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Comparative Analysis of MHD Boundary-Layer Flow of Viscoelastic Fluid in Permeable Channel with Slip Boundaries by using HAM, VIM, HPM

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Abstract

In this present study, the problem of two-dimensional magnetohydrodynamic (MHD) boundary layer flow of steady, laminar flow of an incompressible, viscoelastic fluid in a parallel plate channel with slip at the boundaries is presented. The upper convected Maxwell model is implemented due to its accuracy in simulating highly elastic fluid flows at high Deborah numbers. Moreover, this paper deals with the solution of third order of nonlinear ordinary differential equations which are solved by using three analytical approximate methods, namely the Homotopy Analysis Method (HAM), Homotopy Perturbation Method (HPM), and Variational Iteration Method (VIM). The comparisons of these results reveal that HAM is very effective, convenient and quite accurate for non-linear ordinary differential equation. In addition, this work demonstrates that HAM is able to solve problems with mixed (Robin) as well as other boundary conditions.

Keywords: Homotopy Analysis Method, Homotopy Perturbation Method, Variation Iteration Method, slip condition, magneto hydrodynamic, UCM.

Introduction

The flow problem in porous tubes or channels has received much attention in recent years because of its various applications in biomedical engineering, for example, in the dialysis of blood in artificial kidneys, in the flow of blood in the capillaries, and in the flow of blood oxygenators, as well as in many other engineering areas, such as the design of filters, in transpiration cooling boundary layer control, and in gaseous diffusion. Because of its relevance to a variety of situations, convection in porous media is a well-developed field of investigation. Over recent decades it has generally been recognized that some rheological complex fluids such as polymer solutions, blood, paints, butter, synovial fluid, salvia, soups, jams, ice-creams and certain oils cannot be adequately described by the Navier-Stokes theory. Because of this, several constitutive equations and flows for non-Newtonian fluids have been developed. Undoubtedly, the equations of motion of these fluids are highly nonlinear and of a higher order than the Navier-Stokes equations. Explicit solutions to the nonlinear equations are of fundamental interest. As only a limited number of these problems have precise and standard analytical solutions, the other ones should therefore be solved using alternative methods. In recent decades many attempts have been made to develop analytical methods for solving such nonlinear equations. One of them is the perturbation method [1], which is strongly dependent on a so called small parameter to be defined according to the physics of the problem. Thus, it is worth developing some new analytical techniques which are independent of defining a small parameter, such as the Homotopy Perturbation Method (HPM) [2-5], or the Variational Iteration Method (VIM) [6-11]. In fact the perturbation method cannot provide a simple way to adjust and control the region and rate of convergence of a particular approximated series. Liao [12] introduced the basic idea of Homotopy in topology to propose a general analytical method for nonlinear problems, namely the Homotopy Analysis Method (HAM) [12-19], that does not need any small parameter. This method has been successfully applied to solve many types of nonlinear problems [20-25]. One of the difficulties in the HAM is to prove the convergence of the homotopy series obtained. The recent study in [26] presents a rigorous mathematical approach to show the convergence of the obtained series. It is worth mentioning that there is still unknown auxiliary parameter h in the series solution which should be determined. Liao [13] interpreted this series solution as the generalized Taylor series solution in his book [13]. Liu [27-30] has proven that the essence of the HAM is just the usual series expansion at another point which gives a relation with the auxiliary parameter h in HAM. In [27], Liu proved that the generalized Newton binomial theorem is essentially the usual Newton binomial theorem at another point. In [28], Liu pointed out, without detailed proof, that the generalized Taylor theorem is essentially the usual Taylor theorem at another point. In [29], Liu Cheng-shi and Liu Yang proposed a general series expansion method and compared it with Liao's Homotopy Analysis Method, and showed that one could use their method to obtain the same result as Liao's [19]. Therefore, essentially, the HAM is exactly the usual series expansion method at another point. In [30], Liu proved that the generalized Taylor series is just the usual Taylor expansion at another point. This means that the meaning of the auxiliary parameter h in the HAM is clarified and the essence of the generalized Taylor theorem is uncovered.

In this paper, the authors present approximate methods namely HPM, VIM and HAM to solve the MHD boundary layer flow of an UCM fluid in a permeable channel with slip boundaries. Comparison between presented results and numerical results which are in full agreement shows that HAM is useful in solving a large number of Linear or nonlinear differential equations and give rapidly convergent successive approximations.

Nomenclature

HAM	Homotopy Analysis Method
HPM	Homotopy Perturbation Method
VIM	Variation Iteration Method
NUM	Numerical Method
M	Hartman Number
$\mathrm{Re}\,w$	Reynolds number(wall)
k	Slip condition
De	Deborah number
$\hbar$	Auxiliary parameter
$\mathcal{H}$	Auxiliary function
$\mathcal{L}$	Linear operator of HAM
$\mathcal{N}$	Non-linear operator
u^*	Velocity component in x-direction
v^*	Velocity component in y-direction
x	dimensionless horizontal coordinate
y	dimensionless vertical coordinate
x^*	distance in x direction parallel to the plates
y^*	distance in y direction parallel to the plates

Greek symbols

ρ	Density of the fluid
λ	relaxation time
υ	Kinematic viscosity

Governing equations

Consider a two-dimensional incompressible UCM fluid in a permeable channel. The x^* −axis is taken along the centerline of the channel parallel to the channel surfaces, and the y^* −axis transverse to these. The flow is symmetric about both axes. The permeable walls of the channel are at $y^* = H$ and $y^* = -H$ (where $2H$ is the channel width). The fluid injection or extraction takes place through the permeable walls with velocityV_w. Here, $V_w > 0$ and $V_w < 0$ stands for suction and injection, respectively. Let u^* and v^* be the velocity components along the x^* − and y^* − axes, respectively (see **Figure 1**). The constitutive equation for a Maxwell fluid is [31];

$$\tau + \lambda \hat{\tau} = \mu_0 \gamma \tag{1}$$

where τ is the extra stress tensor and the upper convected time derivative of the stress tensor $\hat{\tau}$ satisfies;

$$\hat{\tau} = \frac{\partial \tau}{\partial t} + v.\nabla \tau - \left(\nabla v\right)^T .\tau - \tau.\nabla v \tag{2}$$

In which μ_0 is the low-shear viscosity, λ is the relaxation time, γ is the rate-of-strain tensor, t denotes time, v is the velocity vector, $(.)^T$ is the transpose of the tensor and ∇v represents the fluid velocity gradient tensor.

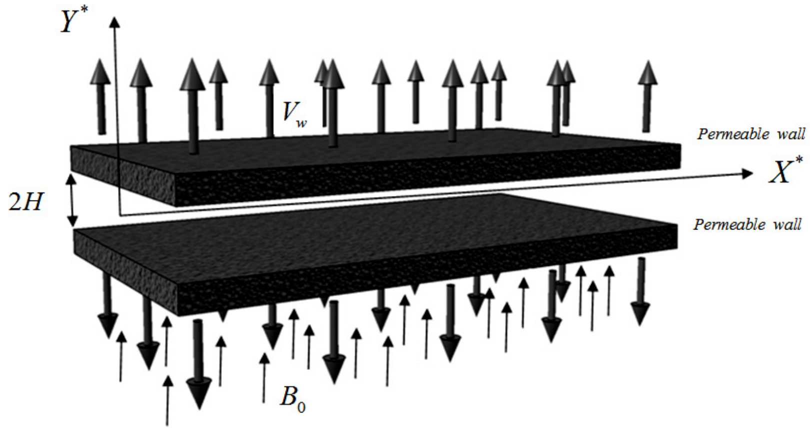

Figure 1 Schematic diagram of the physical system.

Implementing the shear-stress strain tensor for a UCM liquid from Eqs. (1) and (2), in the absence of a pressure gradient, the steady two-dimensional boundary layer equations for this flow in usual notation are [32];

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{3}$$

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} + \lambda \left[u^{*2} \frac{\partial^2 u^*}{\partial x^{*2}} + v^{*2} \frac{\partial^2 u^*}{\partial y^{*2}} + 2u^* v^* \frac{\partial^2 u^*}{\partial x^* \partial y^*} \right] = \upsilon \frac{\partial^2 u^*}{\partial y^{*2}} - \frac{\sigma B_0^2}{\rho} u^*, \tag{4}$$

Where υ is the kinematic viscosity and λ is the relaxation time. In order to complete the formulation of the problem, the boundary conditions have to be specified. The appropriate boundary conditions for the velocity are symmetry about the x*-axis and slip conditions at x* = H yield;

$$y^* = 0: \qquad \frac{\partial u^*}{\partial y^*} = 0, \qquad v^* = 0, \tag{5}$$

$$y^* = H: \qquad -\beta u^* = \frac{\partial u^*}{\partial y^*}, \qquad v^* = V_W. \tag{6}$$

The boundary condition (6) is the well known slip condition, μ the dynamic viscosity coefficient, and β the coefficient of sliding friction. The following dimensionless variables are introduced;

$$x = \frac{x^*}{H}, \qquad y = \frac{y^*}{H}, \qquad k = \frac{\mu}{H\beta}, \qquad u^* = -V_W x f'(y), \qquad v^* = V_w f(y) \tag{7}$$

Eq. (3) is automatically satisfied and Eqs. (3) - (6) may be written as;

$$f''' - M^2 f' + \operatorname{Re} w\left(f'^2 - f f''\right) + De\left(2 f f' f'' - f^2 f'''\right) = 0 \tag{8}$$

The boundary conditions become;

$$\begin{aligned} &y = 0: \qquad f'' = 0; \qquad f = 0, \\ &y = 1: \qquad f' = -k f''; \qquad f = 1. \end{aligned} \tag{9}$$

The differential equation of the model is in third order, but there are four boundary conditions for the problem. Some authors satisfy boundary conditions in the initial guess function. It is possible to work creatively with the derivation of Eq. (8) and introduce fourth order differential equation all of the boundary conditions can be satisfied in the main equation. This gives;

$$f'''' - M^2 f'' + \operatorname{Re}_W\left(f' f'' - f f'''\right) + De\left(2 f'^2 f'' - 2 f f''^2 + f^2 f''''\right) = 0 \tag{10}$$

Here, $Re_w = \frac{V_w H}{v}$ is the Reynolds number, $De = \frac{\lambda V_w^2}{\upsilon}$ is the Deborah number, and $M^2 = \frac{\sigma B_0^2 H^2}{\mu}$ is the Hartman number, where $Re_w > 0$ corresponds to suction and $Re_w < 0$ for injection.

Application of Homotopy Analysis Method

For HAM solutions, the initial guess and auxiliary linear operator is given in the following form;

$$f_0(\mathrm{y}) = -\frac{1}{2(3k+1)} y^3 + \frac{3(2k+1)}{2(3k+1)} y, \tag{11}$$

$$L(f) = f'''' , \tag{12}$$

$$L\left(\frac{1}{6} c_1 y^3 + \frac{1}{2} c_2 y^2 + c_3 y + c_4\right) = 0, \tag{13}$$

Where ($C_i = 1 - 4$) are constants. Let $p \in [0,1]$ denote the embedding parameter and $\hbar$ indicate non-zero auxiliary parameters. The following Zeroth-order deformation equations are then constructed;

$$(1-P)L\left[F(y;p)-f_0(y)\right] = p\hbar H(y)N\left[F(y;p)\right] \tag{14}$$

$$F(0;p)=0, \qquad F''(0;p)=0, \qquad F(1;p)=1, \qquad k\,F''(1;p)+F'(1;p)=0 \tag{15}$$

$$\begin{aligned} N[F(y;p)] &= \frac{d^4F(y;p)}{dy^4} - M^2\frac{d^2F(y;p)}{dy^2} + \mathrm{Re}_w\left[\frac{dF(y;p)}{dy}\frac{d^2F(y;p)}{dy^2} - F(y;p)\frac{d^3F(y;p)}{dy^3}\right] \\ &+De\left[2\left(\frac{dF(y;p)}{dy}\right)^2\frac{d^2F(y;p)}{dy^2} - 2F(y;p)\left(\frac{d^2F(y;p)}{dy^2}\right)^2 + \left(F(y;p)\right)^2\frac{d^4F(y;p)}{dy^4}\right] \end{aligned} \tag{16}$$

For $p = 0$ and $p = 1$;

$$F(y;0) = f_0(y), \qquad F(y;1) = f(y)\cdot \tag{17}$$

When p increases from 0 to 1, $f(y;p)$ varies from $f_0(y)$ to $f(y)$. By Taylor's theorem, and using Eq. (17), $f(y;p)$ can be expanded in a power series of p as follows;

$$F(y;p) = f_0(y) + \sum_{m-1}^{\infty} f_m(y)p^m, \qquad f_m(y) = \frac{1}{m!}\frac{\partial^m(F(y;p))}{\partial p^m}\bigg|_{p=0} \tag{18}$$

In which $\hbar$ is chosen in such a way that this series is convergent at $p = 1$; therefore Eq. (18) shows;

$$f(y) = f_0(y) + \sum_{m-1}^{\infty} f_m(y), \tag{19}$$

Mth-order deformation equations;

$$L\left[f_m(y) - \chi_m f_{m-1}(y)\right] = \hbar H(y)R_m(y) \tag{20}$$

$$F(0;p)=0; \qquad F''(0;p)=0, \qquad F(1;p)=0, \qquad k\,F''(1;p)+F'(1;p)=0 \tag{21}$$

$$\begin{aligned} R_m(\mathrm{y}) &= f''''_{m-1} + \sum_{k=0}^{m-1}\left[\mathrm{Re}_w\left(f'_{m-1-k}\,f''_k - f_{m-1-k}\,f'''_k\right) + De\,f'_{m-1-k}\left(\sum_{l=0}^{k}\left(2\,f'_{k-l}\,f''_l\right)\right)\right. \\ &\left.- De\,f_{m-1-k}\left(\sum_{l=0}^{k}\left(2\,f''_{k-l}\,f''_l - f_{k-l}\,f''''_l\right)\right)\right] - M^2 f''_{m-1} \end{aligned} \tag{22}$$

Now convergency of the result, the differential equation, and the auxiliary function according to the solution expression are determined. Assuming that;

$$H(y) = 1 \tag{23}$$

Hence, the answer is found by use of a Maple Analytic Solution device. Firstly, deformation of the solution is presented below;

$$f_1\left(\mathrm{y}\right)=-\frac{5\hbar}{672}\frac{De\,y^9}{\left(3k+1\right)^3}-\frac{3\hbar}{2}\frac{\left(-0.0071429\,\mathrm{Re}\,wk-0.042857De-0.0023810\,\mathrm{Re}\,w-0.085714Dek\right)\mathrm{y}^7}{27k^3+27k^2+9k+1}$$
$$-\frac{3\hbar}{2}\frac{\left(-0.15000M^2k^2-0.10000M^2k-0.016667M^2+0.30000Dek^2+0.30000Dek+0.075000De\right)\mathrm{y}^5}{27k^3+27k^2+9k+1}$$
$$+\frac{\hbar}{840}\left[\frac{\left(3780Dek^3-1890M^2k^3-189\,\mathrm{Re}_w k^2+2268Dek^2-1638M^2k^2-90\,\mathrm{Re}_w k-462M^2k+468Dek\right)\mathrm{y}^3}{1+54k^2+12k+108k^3+81k^4}\right. \tag{24}$$
$$\left.+\frac{\left(-9\,\mathrm{Re}_w+52De-42M^2\right)\mathrm{y}^3}{1+54k^2+12k+108k^3+81k^4}\right]-\frac{\hbar}{1120}\left[\frac{\left(-8\,\mathrm{Re}_w-96\,\mathrm{Re}_w k-1764M^2k^3 1440Dek^2+77Dek-364M^2k\right)\mathrm{y}}{1+54k^2+12k+108k^3+81k^4}\right.$$
$$\left.+\frac{\left(-1428M^2k^2+3528Dek^2-216\,\mathrm{Re}\,wk^2-28M^2+7De\right)\mathrm{y}}{1+54k^2+12k+108k^3+81\,k^4}\right]$$

The solutions $f(y)$ were too long to be mentioned here; therefore, they are shown graphically.

Application of Variational Iteration Method

Initially, a correction functional is constructed, which reads;

$$f_{n+1}\left(y\right)=f_n\left(y\right)+\int_0^y \lambda\Big\{f''''(\tau)-M^2f''(\tau)+\mathrm{Re}\,w\left(f'\left(\tau\right)f''\left(\tau\right)-f\left(\tau\right)f'''\left(\tau\right)\right)$$
$$+De\left(2f'\left(\tau\right)^2f''\left(\tau\right)-2f\left(\tau\right)f''\left(\tau\right)^2\right)+De\left(f(\tau)^2f''''(\tau)\right)\Big\}d\tau \tag{25}$$

where λ is a general Lagrange multiplier. The Lagrange multiplier can be identified as;

$$\lambda=-\frac{1}{6}\left(y-\tau\right)^3; \tag{26}$$

As a result, the following iteration formula is obtained;

$$f_{n+1}\left(y\right)=f_n\left(y\right)+\int_0^y \lambda\Big\{f''''(\tau)-M^2f''(\tau)+\mathrm{Re}\,w\left(f'\left(\tau\right)f''\left(\tau\right)-f\left(\tau\right)f'''\left(\tau\right)\right)$$
$$+De\left(2f'\left(\tau\right)^2f''\left(\tau\right)-2f\left(\tau\right)f''\left(\tau\right)^2\right)+De\left(f(\tau)^2f''''(\tau)\right)\Big\}d\tau \tag{27}$$

Starting with an arbitrary initial approximation that satisfies the initial condition;

$$f_0\left(\mathrm{y}\right)=-\frac{1}{2\left(3k+1\right)}y^3+\frac{3\left(2k+1\right)}{2\left(3k+1\right)}y, \tag{28}$$

and using the above variational formula (27), after some simplifications (for example $M = 4$, $\mathrm{Re}_w = 4$, $De = 0.1$, $K = 0.9$);

$$f_1(y) = 0.000014689y^9 - 0.0013988y^7 - 0.106366y^5 - 0.135135y^3 + 1.13513y \tag{29}$$

$$\begin{aligned} f_2(y) = & -1.43032\,10^{-17}y^{27} + 4.02884\,10^{-15}y^{25} - 1.037\,10^{-13}y^{23} - 4.1401510^{-11}y^{21} \\ & +9.570510^{-10}y^{19} + 2.21410^{-7}y^{17} + 0.00000662x^{15} + 0.000017y^{13} - 0.00034493y^{11} \\ & -0.001474y^9 - 0.0662231y^7 - 0.1063668y^5 - 0.135135135y^3 + 1.135135y \end{aligned} \tag{30}$$

$$\begin{aligned} f_3(y) = & 1.163510^{-53}y^{81} - 9.81510^{-51}y^{79} + 3.0199210^{-48}y^{77} - 3.050112410^{-46}y^{75} \\ & -3.597893010^{-44}y^{73} + 9.6400010^{-42}y^{71} - 7.043427910^{-41}y^{69} - 1.1966310^{-37}y^{67} \\ & +3.80007110^{-36}y^{65} + 9.99151510^{-34}y^{63} - 2.79242910^{-32}y^{61} - 6.445163210^{-30}y^{59} \\ & +2.862068510^{-29}y^{57} + 2.90612310^{-26}y^{55} + 8.450113510^{-25}y^{53} - 5.48601410^{-23}y^{51} \\ & +2.735185110^{-18}y^{43} + 2.74419810^{-16}y^{41} + 3.474775610^{-15}y^{39} + 3.06017010^{-14}y^{37} \\ & +1.68080610^{-14}y^{35} - 3.2692140\,10^{-12}y^{33} - 2.8351750\,10^{-11}y^{31} - 1.84312240\,10^{-10}y^{29} \\ & +0.000006584130y^{19} + 0.000018240590y^{17} - 0.000050540840y^{15} - 0.000350710800y^{13} \\ & -0.00078762600y^{11} - 4.5670896810^{-21}x^{49} - 1.336201240^{-19}x^{47} - 1.606769710^{-18}x^{45} \\ & -0.03431980525y^9 - 0.0662231205874y^7 - 0.106366849y^5 - 2.11499142010^{-10}y^{27} \\ & +1.8815023310^{-8}y^{25} + 1.243623910^{-7}y^{23} + 0.00000163725y^{21} - 0.13513513y^3 \\ & +1.1351351y \end{aligned} \tag{31}$$

Application of Homotopy Perturbation Method

In this section, the HPM is employed to solve Eq. (10) subject to boundary conditions from Eq. (9). The Homotopy function of Eq. (10) is constructed as described in [2];

$$\begin{aligned} H(f,p) = & (1-P)\left[f''''(y) - g_0(y)\right] + p\left[f'''' - M^2 f'' + \mathrm{Re}\,w\left(f'f'' - ff'''\right)\right. \\ & \left. +De\left(2f'^2 f'' - 2ff''^2 + f^2 f''''\right)\right], \end{aligned} \tag{32}$$

Where $p \in [0,1]$ is an embedding parameter. For $p = 0$ and $p = 1$;

$$f(y,0) = f_0(y) \quad , \quad f(y,1) = f(y) \tag{33}$$

Note that when p increases from 0 to 1, $f(y,p)$ varies from $f_0(y)$ to $f(y)$. By substituting;

$$f(y) = f_0(y) + p f_1(y) + p^2 f_2(y) + \cdots = \sum_{i=0}^{n} p^i f_i(y), \;\; g_0 = 0 \tag{34}$$

Into Eq. (32) and rearranging the result based on powers of p-terms (for example $M = 4$, $\mathrm{Re}_w = 4, De = 0.1, K = 0.9$);

$$\begin{aligned} P^0: \quad & f_0''''(y) = 0 \\ & f_0(0) = 0, \quad f_0''(0) = 0, \quad f_0(1) = 1, \quad k f_0''(1) + f_0'(1) = 0 \end{aligned} \tag{35}$$

$$\begin{aligned} P^1: \quad & f_1''''(y) - 4 f_0(y) f_0'''''(y) + 0.2 f_0(y) f_0''(y)^2 + 4 f_0'(y) f_0''(y) + 0.2 f_0'(y)^2 f_0''(y) \\ & -16 f_0''(y) - 0.1 f_0(y)^2 f_0''''(y) = 0 \\ & f_1(0) = 0, \quad f_1''(0) = 0, \quad f_1(1) = 0, \quad k f_1''(1) + f_1'(1) = 0 \end{aligned} \tag{36}$$

$$\begin{aligned} P^2: \quad & f_2''''(y) - 0.2 f_0(y) f_1(y) f_0''''(y) + 4 f_0'(y) f_1''(y) + 0.2 f_0'(y)^2 f_1''(y) - 0.1 f_0'(y)^2 f_1''''(y) \\ & + 4 f_1'(y) f_0''(y) + 0.4 f_0(y) f_0''(y) f_1''(y) - 16 f_1''(y) + 0.2 f_1(y) f_0''(y)^2 - 4 f_1(y) f_0'''(y) \\ & + 0.4 f_0'(y) f_1'(y) f_0''(y) - 4 f_0(y) f_1'''(y) = 0 \\ & f_2(0) = 0; \quad f_2''(0) = 0, \quad f_2(1) = 0, \quad k f_2''(1) + f_2'(1) = 0 \end{aligned} \tag{37}$$

$\vdots$

Solving Eqs. (35) - (37) with boundary conditions;

$$f_0(y) = -0.13514 y^3 + 1.1351 y \tag{38}$$

$$f_1(y) = 0.00001469 y^9 - 0.00139 y^7 - 0.10636 y^5 + 0.3243 y^3 - 0.2166 y \tag{39}$$

$$\begin{aligned} f_2(y) = & -5.896710^{-10} y^{15} + 6.970910^{-8} y^{13} + 0.000039301 y^{11} - 0.00159472121 y^9 \\ & -0.058036 y^7 + 0.25464 y^5 - 0.39843 y^3 + 0.20338 y \end{aligned} \tag{40}$$

The solution of this equation, when $p \to 1$, will be as follows;

$$\begin{aligned} f(y) = & \sum_{i=0}^{4} \lim_{p \to 1} p^i f_i(y) \\ = & \; 4.60021410^{-17} y^{27} - 1.35500010^{-14} y^{25} + 2.7211010^{-12} y^{23} - 2.94301010^{-10} y^{21} \\ & + 1.446710^{-8} y^{19} - 4.849010^{-7} y^{17} + 0.00006015 y^{15} - 0.00060475 y^{13} - 0.010333 y^{11} \\ & + 0.038996 y^9 - 0.094165 y^7 + 0.09999 y^5 - 0.0956 y^3 + 1.0617 y \end{aligned} \tag{41}$$

Convergence of the HAM solution

As pointed out by Liao, the convergence region and rate of solution series can be adjusted and controlled by means of the auxiliary parameter ħ. In general, by means of the so-called ħ-curve, it is straightforward to choose an appropriate range for ħ which ensures the convergence of the solution series.

To influence of ħ on the convergence of solution, we plot the so-called ħ-curve of $f'''(0)$ by 11th-order approximation, as shown in **Figures 2 - 5**. According to **Figures 2 - 5**, for $M = 2,\ k = 0.2,\ \mathrm{Re}_w = 1$ and $0 < De < 0.9$, the ranges are $-1.5 < \hbar < -0.3$, for $M = 2, De = 0.1$, $\mathrm{Re}_w = 1$ and $0 < k < 0.9$, the ranges are $-1.5 < \hbar < -0.3$, for $k = 0.2,\ De = 0.1,\ \mathrm{Re}_w = 1$ and $0 < M < 4$, the ranges are $-0.6 < \hbar < -0.2$, and for $M = 2, De = 0.1, k = 0.2$ and $-5 < \mathrm{Re}_w < 5$, the ranges are $-1 < \hbar < -0.3$. Then $\hbar = -0.5$ is a suitable value for ranges $0 < k < 0.9, -5 < \mathrm{Re}_w < 5, 0 < M < 4$ and $0 < De < 0.9$, which is used for the solution.

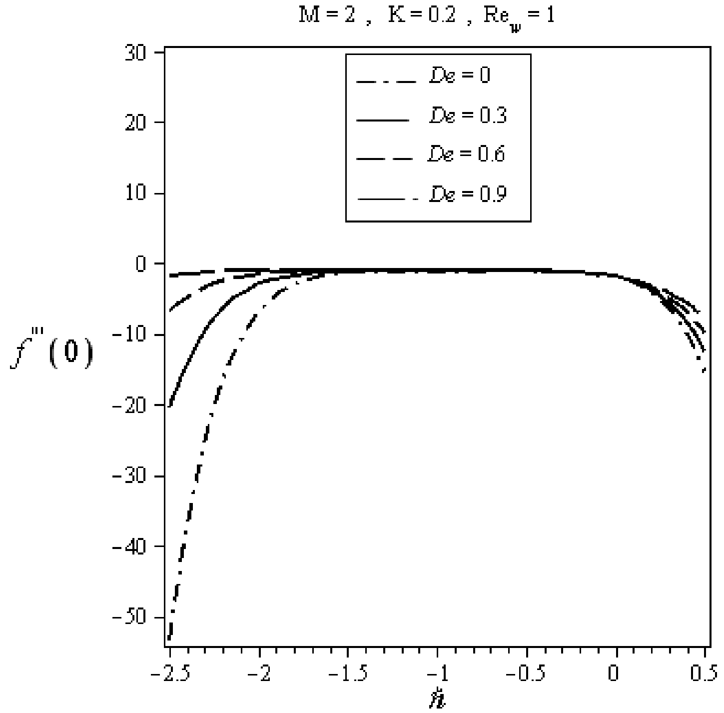

Figure 2 The $\hbar$ - validity for $M = 2, k = 0.2$, $\mathrm{Re}_w = 1$ and different values of De.

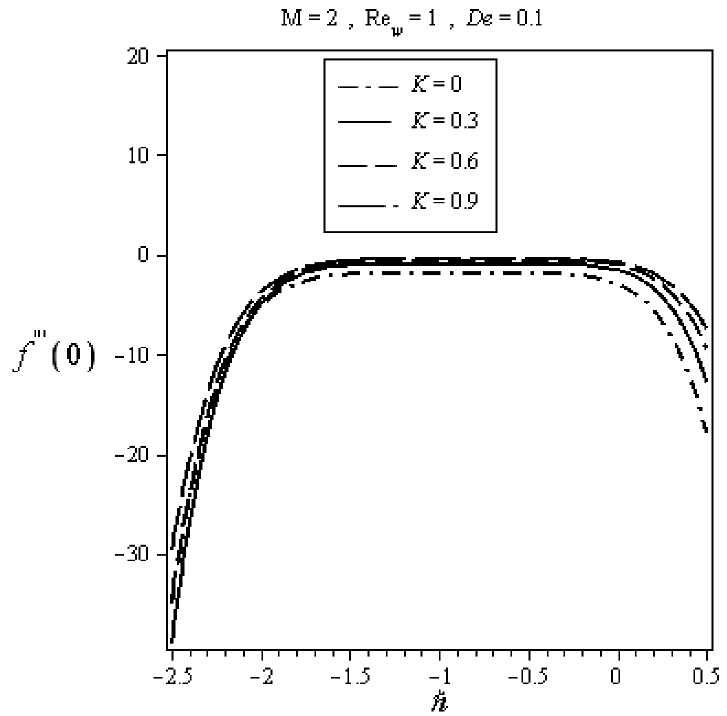

Figure 3 The $\hbar$ - validity for $M = 2, De = 0.1$, $\mathrm{Re}_W = 1$ and different values of k .

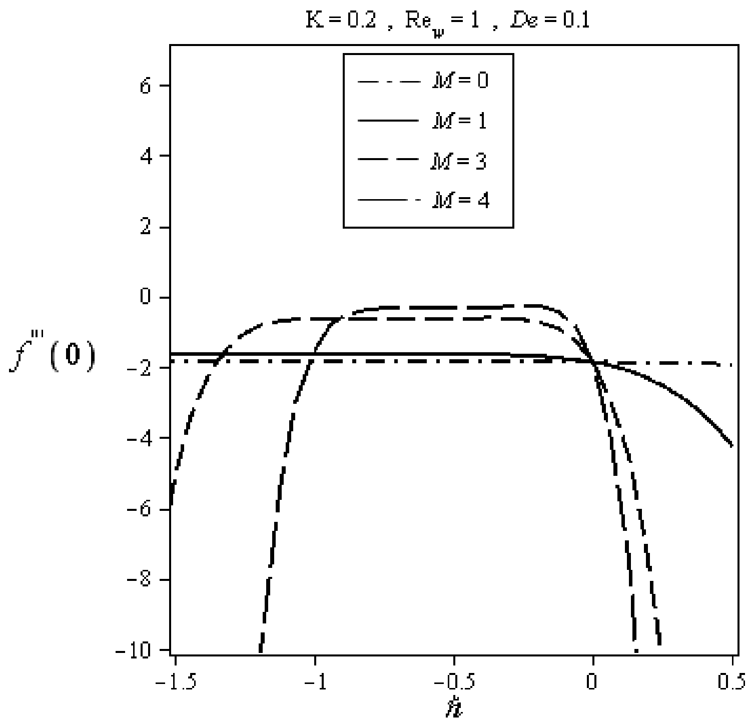

Figure 4 The $\hbar$ - validity for $De = 0.1, k = 0.2$, $\mathrm{Re}_W = 1$ and different values of M .

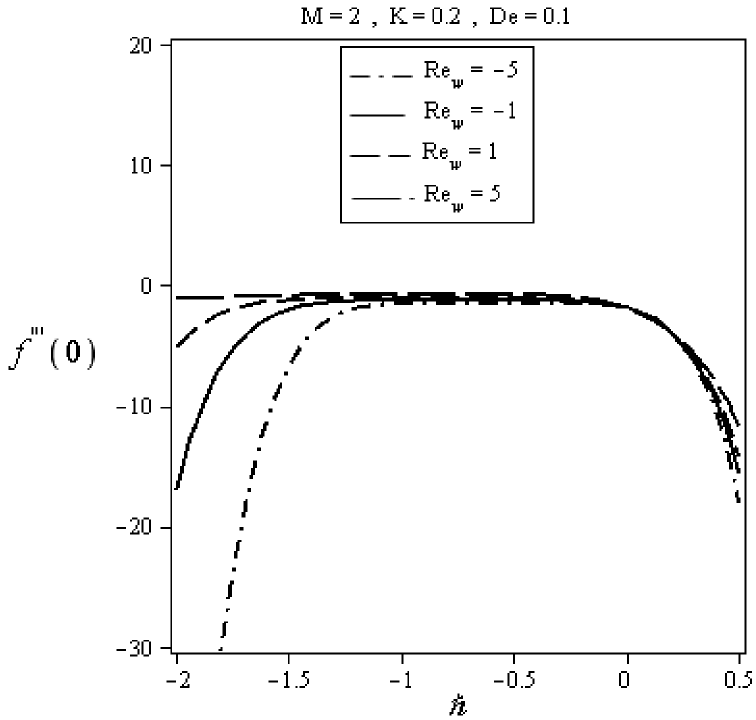

Figure 5 The $\hbar$ - validity for $M = 2, k = 0.2$, $De = 0.1$ and different values of Re_W .

Results and discussion

In this paper, we have used 3 analytical techniques to achieve approximate solutions for solving nonlinear equations of MHD boundary layer flow of an UCM fluid in a permeable channel with slip boundaries. The plots for f' and f variations with y for different values of De , k , M and Rew parameter have been achieved. For verification purpose, a numerical approach has also been implemented. For every case investigated **Figures 6 - 11**, the analytical method predictions have been compared with the corresponding direct numerical solutions NSs obtained by using Maple 15 software. This software uses a Fehlberg fourth-fifth order Runge-Kutta finite-difference method for the numerical solution of the boundary value problem [33]. For more clearance we compare these methods for $De = 0.9, k = 0.1, M = 0$ and $Rew = -4$ in **Table 1**. It can be observed that there is an excellent agreement between the results obtained from the HAM with those of Runge-Kutta. Many of the results attained in this paper confirm the idea that HAM is a powerful mathematical tool for solving different kinds of nonlinear problems arising in various fields of science and engineering.

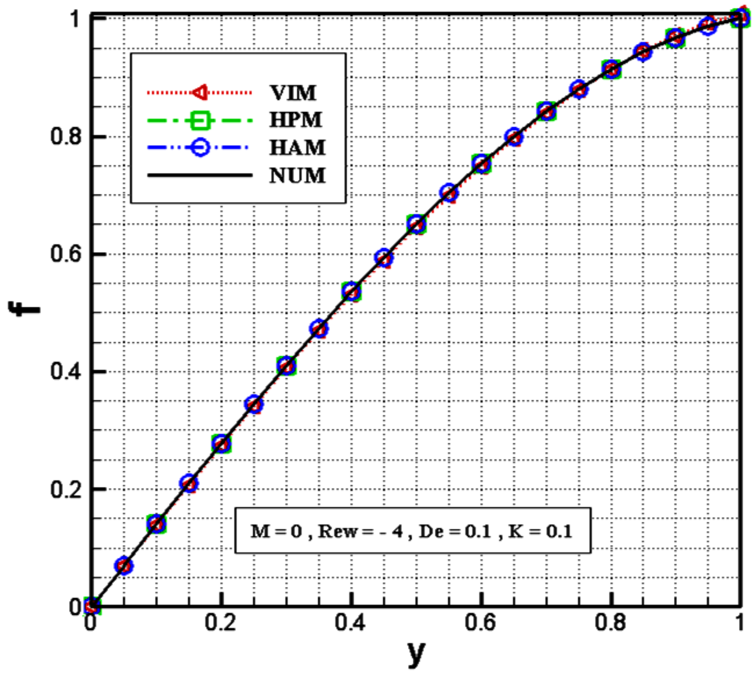

Figure 6 The comparison between the HAM, VIM, HPM and numerical solutions for $f(y)$ when $M=0$, $\mathrm{Re}_w=-4$, $De=0.1$, $K=0.1$.

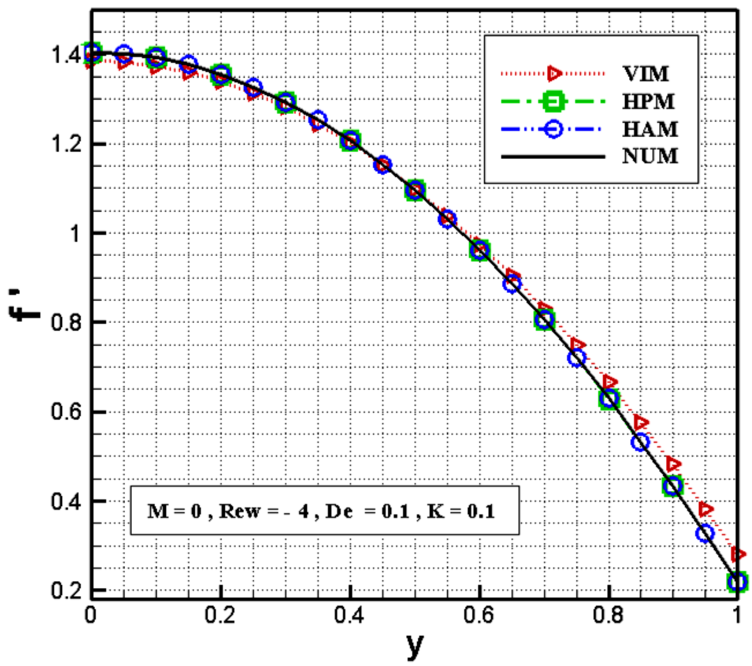

Figure 7 The comparison between the HAM, VIM, HPM and numerical solutions for $f'(y)$ when $M=0$, $\mathrm{Re}_w=-4$, $De=0.1$, $K=0.1$.

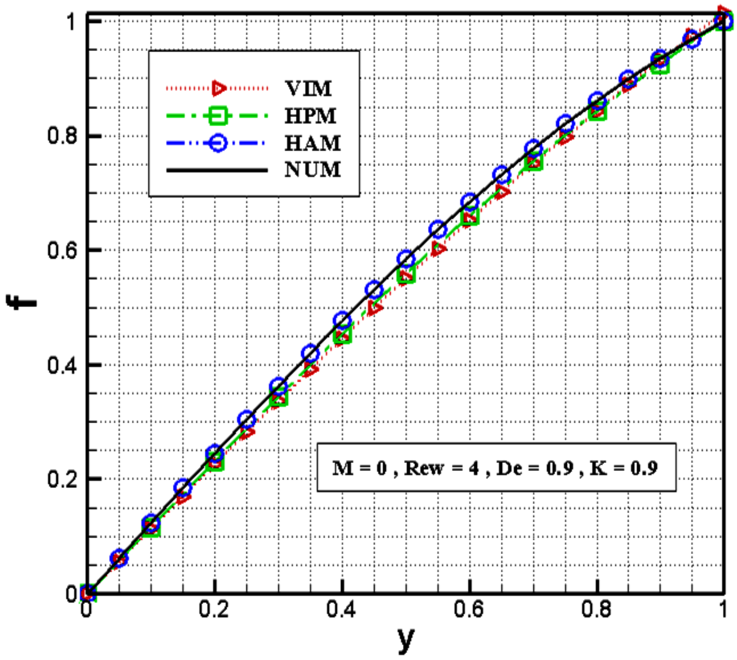

Figure 8 The comparison between the HAM, VIM, HPM and numerical solutions for $f(y)$ when $M = 0, \mathrm{Re}_w = 4, De = 0.9, K = 0.9$.

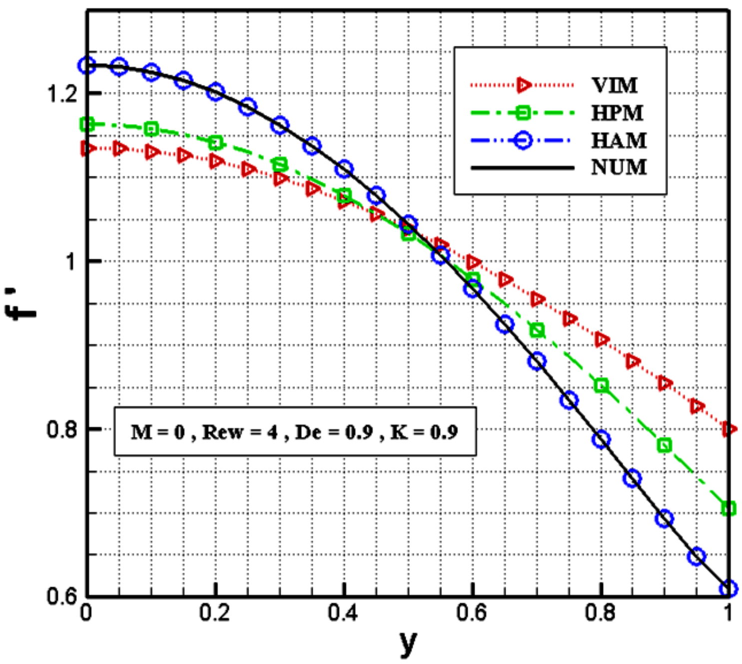

Figure 9 The comparison between the HAM, VIM, HPM and numerical solutions for $f'(y)$ when $M = 0, \mathrm{Re}_w = 4, De = 0.9, K = 0.9$.

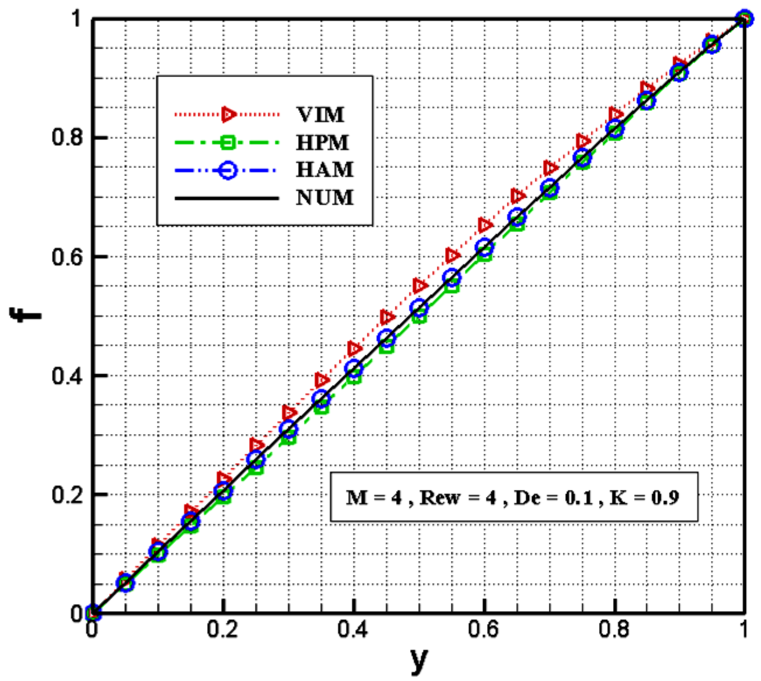

Figure 10 The comparison between the HAM, VIM, HPM and numerical solutions for $f(y)$ when $M = 4$, $\mathrm{Re}_w = 4$, $De = 0.1$, $K = 0.9$.

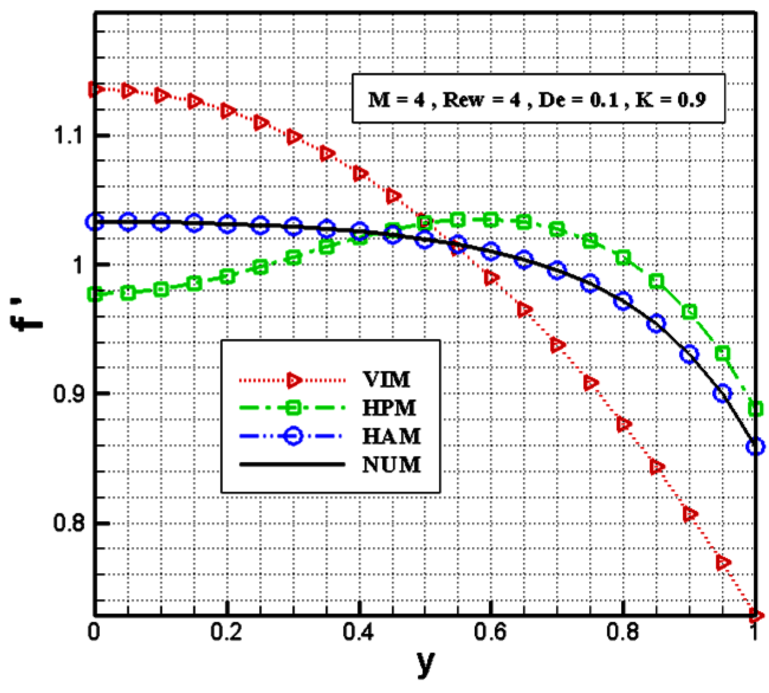

Figure 11 The comparison between the HAM, VIM, HPM and numerical solutions for $f'(y)$ when $M = 4$, $\mathrm{Re}_w = 4$, $De = 0.1$, $K = 0.9$.

Table 1 The results of HAM, HPM, VIM and Numerical methods for $f'(y)$ for De = 0.9, k = 0.1, M = 0 and $Re_w = -4$.

y	HPM	VIM	HAM	NUM	Error of HAM	Error of HPM	Error of VIM
0	1.405607800	1.384615385	1.405943339	1.405943328	0.000000011	0.000335528	0.021327943
0.05	1.402208557	1.381732841	1.402548122	1.402548111	0.000000011	0.000339554	0.02081527
0.10	1.392041516	1.373109898	1.392392426	1.392392415	0.000000011	0.000350899	0.019282517
0.15	1.375197098	1.358819470	1.375564531	1.375564519	0.000000012	0.000367421	0.016745049
0.20	1.351820543	1.338979249	1.352206323	1.352206310	0.000000013	0.000385767	0.013227061
0.25	1.322103937	1.313746092	1.322505609	1.322505595	0.000000014	0.000401658	0.008759503
0.30	1.286275432	1.283308441	1.286685726	1.286685711	0.000000015	0.000410279	0.00337727
0.35	1.244586064	1.247877005	1.244992790	1.244992774	0.000000016	0.000406710	0.002884231
0.40	1.197294659	1.207674030	1.197681040	1.197681024	0.000000016	0.000386365	0.009993006
0.45	1.144651401	1.162921528	1.144996805	1.144996789	0.000000016	0.000345388	0.017924739
0.50	1.086880739	1.113828803	1.087161720	1.087161704	0.000000016	0.000280965	0.026667099
0.55	1.024164432	1.060579797	1.024355925	1.024355908	0.000000017	0.000191476	0.036223889
0.6	0.956625495	1.003320675	0.956702118	0.956702101	0.000000017	0.000076605	0.046618575
0.65	0.884314055	0.942148179	0.884251492	0.884251474	0.000000018	0.000062581	0.057896705
0.70	0.807196043	0.877099391	0.806972797	0.806972775	0.000000022	0.000223268	0.070126616
0.75	0.725145778	0.808143451	0.724745979	0.724745956	0.000000023	0.000399823	0.083397495
0.80	0.637943524	0.735175985	0.637362093	0.637362066	0.000000026	0.000581458	0.097813919
0.85	0.545279074	0.658016950	0.544531083	0.544531069	0.000000014	0.000748006	0.113485881
0.90	0.446762388	0.576412650	0.445898828	0.445898859	0.000000031	0.000863529	0.130513791
0.95	0.341942187	0.490042691	0.341073641	0.341073783	0.000000142	0.000868404	0.148968909
1.00	0.230333226	0.398532584	0.229661062	0.229661260	0.000000197	0.000671966	0.168871324

Conclusions

In this present work, the approximate analytical solution of the MHD boundary layer flow of an upper-convected Maxwell (UCM) fluid in a permeable channel with slip at the boundaries has been obtained by employing the HAM, HPM and the VIM. The approximate solutions have been compared with the direct numerical solutions generated by the symbolic algebra package Maple 15 which uses a Fehlberg fourth-fifth order Runge-Kutta finite-difference method for solving nonlinear boundary value problems. The comparison showed that the HAM solutions are highly accurate and provide a rapid means of computing the flow velocities.

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