Modification of the Homotopy Perturbation Method and It's Convergence

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Abstract

There are some methods to obtain approximate solutions of functional equations. One of them is the homotopy perturbation method. However, computing coefficients of p^j, is time-consuming and challenging. In order to deal with this problem, a new modified homotopy perturbation method is introduced, which doesn't need computations of the coefficients of p^j. Here, the method has been applied to solve some examples and the results have been compared with those obtained from the homotopy perturbation method. Moreover, convergence of the method has been discussed.

Keywords: Homotopy perturbation method, functional equation, new Iterative method, convergence.

Introduction

Mathematical modeling of many systems in various fields of physics and engineering leads to functional equations. However, it is difficult to obtain analytic solutions, especially for nonlinear equations. In most cases, only approximate solutions (either analytical or numerical) can be expected. There are some methods to obtain approximate solutions of functional equations. One of them is homotopy perturbation method (HPM). The method introduced by He in 1998 ,well addressed in [1], has been known as a powerful device for solving different kinds of equations, this is because of further developments and improvement applied by himself and other researchers. This method has been used to solve many equations such as non-linear wave equations, non-linear Schrodinger equations, fractional KdV-Burgers equation, and many other equations [2-12]. In this method the solution is considered as the summation of an infinite series, which usually converges rapidly to the solution. Recently a new technique has been proposed for solving linear or nonlinear functional equations [13,14]. In this paper, this method is used to present a new modification of HPM. The method has been applied to solve some examples and the results are compared with those obtained from the HPM. Moreover the convergence of the method has also been addressed.

New modified HPM

To illustrate the basic concept of modified HPM, consider the following nonlinear differential equation;

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{1}$$

with boundary conditions;

$$B(u, \partial u / \partial n) = 0, \quad r \in \Gamma. \tag{2}$$

Here A is a general differential operator, B is a linear boundary operator, $f(r)$ is a known analytic function, and Γ is the boundary of the domain Ω. Generally speaking the operator A can be divided into two parts L and N, where L is a linear, while N is a nonlinear operator. Eq. (1), therefore, can be rewritten as follows.

$$L(u) + N(u) - f(r) = 0. \tag{3}$$

According to HPM, a homotopy $v(r, p)$ should be considered, which satisfies;

$$H(v, p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \tag{4}$$

or

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0. \tag{5}$$

Let's rewrite Eq. (5) in the following form;

$$L(v) - L(u_0) = p[f(r) - L(u_0) - N(v)]. \tag{6}$$

Here $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation for the solution of Eq. (1), which satisfies the boundary conditions. According to HPM, we can first use the embedding parameter p as a small parameter, and assume that the solution of Eq. (5) can be written as a power series in p, say;

$$v = \sum_{i=0}^{\infty} v_i p^i. \tag{7}$$

By substituting (7), into (6), the following equation will be derived.

$$L(\sum_{i=0}^{\infty} v_i p^i) - L(u_0) = p[f(r) - L(u_0) - N(\sum_{i=0}^{\infty} v_i p^i)]. \tag{8}$$

If $p = 1$, the nonlinear operator N can be decomposed as;

$$N(\sum_{i=0}^{\infty} v_i) = N(v_0) + \sum_{i=1}^{\infty}\left[N\left(\sum_{j=0}^{i} v_j\right) - N\left(\sum_{j=0}^{i-1} v_j\right)\right]. \tag{9}$$

So

$$\sum_{i=0}^{\infty} L(v_i) - L(u_0) = f(r) - L(u_0) - N(v_0) - \sum_{i=1}^{\infty}\left[N\left(\sum_{j=0}^{i} v_j\right) - N\left(\sum_{j=0}^{i-1} v_j\right)\right]. \tag{10}$$

By following the iterative scheme, v_i's will be obtained.

$$L(v_0) = L(u_0),$$
$$L(v_1) = f(r) - L(u_0) - N(v_0), \tag{11}$$
$$L(v_{i+1}) = N\left(\sum_{j=0}^{i} v_j\right) - N\left(\sum_{j=0}^{i-1} v_j\right).$$

So, $u = \sum_{i=0}^{n-1} v_i$ as an n-terms approximation can be used to approximate the solution.

Numerical Examples

To explain the proposed technique, an integro-differential equation and a non-linear system of partial differential equation are solved as two illustrative examples.

Example 1: Let us consider the following non-linear integral-differential equation;

$$u'(x) = -2\sin x - \frac{1}{3}\cos x - \frac{2}{3}\cos 2x + \int_0^x \cos(x-t)u^2(t)dt, \qquad u(0) = 1. \tag{12}$$

With the exact solution $u(x) = \cos x - \sin x$.

HPM consists of the following scheme;

$$v'(x) - u_0{}'(x) = p[-2\sin x - \frac{1}{3}\cos x - \frac{2}{3}\cos 2x + \int_0^x \cos(x-t)u^2(t)dt - u_0{}'(x)\,]. \tag{13}$$

Suppose the solution of Eq. (12), has the following form;

$$u(x) = \sum_{i=0}^{\infty} u_i(x)p^i. \tag{14}$$

Substitution of this solution into the last equation, and equating terms with identical powers of *p*, leads to;

$$p^0: \; v_0' = u_0{}', \; v_0(0) = 1,$$
$$p^1: \; v_1' = -2\sin x - \frac{1}{3}\cos x - \frac{2}{3}\cos 2x + \int_0^x \cos(x-t)v_0{}^2(t)dt - u_0{}'(x),$$
$$p^{n+1}: v_{n+1}' = -\int_0^x \left(\cos(x-t)\sum_{i=0}^{n} v_i(t)v_{n-i}(t)\right)dx, \;\; n = 1, 2, \ldots. \tag{15}$$

Let's take $v_0 = u_0 = 1$, so;

$$v_1 := -1 + \cos(x) - \frac{1}{3}\sin(x) - \frac{1}{3}\sin(2x)$$
$$v_2 := -2 - \frac{7}{9}\sin(x) + 2\cos(x) + \frac{4}{9}\cos(x)\sin(x) + \frac{1}{3}\cos(x)\,x + \sin(x)\,x$$
$$\begin{aligned} v_3 := &-\frac{302}{135} + \frac{223}{90}\cos(x) + \frac{14}{9}\sin(x)\,x + \frac{4}{135}\cos(x)^4 \\ &- \frac{44}{135}\cos(x)^2 - \frac{23}{54}\sin(x) + \frac{1}{6}\cos(x)^2\sin(x) \\ &+ \frac{1}{18}\cos(x)^3 - \frac{14}{27}\cos(x)\sin(x) + \frac{1}{6}\sin(x)\,x^2 \\ &+ \frac{7}{9}\cos(x)\,x - \frac{1}{2}\cos(x)\,x^2 \end{aligned} \tag{16}$$

Applying (11) to this example, leads to;

$$\begin{aligned} &v_0 = 1, \\ &v_1' = -2\sin x - \frac{1}{3}\cos x - \frac{2}{3}\cos 2x + \int_0^x \cos(x-t)v_0^{\,2}(t)dt - u_0{}'(x), \\ &v_{n+1}' = -\int_0^x \cos(x-t)\left((\sum_{j=0}^{n} v_j)^2 - (\sum_{j=0}^{n-1} v_j)^2 \right)dx,\ \ n = 1, 2, \ldots. \end{aligned} \tag{17}$$

So

$$v_1 := -1 + \cos(x) - \frac{1}{3}\sin(x) - \frac{1}{3}\sin(2x)$$
$$\begin{aligned} v_2 := &-\frac{5}{9}\sin(x) + \frac{1}{18}\sin(x)\,x + \frac{2}{9}\cos(x)\sin(x) \\ &+ \frac{1}{6}\cos(x)^2\sin(x) - \frac{32}{135} + \frac{4}{135}\cos(x)^4 + \frac{1}{18}\cos(x)^3 \\ &- \frac{44}{135}\cos(x)^2 + \frac{43}{90}\cos(x) + \frac{1}{6}\cos(x)\,x \end{aligned}$$

$$
\begin{aligned}
v_3 := &\frac{128}{405}\,x - \frac{61417}{127575}\sin(x) - \frac{40759337}{36741600}\cos(x) \\
&- \frac{80971}{116640}\sin(x)\,x - \frac{12019}{38880}\cos(x)\,x - \frac{819761}{4592700}\cos(x)^2 \\
&- \frac{7427}{874800}\cos(x)^5 + \frac{33238}{42525}\cos(x)\sin(x) \\
&- \frac{1}{162}x^2\cos(x)\sin(x) - \frac{1}{24}\sin(x)\,x^2 + \frac{35}{1944}\cos(x)\,x^2 \\
&- \frac{2207}{38880}\cos(x)^2\sin(x) - \frac{11551}{255150}\cos(x)^3\sin(x) \\
&+ \frac{317507}{1749600}\cos(x)^3 + \frac{268349}{229635} - \frac{252059}{4592700}\cos(x)^4 \\
&+ \frac{19}{85050}\cos(x)^5\sin(x) - \frac{1}{4860}\cos(x)^6\sin(x) \\
&+ \frac{169}{19440}\cos(x)^4\sin(x) + \frac{421}{7290}\,x\cos(x)\sin(x) \\
&+ \frac{37}{1620}\,x\cos(x)^2\sin(x) - \frac{89}{405}\,x\cos(x)^2 + \frac{31}{4860}\,x\cos(x)^3 \\
&+ \frac{1418}{1148175}\cos(x)^6 - \frac{1}{14580}\cos(x)^7 - \frac{16}{1148175}\cos(x)^8 \\
&- \frac{1}{2430}\cos(x)^5\,x - \frac{2}{243}\cos(x)^2x^2 + \frac{19}{972}x^2 \\
&- \frac{1}{7290}\sin(x)\cos(x)^4\,x - \frac{1}{243}\sin(x)\,x\cos(x)^3
\end{aligned}
\tag{18}
$$

Taking a 5 terms approximation, $u(x)=\sum_{i=0}^{4} v_i(x).$ comparison of HPM, modified HPM and exact solutions are shown in **Figure 1**.

Example 2: Consider the following system of non-linear Partial Differential Equations.

$$
\begin{cases}
u_t - u_{xx} - 2uu_x + (uv)_x = 0, \\
v_t - v_{xx} - 2vv_x + (uv)_x = 0.
\end{cases}
\tag{19}
$$

with the initial condition;

$$
u(x,0) = \sin x,\; v(x,0) = \sin x.
\tag{20}
$$

In order to apply the modified HPM, by using (11), the following iterative scheme will be obtained.

$$
\begin{aligned}
&u_0 = \sin x, \quad v_0 = \sin x, \\
&u_{1t} = u_{0xx} + 2u_0u_{0x} - (u_0v_0)_x, \\
&v_{1t} = v_{0xx} + 2v_0v_{0x} - (u_0v_0)_x,
\end{aligned}
\tag{21}
$$

$$(u_{i+1})_t = (u_i)_{xx} + 2\left\{(\sum_{j=0}^{i} u_j)(\sum_{j=0}^{i} u_j)_x\right\} - \left\{(\sum_{j=0}^{i} u_j)(\sum_{j=0}^{i} v_j)\right\}_x - 2\left\{(\sum_{j=0}^{i-1} u_j)(\sum_{j=0}^{i-1} u_j)_x\right\} + \left\{(\sum_{j=0}^{i-1} u_j)(\sum_{j=0}^{i-1} v_j)\right\}_x, \tag{22}$$

$$(v_{i+1})_t = (v_i)_{xx} + 2\left\{(\sum_{j=0}^{i} v_j)(\sum_{j=0}^{i} v_j)_x\right\} - \left\{(\sum_{j=0}^{i} u_j)(\sum_{j=0}^{i} v_j)\right\}_x - 2\left\{(\sum_{j=0}^{i-1} v_j)(\sum_{j=0}^{i-1} v_j)_x\right\} + \left\{(\sum_{j=0}^{i-1} u_j)(\sum_{j=0}^{i-1} v_j)\right\}_x, \qquad i = 1, 2, ... \tag{23}$$

Therefore

$$\begin{aligned} u_1 &= v_1 = -t\sin(x), \\ u_2 &= v_2 = \frac{1}{2}t^2\sin(x), \\ u_3 &= v_3 = -\frac{1}{6}t^3\sin(x), \\ u_n &= v_n = -\frac{1}{n!}t^n\sin(x). \end{aligned} \tag{24}$$

So, the following exact solution will be obtained.

$$\begin{aligned} u(x,t) = v(x,t) &= \sin x(-t + \frac{1}{2}t^2 - \frac{1}{6}t^3 + ...) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} t^n \sin(x) \\ &= e^{-t}\sin x \end{aligned} \tag{25}$$

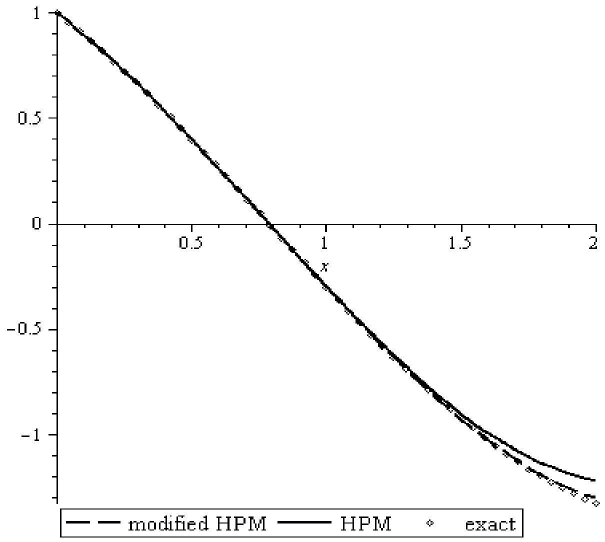

Figure 1 Plot of the solutions of example 1.

The convergence of New modified HPM

The convergence of the modified HPM has been proved by the following theorems.

Theorem 1: The new modified HPM for finding the solution of Eq. (1) is equivalent to determining the following sequence;

$$s_n = v_1 + ... + v_n,$$
$$s_0 = 0. \tag{26}$$

By using the iterative scheme;

$$s_{n+1} = -L^{-1}N(s_n + v_0) + L^{-1}(f(r)) - u_0. \tag{27}$$

Proof: In the new modified HPM, v_i 's have been obtained by the following iterative scheme.

$$L(v_1) = f(r) - L(u_0) - N(v_0),$$
$$L(v_2) = -N(v_0 + v_1) + N(v_0),$$
$$L(v_{i+1}) = -N\left(\sum_{j=0}^{i} v_j\right) + N\left(\sum_{j=0}^{i-1} v_j\right), \quad i = 1, 2, ..., n. \tag{28}$$

Then;

$$L(\sum_{i=1}^{n+1} v_i) = -N\left(\sum_{i=0}^{n} v_i\right) + f(r) - L(u_0). \tag{29}$$

By using (26), the following result will be obtained.

$$L(s_{n+1}) = -N\left(s_n + v_0\right) + f(r) - L(u_0). \tag{30}$$

So

$$s_{n+1} = -L^{-1}N(s_n + v_0) + L^{-1}(f(r)) - u_0. \tag{31}$$

Theorem 2: The following statements are valid in a Banach space B.

a) $\sum_{i=0}^{\infty} v_i$ obtained by (27), converges to $s \in \mathrm{B}$, if ;

$$\exists \ (0 \le \lambda < 1), \ s.t \ \left(\forall n \in \mathbf{N} \Rightarrow \|v_n\| \le \lambda \|v_{n-1}\| \right). \tag{32}$$

b) If N is continuous, then $s = \sum_{n=1}^{\infty} v_n$, satisfies in;

$$s = -L^{-1}N(s + v_0) - u_0 + L^{-1}(f(r)). \tag{33}$$

Proof:

a) $$\|s_{n+1} - s_n\| = \|v_{n+1}\| \le \lambda \|v_n\| \le \lambda^2 \|v_{n-1}\| \le \cdots \le \lambda^{n+1} \|v_0\|. \tag{34}$$

For any natural number such as n, and m, where $n \ge m$, the following result will be obtained.

$$\begin{aligned}
\|s_n - s_m\| &= \|(s_n - s_{n-1}) + (s_{n-1} - s_{n-2}) + \cdots + (s_{m+1} - s_m)\|, \\
&\le \|s_n - s_{n-1}\| + \|s_{n-1} - s_{n-2}\| + \cdots + \|s_{m+1} - s_m\|, \\
&\le \lambda^n \|v_0\| + \lambda^{n-1} \|v_0\| + \cdots + \lambda^{m+1} \|v_0\|, \\
&\le \left(\lambda^n + \lambda^{n-1} + \cdots + \lambda^{m+1}\right) \|v_0\|, \\
&\le \left(\lambda^{m+1} + \cdots + \lambda^n + \cdots\right) \|v_0\|, \\
&\le \lambda^{m+1} \left(1 + \lambda + \cdots + \lambda^n + \cdots\right) \|v_0\|, \\
&\le \frac{\lambda^{m+1}}{1-\lambda} \|v_0\|.
\end{aligned} \tag{35}$$

So

$$\lim_{n,m\to\infty} \|s_n - s_m\| = 0. \tag{36}$$

Then $\{s_n\}$ is the Cauchy sequence in a Banach space and so it is convergent, i.e.,

$$\exists s \in \mathrm{B},\ s.t\ \lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} v_n = s.$$

b) From Eq. (27) and considering the continuity of N, we derive;

$$\begin{aligned}
&\lim_{x\to\infty} s_{n+1} = -L^{-1}N(\lim_{x\to\infty} s_n + v_0) - u_0 + L^{-1}(f(r)), \\
&s = -L^{-1}N(\sum_{i=0}^{\infty} v_i) - u_0 + L^{-1}(f(r)).
\end{aligned} \tag{37}$$

So

$$s = -L^{-1}N(s + v_0) - u_0 + L^{-1}(f(r)). \tag{38}$$

Lemma. Eq. (33) is equivalent to;

$$L(u) + N(u) - f(r) = 0. \tag{39}$$

Proof:

Eq. (33) can be rewritten as the following;

$$s+u_0=-L^{-1}N(s+v_0)+L^{-1}(f(r)). \tag{40}$$

Applying the operator L on the Eq. (40), leads to;

$$L\left(s+u_0\right)=-N(s+v_0)+(f(r)). \tag{41}$$

But $u_0=v_0$, then;

$$L\left(s+v_0\right)+N(s+v_0)=(f(r)). \tag{42}$$

Eq. (33), which is equivalent to the original equation, has been resulted, by considering $u=s+v_0=\sum_{n=0}^{\infty}v_n$. So the solution of Eq. (27) is the same as solution of $A(u)-f(r)=0$.

Conclusions

The modified HPM is a simple tool for obtaining the solution of a functional. Easy procedures used for solving some examples, show that this method is simpler than the HPM. In this article, we have proved the convergence for this method. As a result from this paper, under assumption of theorem 3.2, the modified HPM converges to the exact solution of the problem. The method can be extended easily to other nonlinear functional equations.

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