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A Collocation Method for Numerical Solution of the Generalized Burgers-Huxley Equation

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Abstract

In this paper, we use a collocation method to solve the Burgers-Huxley equation. To achieve this aim, we use mesh free technique based on sinc functions. The stability analysis is discussed. Some numerical examples are provided to illustrate the accuracy and fluency of the method.

Keywords: Collocation method, Burgers-Huxley equation, Sinc function, Stability analysis

Introduction

Nonlinear phenomena play a crucial role in applied mathematics and physics. The Burgers-Huxley equations arise from mathematical modeling of many scientific phenomena. The generalized Burger's-Huxley equation shows a prototype model for describing the interaction between reaction mechanisms, convection effects and diffusion transports.

Let α be a non-negative real number, β, κ and δ be positive numbers with $\delta \geq 1$, and γ be a real number in $(0,1)$. Suppose that I is a (bounded or unbounded) interval in the set of real numbers. Also, u will be a function that depends on the spatial variable $x \in I$ and the temporal variable t > 0, which satisfies the advection-diffusion equation with nonlinear reaction term;

$$\frac{\partial u}{\partial t} + \alpha u^{\delta} \frac{\partial u}{\partial x} - \kappa \frac{\partial^2 u}{\partial x^2} = \beta u f(u), \tag{1}$$

for every $x \in I$ and every $t \geq 0$, where;

$$f(u) = (1-u^{\delta})(u^{\delta} - \gamma). \tag{2}$$

In the present work, this model is called the generalized Burgers-Huxley equation, and it is a quantitative paradigm which describes the interaction between reaction mechanisms, convection effects and diffusion transport. The constant κ is immediately identified as the coefficient of diffusivity, while α is the advection coefficient and β is the coefficient of reaction. When $\alpha = 0, \delta = 1, \kappa = 1$, Eq. (1) is reduced to the Huxley equation that describes nerve pulse propagation in nerve fibers and wall motion in liquid crystals [1,2]. When $\beta = 0, \delta = 1, \kappa = 1$, Eq. (1) is reduced to the Burger's equation which describes the far field of wave propagation in nonlinear dissipative systems [2]. When $\alpha = 0, \beta = 1, \delta = 1, \kappa = 1$, it is the Fitzhugh-Nagoma equations [3,4]. At $\delta = 1$ and $\alpha, \kappa, \beta \neq 0$, Eq. (1) is turned into the Burgers-Huxley equation.

The importance of the study of Burgers' equation is evident from the fact that it has several applications in engineering and environmental sciences. So far, there exists no general method for finding solutions of nonlinear diffusion equations. The Burgers-Huxley equation has been studied by a number of authors from various viewpoints [5-17]. For instance, Ismail, Raslan, and Rabboh [5] used the Adomian decomposition method, Javidi [6] used the spectral collocation method, and Deng [7] employed the first integral method to solve the generalized Burgers-Huxley equation.

The present paper is divided into several sections. The following section outlines some of the main properties of the sinc functions and the sinc method that are necessary for the formulation of the discrete Burgers-Huxley equation. The next section is concerned with the sinc-collocation discretization for the Burgers-Huxley equation and then the stability analysis of the method is discussed. Finally, numerical results are reported which demonstrate the efficiency and accuracy of the proposed numerical scheme.

Materials and methods

In this section, we present the notations and definitions of the sinc function which are discussed thoroughly in [8]. These properties will be used in the next Section to solve Burgers-Huxley Eq. (1).

The sinc function is defined on the whole real line, $-\infty < x < \infty$, by;

$$\mathrm{sinc}(x) = \begin{cases} \dfrac{\sin(\pi x)}{\pi x}, & x \neq 0; \\ 1, & x = 0. \end{cases} \tag{3}$$

For any $h > 0$, the translated sinc functions with evenly spaced nodes are given as;

$$S(j,h)(x) = \mathrm{sinc}\left(\frac{x - jh}{h}\right), \quad j = 0, \pm 1, \pm 2, \ldots \tag{4}$$

which are called the j th sinc functions. If f is defined on the real line, then for $h \geq 0$ the series;

$$C(f,h)(x) = \sum_{j=-\infty}^{\infty} f(jh)\, \mathrm{sinc}\left(\frac{z - jh}{h}\right)(x), \tag{5}$$

is called the Whittaker cardinal expansion of f, and whenever this series converges, f is approximated by using the finite number of terms in Eq. (5). For a confined class of functions known as the Paley-Weiner class, which are entire functions, the sinc interpolation and quadrature formulas are stated by Paley *et al*. [9]. A less confining class of functions, which is analyzed only on an infinite strip containing the real line and permit specific growth restrictions, has exponentially decaying absolute errors in the sinc approximation.

Definition1 Let D_S denote the infinite strip domain of width $2d, d > 0,$ given by;

$$D_s = \{z \in C : z = x + iy : |y| < d\}, \tag{6}$$

$B(D_S)$ is the class of functions f that are analytic in D_S such that;

$$\int_{-d}^{d} |f(t + iy)|\, dy \to 0, \quad t \to \pm\infty, \tag{7}$$

and satisfy;

$$\mathrm{N}\ (F, D_s) = \int_{\partial D_s} |f\ (z\ )dz\ | < \infty, \tag{8}$$

where ∂D_s represents the boundary of D_S .

According to [8], the derivatives of sinc functions evaluated at the nodes as follows;

$$\delta_{jk}^{(0)} = [S(j,h)o\phi(x)]|_{x=x_k} = \begin{cases} 1, & k = j; \\ 0, & k \neq j, \end{cases} \tag{9}$$

$$\delta_{jk}^{(1)} = h\frac{d}{d\phi}[S(j,h)o\phi(x)]|_{x=x_k} = \begin{cases} 0, & k = j; \\ \dfrac{(-1)^{(k-j)}}{(k-j)}, & k \neq j, \end{cases} \tag{10}$$

and

$$\delta_{jk}^{(2)} = h^2\ \frac{d^2}{d\phi^2}[\ S(\ j,h\ )o\phi(\ x\ )]\ |_{x=x_k} = \begin{cases} \dfrac{-\pi^2}{3}, & k = j; \\ \dfrac{-2(-1)^{(k-j)}}{(k-j)^2}, & k \neq j. \end{cases} \tag{11}$$

where the symbol "*o*" denotes function composition. Higher-order coefficients can be computed by formulae which exploit recurrence rules. The following result is proposed.

$$\delta_{ji}^{(2r)} = \frac{d^{2r}}{dx^{2r}}[\ S(\ j,h\ )(\ x\ )]\Big|_{x=x_i} = \begin{cases} \dfrac{\pi}{h}^{2r}\ \dfrac{(-1)^r}{2r+1}, & j = i; \\ \dfrac{(-1)^{(j-i)}}{h^{2r}(\ j-i\ )^{2r}} \displaystyle\sum_{l=0}^{r-1}(-1)^{l+1}\ \frac{2r\,!}{(2l+1)!}\pi^{2l}(\ i-j\ )^{2l}, & j \neq i, \end{cases} \tag{12}$$

for even coefficients, where $r = 1, 2, \ldots,$ and

$$\delta_{ji}^{(2r+1)} = \frac{d^{2r}}{dx^{2r+1}}[\ S(\ j,h\ )(\ x\ )]\Big|_{x=x_i} =$$
$$\begin{cases} 0 & j = i; \\ \dfrac{(-1)^{(j-i)}}{h^{2r+1}(\ j-i\ )^{2r+1}} \displaystyle\sum_{l=0}^{r-1}(-1)^l\ \frac{(2r+1)!}{(2l+1)!}\pi^{2l}(\ i-j\ )^{2l}, & j \neq i, \end{cases} \tag{13}$$

for odd ones.

Construction of the method

Consider the generalized nonlinear Burgers- Huxley equation as;

$$\frac{\partial u}{\partial t}+\alpha u^{\delta}\frac{\partial u}{\partial x}-\kappa\frac{\partial^2 u}{\partial x^2}=\beta u(1-u^{\delta})(u^{\delta}-\gamma),\ x\in I,\quad t\geq 0, \tag{14}$$

with the boundary conditions;

$$u(a,t)=g_a(t),\qquad u(b,t)=g_b(t),\qquad t\geq 0, \tag{15}$$

and the initial condition;

$$u(x,0)=f(x),\quad x\in I, \tag{16}$$

where α is a non-negative real number. β,δ,κ are positive numbers with $\delta\geq 0,$ and γ is a real number in (0, 1).

Let $f(x), g_a(t)$ and $g_b(t)$ be known functions. The time derivative is discretized in the usual finite difference way and the Crank-Nicolson scheme is applied to Eq. (14), to get;

$$\begin{aligned}&[\frac{u^{n+1}-u^n}{\Delta t}]+\alpha[\frac{(u^{\delta}u_x)^{n+1}+(u^{\delta}u_x)^n}{2}]\\&-\kappa[\frac{(u_{xx})^{n+1}+(u_{xx})^n}{2}]-\beta[\frac{(u(1-u^{\delta})(u^{\delta}-\gamma))^{n+1}+(u(1-u^{\delta})(u^{\delta}-\gamma))^n}{2}]\\&=0,\end{aligned} \tag{17}$$

where $u^n=u(x,t^n), t^n=t^{n-1}+\Delta t$ and Δt is the time step. To linearize the nonlinear terms in Eq. (17) the following formula is used, which is acquired by applying the Taylor expansions;

$$(u^{\delta})^{n+1}\approx(u^{\delta})^n+\Delta t((u^{\delta})_t)^n\approx(u^{\delta})^n+\delta\Delta t(u^{\delta-1})^n(\frac{u^{n+1}-u^n}{\Delta t})+O(\Delta t^2), \tag{18}$$

and

$$\begin{aligned}&(u^{\delta}u_x)^{n+1}\approx(u^{\delta}u_x)^n+\Delta t((u_t^{\delta})^n u_x^n+(u^{\delta})^n u_{xt}^n)+O(\Delta t^2),\\&=(u^{\delta})^n u_x^{n+1}+\delta(u^{\delta-1})^n u_x^n u^{n+1}-\delta(u^{\delta})^n u_x^n+O(\Delta t^2).\end{aligned} \tag{19}$$

Rearranging the terms and simplifying;

$$\begin{aligned}&u^{n+1}+\frac{\Delta t}{2}(\alpha((u^n)^{\delta}u_x^{n+1}+\delta(u^n)^{\delta-1}u_x^n u^{n+1})-\kappa u_{xx}^{n+1}-\beta((\gamma+1)(\delta+1)(u^n)^{\delta}u^{n+1}\\&-(2\delta+1)(u^n)^{2\delta}u^{n+1}-\gamma u^{n+1})=u^n+\frac{\Delta t}{2}(\alpha(\delta-1)(u^n)^{\delta}u_x^n\end{aligned}$$

$$+\kappa u_{xx}^{n}+\beta((\gamma+1)[2(u^{n})^{\delta+1}-(\delta+1)(u^{n})^{\delta}u^{n}]-[2(u^{n})^{2\delta+1}-(2\delta+1)(u^{n})^{2\delta}u^{n}]-\gamma u^{n})). \quad (20)$$

For the rest of this work, let M and N be positive integers, let a and b be real numbers such that $a<b$, and let T be a positive real number. In order to approximate the solutions of the partial differential Eq. (14) in the special interval $I=[a,b]$ over the time period T, uniform partitions are fixed of $a=x_1<\ldots<x_i=a+(i-1)h<\ldots<x_N=b$ and $0=t_0<t_1<\ldots<t_M=T$ of $[a,b]$ and $[0,T]$ respectively, each of them having a norm equal to $h=(b-a)/(N-1)$ and $\Delta t=T/M$. The solution of Eq. (14) is interpolated and approximated by means of the sinc functions as follows;

$$u(x,t^{n})=u^{n}(x)\approx\sum_{j=1}^{N}u_j^{n}S_j(x), \quad (21)$$

where

$$S_j(x)=\operatorname{sinc}\Big(\frac{x-(j-1)h-a}{h}\Big), \quad (22)$$

The unknown parameters u_j in Eq. (21) are to be determined by the collocation method. Therefore for each collocation point x_i, Eq. (21) can be written as;

$$u^{n}(x_i)=\sum_{j=1}^{N}u_j^{n}S_j(x_i),\ i=1,\ldots N. \quad (23)$$

By substituting Eq. (23) into Eqs. (20) and (15) and using the collocation points $x_i, i=1,\ldots,N$, the following equations are obtained;

$$\sum_{j=1}^{N}u_j^{n+1}S_j(x_i)+\frac{\Delta t}{2}\Big\{\alpha\Big[\Big(\sum_{j=1}^{N}u_j^{n}S_j(x_i)\Big)^{\delta}\Big(\sum_{j=1}^{N}u_j^{n+1}S_j'(x_i)\Big)$$
$$+\delta\Big(\sum_{j=1}^{N}u_j^{n}S_j(x_i)\Big)^{\delta-1}\times\Big(\sum_{j=1}^{N}u_j^{n}S_j'(x_i)\Big)\Big(\sum_{j=1}^{N}u_j^{n+1}S_j(x_i)\Big)\Big]-\kappa\sum_{j=1}^{N}u_j^{n+1}S_j''(x_i)$$
$$-\beta\Big[(\gamma+1)(\delta+1)\times\Big(\sum_{j=1}^{N}u_j^{n}S_j(x_i)\Big)^{\delta}\Big(\sum_{j=1}^{N}u_j^{n+1}S_j(x_i)\Big)-(2\delta+1)\Big(\sum_{j=1}^{N}u_j^{n}S_j(x_i)\Big)^{2\delta}$$
$$\Big(\sum_{j=1}^{N}u_j^{n+1}S_j(x_i)\Big)-\gamma\sum_{j=1}^{N}u_j^{n+1}S_j(x_i)\Big]\Big\}$$
$$=\sum_{j=1}^{N}u_j^{n}S_j(x_i)+\frac{\Delta t}{2}\Big\{\alpha(\delta-1)\Big(\sum_{j=1}^{N}u_j^{n}S_j(x_i)\Big)^{\delta}\Big(\sum_{j=1}^{N}u_j^{n}S_j'(x_i)\Big)+\kappa\sum_{j=1}^{N}u_j^{n}S_j''(x_i)$$
$$+\beta\Big[2(\gamma+1)\Big(\sum_{j=1}^{N}u_j^{n}S_j(x_i)\Big)^{\delta+1}-(\gamma+1)(\delta+1)\Big(\sum_{j=1}^{N}u_j^{n}S_j(x_i)\Big)^{\delta}\Big(\sum_{j=1}^{N}u_j^{n}S_j(x_i)\Big)$$

$$-2\Big(\sum_{j=1}^{N}u_j^nS_j(x_i)\Big)^{2\delta+1}+(2\delta+1)\Big(\sum_{j=1}^{N}u_j^nS_j(x_i)\Big)^{2\delta}\Big(\sum_{j=1}^{N}u_j^nS_j(x_i)\Big)-\gamma\sum_{j=1}^{N}u_j^nS_j(x_i)\Big]\Big\},$$
$$i=2,\ldots,N-1, \tag{24}$$

and

$$\sum_{j=1}^{N}u_j^{n+1}S_j(x_1)=g_a(t^{n+1}),$$
$$\sum_{j=1}^{N}u_j^{n+1}S_j(x_N)=g_b(t^{n+1}). \tag{25}$$

To obtain a matrix representation of the above equations, the $N\times N$ matrices $I^{(i)}=[\delta_{jk}^{i}],\ 0\le i\le 2$ defined by Eqs. (9) - (11) are used. Note that the matrix $I^{(2)}$ is a symmetric matrix, the matrix $I^{(1)}$ is a skew symmetric matrix, and the matrix $I^{(0)}$ is an identity matrix, that is $\delta_{ij}^{(2)}=\delta_{ji}^{(2)},\ \delta_{ij}^{(1)}=-\delta_{ji}^{(1)},\ \delta_{ij}^{(0)}=\delta_{ji}^{(0)}$. Thus, the following system of N linear equations in N unknown parameters u_j^{n+1} is obtained, which can be expressed in a matrix form as follows;

$$Mu^{n+1}=P, \tag{26}$$

where

$$M=[A_d+A_b+0.5\Delta t\{\alpha(E+D)-\kappa C-\beta(F-G-\gamma A_d)\}],$$
$$P=[A_d+0.5\Delta t\{\alpha(\delta-1)E+\kappa C+\beta(2(\gamma+1)(u^n)^{\delta}-F-2(u^n)^{2\delta}$$
$$+G-\gamma A_d)\}]u^n+H^{n+1},$$
$$A_d=[I_{ij}^{0}:i=2,\ldots,N-1,j=1,\ldots,N\ \ and\ 0\ \ elsewhere]_{N\times N},$$
$$A_b=[I_{ij}^{0}:i=1,N,j=1,\ldots,N\ \ \text{and}\ \ 0\ \ \text{elsewhere}]_{N\times N},$$
$$B=[-I_{ij}^{1}:i=2,\ldots,N-1,j=1,\ldots,N\ \ \text{and}\ \ 0\ \ \text{elsewhere}]_{N\times N},$$
$$C=[I_{ij}^{2}:i=2,\ldots,N-1,j=1,\ldots,N\ \ \text{and}\ \ 0\ \ \text{elsewhere}]_{N\times N},$$
$$u_x^n=Bu^n,\quad D=\delta(u^n)^{\delta-1}*u_x^n*A_d,$$
$$E=(u^n)^{\delta}*B,\quad F=(\gamma+1)(\delta+1)(u^n)^{\delta}*A_d,$$
$$G=(2\delta+1)(u^n)^{2\delta}*A_d\quad,$$
$$H^{n+1}=[g_a(t^{n+1}),0,\ldots,0,g_b(t^{n+1})]^T,$$

where the symbol $*$ means the componentwise multiplication. This system can be solved by the Gaussian elimination method.

Stability analysis

Following Mokhtari and Mohammadi [10], this section presents the stability of the approximation Eq. (26) using the matrix method. The Eq. (14) can be linearized; assuming that the quantity u^{δ} in nonlinear terms is locally constant. The error e^n at the n th time level is given by;

$$e^n = u^n_{exact} - u^n_{app}, \tag{27}$$

where u^n_{exact} is the exact solution and u^n_{app} is the numerical solution at the n th time level. The error equation for the linearized Burgers-Huxley equation can be written as;

$$[A + 0.5\Delta t\ R]e^{n+1} = [A_d - 0.5\Delta t\ R]e^n, \tag{28}$$

where $R = [\alpha E + \kappa C - \beta\gamma A_d]$.
Therefore, the Eq. (26) can be written as;

$$e^{n+1} = Ge^n, \tag{29}$$

where $G = [A + 0.5\Delta t\ R]^{-1}[A_d - 0.5\Delta t\ R]$.

The numerical scheme is stable if $\|G\|_2 \le 1,$ which is equivalent to $\rho(G) \le 1,$ where $\rho(G)$ denotes the spectral radius of the matrix G. From Eq. (28), it can be seen that the stability is assured if all the eigenvalues of the matrix $[A + 0.5\Delta t\ R]^{-1}[A_d - \Delta t\ R]$ satisfy the following condition;

$$\left| \frac{\lambda_{A_d} - 0.5\Delta t\,\lambda_R}{\lambda_A + 0.5\Delta t\,\lambda_R} \right| \le 1. \tag{30}$$

where λ_{A_d}, λ_R and λ_A are eigenvalues of the matrices A_d, R and A respectively. In the case of complex eigenvalues $\lambda_A = a_A + ib_A$, $\lambda_{A_d} = a_{A_d} + ib_{A_d}$, $\lambda_R = a_R + ib_R$ where $a_A, a_{A_d}, a_R, b_A, b_{A_d}$ and b_R are any real numbers, the inequality (30) takes the following form;

$$\left| \frac{a_{A_d} - 0.5\Delta t a_R + i(b_{A_d} - 0.5\Delta t b_R)}{a_A - 0.5\Delta t a_R + i(b_A - 0.5\Delta t b_R)} \right| \le 1. \tag{31}$$

The inequality (31) is satisfied if;

$$\Delta t[a_R(a_A + a_{A_d}) + b_R(b_A + b_{A_d})] + (b_A^2 - b_{A_d}^2) \ge 0. \tag{32}$$

For real eigenvalues, the inequality (30) holds true if;

$$0 \le \lambda_{A_d} + \lambda_A \le 2\lambda_A + \Delta t\,\lambda_R . \tag{33}$$

This shows that the scheme (26) is unconditionally stable if;

$$\Delta t\,[a_R\,(a_A + a_{A_d}) + b_R\,(b_A + b_{A_d})] + (b_A^2 - b_{A_d}^2) \ge 0, \tag{34}$$

for complex eigenvalues and if;

$$0 \le \lambda_{A_d} + \lambda_A \le 2\lambda_A + \Delta t\,\lambda_R , \tag{35}$$

for real eigenvalues.

Results and discussion

In order to show the numerical results of solving the Burger-Huxley equation based on the sinc-collocation method (SCM) and discuss the accuracy of the method, the absolute errors of this method are tabulated, compared in tables, and considered. The error function is given by $Error = | u(x_j) - u^n(x_j)|$, $j = -N,\ldots,N$, where u and u^n represent the exact and approximate solutions, respectively.
For $\kappa = 1$, the exact solution of Eq. (14) can be found in [5-7,11];

$$u(x,t) = [\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\{\sigma\gamma(x - \{\frac{\gamma\alpha}{1+\delta} - \frac{(1+\delta-\gamma)(\rho-\alpha)}{2(1+\delta)}\}t)\}]^{1/\delta}, \tag{36}$$

where $\sigma = \delta(\rho-\alpha)/4(1+\delta)$ and $\rho = \sqrt{\alpha^2 + 4\beta(1+\delta)}$. The examples are solved for $I = (0,1)$ and different values of $\gamma, \delta, \beta, \Delta t, x$ and κ. We select the following examples from [6,11-12].

Example 1 The present method is applied to Eq. (14) for $\kappa = 1, \alpha = 1, \beta = 1,\ \gamma = 0.001, t = 0.01$ and the absolute errors are given in **Table 1** for different values of $m = N - 1$. When the exact results are compared with the current ones, the results are very accurate as indicated in the table.

Table 1 Numerical solution of Burger's problem for different values of N.

x	**Exact**	**SCM:** $m = 5$	**SCM:** $m = 10$	**SCM:** $m = 20$
0.0	0.0005000	0.0005000	0.0005000	0.0005000
0.2	0.0005000	0.0005263	0.0005085	0.0005014
0.4	0.0005001	0.0004978	0.0004980	0.0004997
0.6	0.0005001	0.0004978	0.0004980	0.0004998
0.8	0.0005001	0.0005264	0.0005086	0.0005015
1.0	0.0005001	0.0005001	0.0005001	0.0005001

Example 2 In this example, solutions of Eq. (14) at different values of t for $\kappa = 1, \alpha = 1, \delta = 1, \beta = 1, \gamma = 10^{-3}$ and $h = 0.1$ have been obtained and results are shown in **Table 2**. As can be seen from this table, the error decreases when the Δt is decreased.

Table 2 Maximum absolute errors when $\kappa = 1, \alpha = 1, \delta = 1, \beta = 1, \gamma = 10^{-3}$ and $h = 0.1$.

t	$\Delta t = 0.01$	$\Delta t = 0.001$	$\Delta t = 0.0001$
0.2	8.68557E-5	8.68497E-5	8.68496E-5
0.3	8.90784E-5	8.90750E-5	8.90749E-5
0.6	9.02495E-5	9.02492E-5	9.02492E-5
0.9	9.03011E-5	9.0308E-5	9.03080E-5
1	9.03119E-5	9.03119E-5	9.03119E-5
2	9.03353E-5	9.03353E-5	9.03353E-5
3	9.03578E-5	9.03578E-5	9.03578E-5

Example 3 Eq. (14) with $\delta = 1$ is reduced to Burger's-Huxley equation. **Table 3** shows absolute errors for $N = 26, \Delta t = 0.0001$ and different values of δ and x with $\kappa = 1, \alpha = 0.1, \beta = 0.001$ and $\gamma = 0.0001$.

Table 3 The absolute errors for different values of δ and x with $\alpha = 0.1$, $\beta = 0.001$ and $\gamma = 0.0001$.

x	$\delta = 1$	$\delta = 2$	$\delta = 4$
0.12	3.47078E-6	4.90843E-4	5.83414E-3
0.24	6.7432E-7	9.53632E-5	1.13407E-3
0.36	1.17147E-7	1.65671E-5	1.97018E-4
0.48	4.89704E-9	6.92546E-7	8.23580E-6
0.60	2.33626E-8	3.30397E-6	3.92910E-5
0.72	4.47594E-7	6.32993E-5	7.52760E-4
0.84	1.96073E-7	2.77289E-4	3.29754E-3

Example 4 Table 4, presents the numerical solution of Burger's-Huxley equation (BHE) at $t = 0.01$. This table shows absolute errors for various values of γ and x at $\kappa = 1, \delta = 1, \Delta t = 0.0001$, $N = 26$ and $\alpha = 5$. Very accurate results can be seen in the table when the exact and the current results are compared.

Table 4 Errors for the cases $\alpha, \beta, \delta = 1$ and various values of γ and x.

x	$\gamma = 10^{-2}$	$\gamma = 10^{-3}$	$\gamma = 10^{-4}$	$\gamma = 10^{-5}$
0.04	7.43865E-4	7.42953E-5	7.42862E-6	7.42853E-7
0.12	3.48403E-4	3.47207E-5	3.47087E-6	3.47075E-7
0.24	6.83877E-5	6.75229E-6	6.74366E-7	6.74280E-8
0.36	1.25194E-5	1.17905E-6	1.17178E-7	1.17105E-8
0.48	1.19727E-6	5.55794E-8	4.91813E-9	4.85416E-10
0.60	3.07135E-6	2.40508E-7	2.33864E-8	2.33200E-9
0.72	4.53298E-5	4.48118E-6	4.47602E-7	4.47550E-8
0.84	1.96411E-4	1.96103E-5	1.96072E-6	1.96069E-7
0.96	7.4306E-4	7.42873E-5	7.42854E-6	7.42852E-7

Example 5 Table 5 shows absolute errors for various values of β, x with $\kappa = 1$, $\alpha = 1, \Delta t = 0.0001$ and $\gamma = 10^{-3}$.

Table 5 $N = 26, \alpha = 1, \gamma = 10^{-3}$ and $t = 1$.

x	$\beta = 1$	$\beta = 10$	$\beta = 50$	$\beta = 100$
0.04	7.42886E-5	7.42915E-5	7.43070E-5	7.43272E-5
0.12	3.47138E-5	3.47179E-5	3.47317E-5	3.47473E-5
0.24	6.74932E-6	6.75343E-6	6.76260E-6	6.77099E-6
0.36	1.17734E-6	1.18108E-6	1.18843E-6	1.19451E-6
0.48	5.47310E-8	5.86979E-8	6.63072E-8	7.24926E-8
0.60	2.39412E-7	2.43305E-7	2.50887E-7	2.57184E-7
0.72	4.48216E-6	4.49008E-6	4.50539E-6	4.51826E-6
0.84	1.96166E-5	1.96365E-5	1.96772E-5	1.97116E-5
0.96	7.43053E-5	7.43674E-5	7.44920E-5	7.45945E-5

Example 6 Consider Eq. (14) with initial and boundary conditions defined as;

$$u(x,0) = \sin \pi x, \qquad 0 \le x \le 1,$$
$$u(0,t) = 0, \qquad 0 \le t \le 1,$$
$$u(1,t) = 0, \qquad 0 \le t \le 1,$$

The exact solution is not known in this case. In **Figures 1** and **2**, the sinc-collocation method has been plotted over different grids of nodal size $10, 20$ and 30 for the parametric values $\alpha = 1, \beta = 1, \gamma = 0.001, \Delta t = 0.001$, $\delta = 1$ for the singular perturbation parameter $\kappa = 2^{-7}$ at time level $t = 0.1$. The solution plots turn out to be the same over all the different grids of size greater than 20 mesh points. All of the further computations have been carried out on the grid with 25 mesh points.

For different values of the parameter δ and κ the sinc-collocation is presented. In **Figure 3** and **Figure 4**, the sinc-collocation is plotted for $\alpha = 1, \beta = 1, \gamma = 0.001, t = 0.1, \delta = 1, 2$ and different values of the singular perturbation parameter κ. In each plot, for different values of κ, one can notice the development of the boundary layers. As the singular perturbation parameter $\kappa \to 0$, the boundary layer becomes sharper.

In **Figures 5** and **6**, the sinc-collocation is plotted for $\alpha = 1$, $\beta = 1$, $\gamma = 0.001$, $\Delta t = 0.001$, $\kappa = 2^{-7}$, $\delta = 1, 2$ at different time levels.

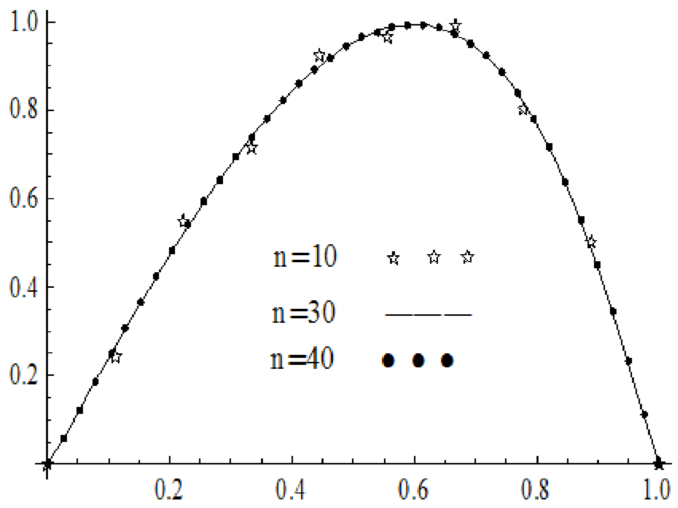

Figure 1 $\delta = 1, \alpha = 1, \beta = 1, \gamma = 0.001, t = 0.1, \kappa = 2^{-7}$ and different n .

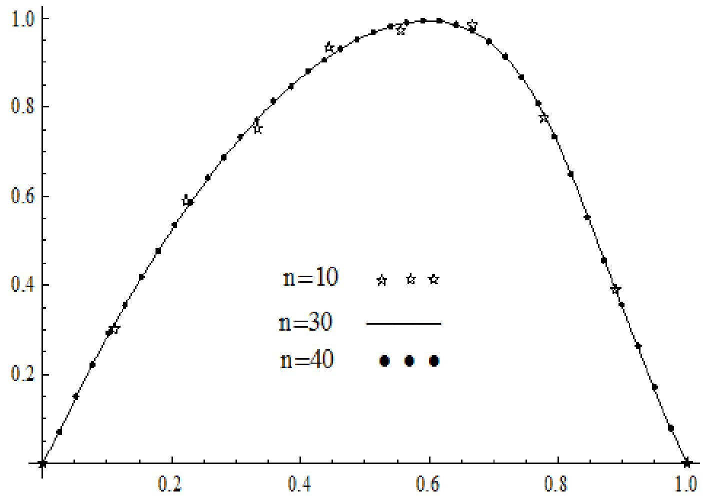

Figure 2 $\delta = 2, \alpha = 1, \beta = 1, \gamma = 0.001, t = 0.1, \kappa = 2^{-7}$ and different n .

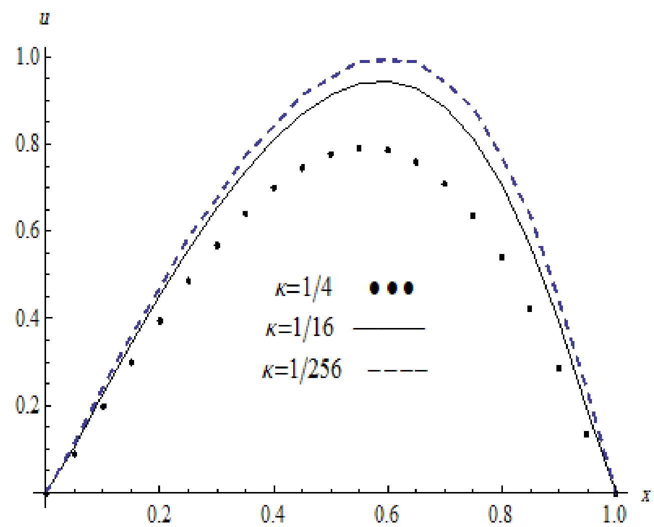

Figure 3 $\delta = 1, \alpha = 1, \beta = 1, \gamma = 0.001, t = 0.1$ and different k .

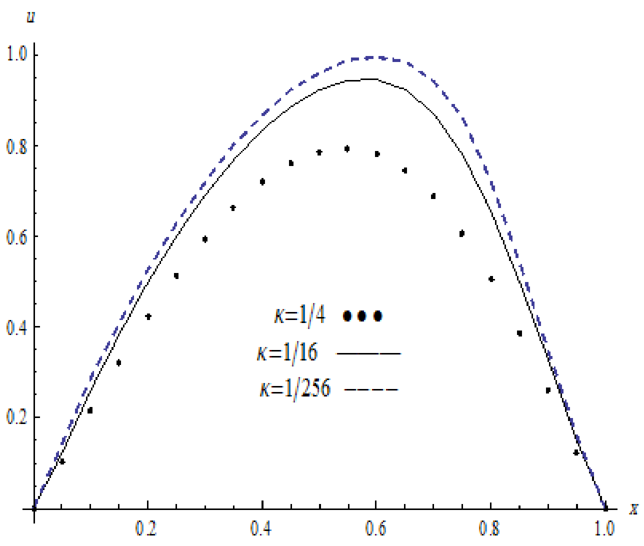

Figure 4 $\delta = 2, \alpha = 1, \beta = 1, \gamma = 0.001, t = 0.1$ and different k .

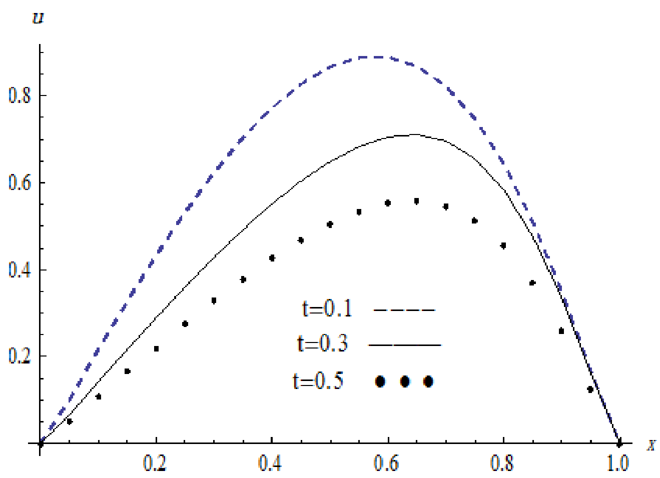

Figure 5 $\delta = 1, \alpha = 1, \beta = 1, \gamma = 0.001, \Delta t = 0.001, \kappa = 2^{-7}$ and different t .

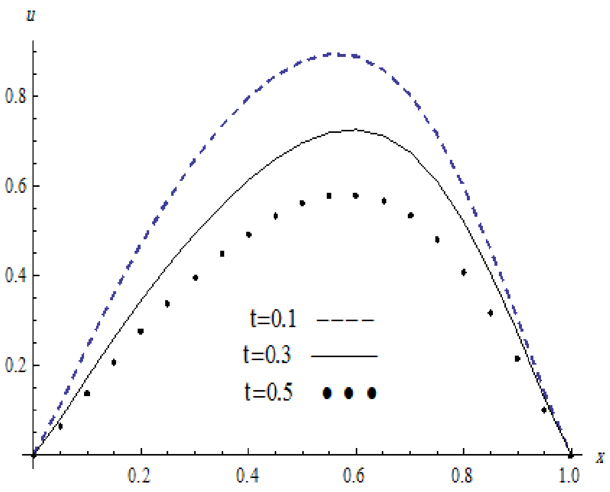

Figure 6 $\delta = 2, \alpha = 1, \beta = 1, \gamma = 0.001, \Delta t = 0.001, \kappa = 2^{-7}$ and different t .

Conclusions

The present study set out to solve the nonlinear Burgers-Huxley equation. To achieve this aim, the sinc-collocation method was applied. The numerical examples were presented and the results obtained were compared with the exact solution. The findings indicate that that the sinc-collocation method can be considered as a beneficial numerical method for solving generalized Burgers-Huxley equation.

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