^{\alpha}}{\Gamma(\alpha+1)}+\frac{x^{\alpha}}{\Gamma(\alpha+1)},$$

$$p^1: u_1(x,t)=\frac{x^{\alpha}}{\Gamma(\alpha+1)}\frac{t^{1-\beta}}{\Gamma(2-\beta)}-\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}\frac{t^{1-\beta}}{\Gamma(2-\beta)}+\frac{tx^{\alpha}}{\Gamma(\alpha+1)}-\frac{2x^{2\alpha+2}}{\Gamma(2\alpha+3)}-\frac{tx^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}, \tag{48}$$

$\vdots$

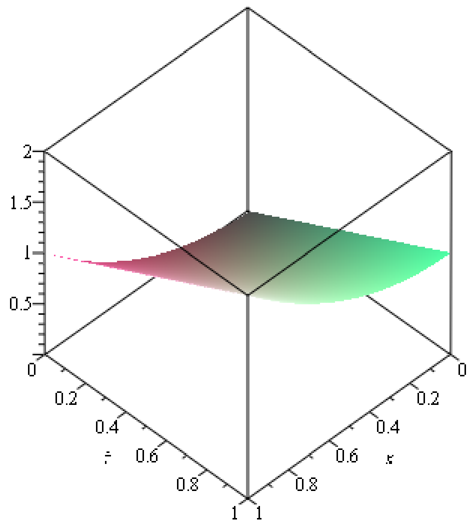
Therefore, the HPTM series solution is;

$$u(x,t)=\left(t-\frac{2x^{\alpha+2}}{\Gamma(\alpha+3)}-\frac{tx^{\alpha}}{\Gamma(\alpha+1)}+\frac{x^{\alpha}}{\Gamma(\alpha+1)}+\frac{x^{\alpha}}{\Gamma(\alpha+1)}\frac{t^{1-\beta}}{\Gamma(2-\beta)}\right.$$
$$\left.-\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}\frac{t^{1-\beta}}{\Gamma(2-\beta)}+\frac{tx^{\alpha}}{\Gamma(\alpha+1)}-\frac{2x^{2\alpha+2}}{\Gamma(2\alpha+3)}-\frac{tx^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots\right). \tag{49}$$

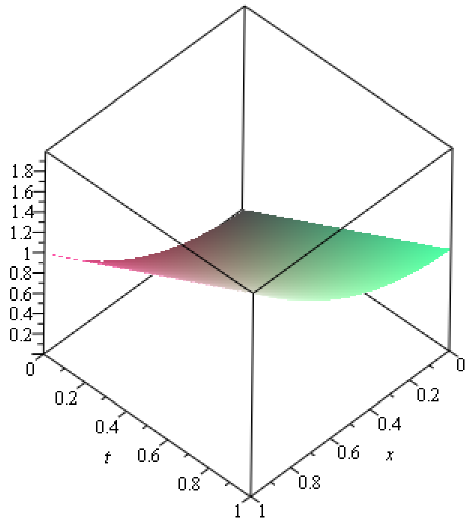
Setting α = 2 and β = 1 in (49), we reproduce the solution of the problem as follows;

$$u(x,t)=\left(t+\frac{2x^2}{2!}-\frac{2x^4}{4!}+\frac{2x^4}{4!}-\frac{2x^6}{6!}-\frac{x^6}{6!}-\frac{2x^8}{8!}+\cdots\right). \quad (50)$$

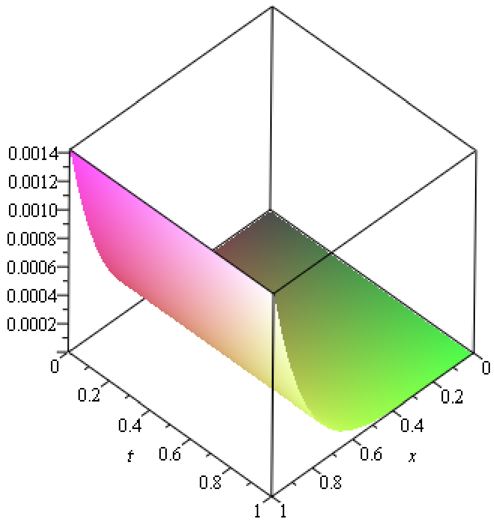
We observe that, setting α = 2 and β = 1 in the nth approximations and canceling noise terms yields the exact solution $u(x,t)=x^2+t$ as $n\to\infty$. The numerical results for the exact solution and the approximate solution (49) obtained by HPTM, for the special case α = 2 and $\beta=1$ are shown in **Figure 5**. It can be seen from **Figure 5** that the solution obtained by the present method is nearly identical with the exact solution. The approximate solutions when α = 1.25, α = 1.75 and β = 1 are shown by **Figures 6a** and **6b** respectively. It is noted that only the third order term of the HPTM is used in evaluating the approximate solutions for **Figures 5** and **6**.



(a)

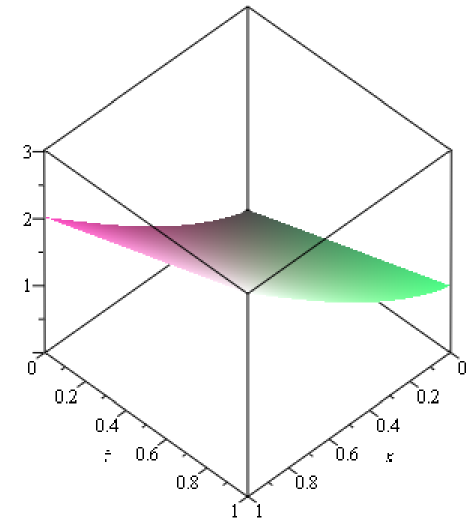


(b)

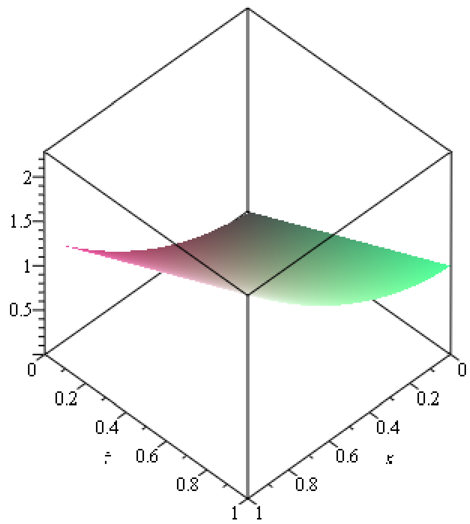


(c)

Figure 5 The surface shows the solution $u(x,t)$ for Eqs. (43) - (44) when α = 2 and β = 1: (a) exact solution (b) approximate solution (c) $|u_{ex} - u_{app}|$.



(a)



(b)

Figure 6 The surface shows the solution $u(x,t)$ for Eqs. (43) - (44), when $\beta = 1$: (a) α = 1.25, (b) α = 1.75.

Conclusions

In this paper, the HPTM has been successfully applied for solving space- time fractional telegraph equations. The method provides the solutions in terms of a convergent series with easily computable components in a direct way without using linearization, perturbation or restrictive assumptions. It is worth mentioning that the HPTM is capable of reducing the volume of the computational work as compared to classical methods while still maintaining the high accuracy of the numerical result; the size reduction amounts to an improvement of the performance of the approach. Hence, we conclude that the HPTM is very powerful and efficient in finding analytical as well as numerical solutions for wide classes of fractional partial differential equations.

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