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Weighted Average Finite Difference Methods for Fractional Reaction-Subdiffusion Equation

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Abstract

In this article, a numerical study for fractional reaction-subdiffusion equations is introduced using a class of finite difference methods. These methods are extensions of the weighted average methods for ordinary (non-fractional) reaction-subdiffusion equations. A stability analysis of the proposed methods is given by a recently proposed procedure similar to the standard John von Neumann stability analysis. Simple and accurate stability criterion valid for different discretization schemes of the fractional derivative, arbitrary weight factor, and arbitrary order of the fractional derivative, are given and checked numerically. Numerical test examples, figures, and comparisons have been presented for clarity.

Keywords: Weighted average, finite difference approximations, fractional reaction-subdiffusion quation, stability analysis

Introduction

In the last few years, there have been extensive studies of fractional order differential equations (FDEs), due to their important applications in many vital areas of research, such as physics, medicine and engineering. Moreover, fractional calculus studies can allow understanding of many fractal phenomena which cannot be studied by ordinary means. There are many applications for FDEs; see [1-7]. The studied models have received a great deal of attention, such as in the fields of viscoelastic materials [8], control theory [9], advection and dispersion of solutes in natural porous or fractured media [1], and anomalous diffusion. Due to the difficulties in claiming the exact solutions for FDEs, approximate and numerical techniques are extensively used (see for example [10] and the references cited therein). Recently, several numerical methods have been adapted to solve fractional differential equations, see [11-17] and the references cited therein.

In this section, the definitions of Riemann-Liouville and the Grünwald-Letnikov fractional derivatives which will be used later are given [18,19].

Definition 1: The Riemann-Liouville derivative of order α of the function $y(x)$ is defined by;

$$D_x^{\alpha} y(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x \frac{y(\tau)}{(x-\tau)^{\alpha-n+1}} d\tau, \quad x > 0 \,, \tag{1}$$

where n is the smallest integer exceeding α and $\Gamma(.)$ is the Gamma function. If $\alpha = m \in \mathrm{N}$, then the above definition coincides with the classical $m-th$ derivative $y^{(m)}(x)$.

Definition 2: The Grünwald-Letnikov definition for the fractional derivatives of order $\alpha > 0$ of the function $y(x)$ is defined by;

$$D^{\alpha} y(x) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{[\frac{x}{h}]} w_k^{(\alpha)} y(x-hk), \quad x \geq 0, \tag{2}$$

where $[\frac{x}{h}]$ means the integer part of $\frac{x}{h}$ and $w_k^{(\alpha)}$ are the normalized Grünwald weights which are defined by $w_k^{(\alpha)} = (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix}$.

The Grünwald-Letnikov definition (2) is simply a generalization of the ordinary discretization formula for integer order derivatives. The Riemann-Liouville and the Grünwald-Letnikov approaches coincide under relatively weak conditions; if $y(x)$ is continuous and $y'(x)$ is integrable in the interval $[0,x]$, then for every order $0<\alpha<1$ both the Riemann-Liouville (1) and the Grünwald-Letnikov derivatives (2) exist and coincide for any value inside the interval $[0,x]$. This fact of fractional calculus ensures the consistency of both definitions for most physical applications, where the functions are expected to be sufficiently smooth [4,9].

Fractional reaction-subdiffusion equation

The standard meanfield model for the evolution of the concentrations $a(x,t)$ and $b(x,t)$ of A and B particles is given by the reaction-diffusion equations;

$$\frac{\partial}{\partial t} a(x,t) = D \frac{\partial^2}{\partial x^2} a(x,t) - \varepsilon a(x,t) b(x,t), \tag{3}$$

$$\frac{\partial}{\partial t} b(x,t) = D \frac{\partial^2}{\partial x^2} b(x,t) - \varepsilon a(x,t) b(x,t), \tag{4}$$

where D is the diffusion coefficient assumed in this paper to equal for species and ε is the rate constant for the bimolecular reaction.

In order to generalize the reaction-diffusion problem to a reaction-subdiffusion problem, the subdiffusive motion of the particles must be dealt with. Seki *et al*. [20] and Yuste *et al*. [21] replaced Eqs. (3) and (4) with a set of reaction-subdiffusion equations in which both the motion and the reaction terms are affected by the subdiffusive character of the process;

$$\frac{\partial}{\partial t} a(x,t) = D_t^{1-\alpha} k_{\alpha} \frac{\partial^2}{\partial x^2} a(x,t) - \varepsilon a(x,t) b(x,t), \tag{5}$$

$$\frac{\partial}{\partial t} b(x,t) = D_t^{1-\alpha} k_{\alpha} \frac{\partial^2}{\partial x^2} b(x,t) - \varepsilon a(x,t) b(x,t), \tag{6}$$

where k_{α} is the generalized diffusion coefficient and $D_t^{1-\alpha}$ is the Riemann-Liouville fractional partial derivative of order $1-\alpha$. The fractional reaction-subdiffusion Eqs. (5) and (6) are decoupled, which is equivalent to solve the following fractional reaction-subdiffusion equation;

$$\frac{\partial}{\partial t} u(x,t) = D_t^{1-\alpha} [k_{\alpha} \frac{\partial^2}{\partial x^2} u(x,t) - \varepsilon u(x,t)] + f(x,t), \quad 0<t\leq T, \quad 0<x<L, \tag{7}$$

where $0 < \alpha < 1$ and ε is a positive constant. The Dirichlet boundary conditions for this problem are assumed as follows;

$$u(0,t) = \phi(t), \quad u(L,t) = \psi(t), \quad 0 < t \le T, \tag{8}$$

with an initial condition;

$$u(x,0) = \omega(x), \quad 0 \le x \le L. \tag{9}$$

In the last few years, many papers have studied the proposed model (7) - (9) [15,20-27]. The main aim of this paper is to adapt the fractional weighted average finite difference method (FDM) to study this model.

The plan of the paper is as follows; in section 3, some approximate formulae of the fractional derivatives and numerical finite difference scheme are given. In section 4, a stability analysis and an accuracy analysis of the presented method are given. In section 5, numerical studies for fractional reaction-subdiffusion model problem are presented. The paper ends with some conclusions in section 6.

Weighted average scheme for the fractional reaction-subdiffusion equation

In this section, the weighted average finite difference method is used to obtain the discretization finite difference formula of the reaction-subdiffusion Eq. (7). For some positive constant numbers M and N, following notations Δt and Δx are used at time-step length and space-step length, respectively. The coordinates of the mesh points are $x_j = j\Delta x \ \ (j = 0,1,...,N)$, and $t_m = m\Delta t$, $(m = 0,1,...,M)$ and the values of the solution $u(x,t)$ on these grid points are $u(x_j, t_m) \equiv u_j^m \cong U_j^m$, where $\Delta x = \frac{L}{N}$, and $\Delta t = \frac{T}{M}$. For more details on the discretization in fractional calculus, see [28,29].

In the first step, the ordinary differential operators are discretized as follows [17];

$$\frac{\partial u}{\partial t}|_{x_j,t_m} = \delta_t u_j^m + O(\Delta t) \equiv \frac{u_j^{m+1} - u_j^m}{\Delta t} + O(\Delta t), \tag{10}$$

and

$$\frac{\partial^2 u}{\partial x^2}|_{x_j,t_m} = \delta_{xx} u_j^m + O((\Delta x)^2) \equiv \frac{u_{j-1}^m - 2u_j^m + u_{j+1}^m}{(\Delta x)^2} + O((\Delta x)^2). \tag{11}$$

In the second step, the Riemann-Liouville operator is discretized as follows;

$$D_t^{1-\alpha} u(x,t)|_{x_j,t_m} = \delta_t^{1-\alpha} u_j^m + O((\Delta t)^p), \tag{12}$$

where p is the order of the approximation which depends on the choice of $w_k^{(1-\alpha)}$, and;

$$\delta_t^{1-\alpha} u_j^m \equiv \frac{1}{(\Delta t)^{1-\alpha}} \sum_{k=0}^{[\frac{t_m}{\Delta t}]} w_k^{(1-\alpha)} u(x_j, t_m - kh) = \frac{1}{(\Delta t)^{1-\alpha}} \sum_{k=0}^{m} w_k^{(1-\alpha)} u_j^{m-k}, \tag{13}$$

where $[\frac{t_m}{\Delta t}]$ means the integer part of $\frac{t_m}{\Delta t}$. There are many choices of the weights $w_k^{(\alpha)}$ [9,28], so the above formula is not unique. Denoting the generating function of the weights $w_k^{(\beta)}$ by $w(z,\beta)$, i.e.;

$$w(z,\beta) = \sum_{k=0}^{\infty} w_k^{(\beta)} z^k.$$

If

$$w(z,\beta) = (1-z)^{\beta}, \tag{14}$$

then (12) gives the backward difference formula of the first order, which is called the Grünwald-Letnikov formula. The coefficients $w_k^{(\beta)}$ can be evaluated by the following formula;

$$w_k^{(\beta)} = (1 - \frac{\beta+1}{k}) w_{k-1}^{(\beta)}, \qquad w_0^{(\beta)} = 1. \tag{15}$$

For $\alpha = 1$ the operator $D_t^{1-\alpha}$ becomes the identity operator, so that the consistency of Eqs. (12) and (13) requires $w_0^{(0)} = 1$, and $w_k^{(0)} = 0$ (from (15)) for $k \geq 1$, which in turn means that $w(z,0) = 1$.
Now, the finite difference scheme of the fractional reaction-subdiffusion Eq. (7) is obtained. In this study, we take $k_\alpha = \varepsilon = 1$.
To achieve this aim, Eq. (7) is evaluated at the points of the grid (x_j, t_m);

$$[u_t(x,t) - D_t^{1-\alpha} u_{xx}(x,t) + D_t^{1-\alpha} u(x,t)]_{x_j,t_m} = f(x_j, t_m). \tag{16}$$

Then, in the above Eq.(16), the first order time-derivative is replaced by the forward difference formula (10) and the second order space-derivative by the three-point centered formula (11) with respect to the weighed average formula (12) at the times t_m and t_{m+1};

$$\delta_t u_j^m - [\lambda \delta_t^{1-\alpha} \delta_{xx} u_j^m + (1-\lambda) \delta_t^{1-\alpha} \delta_{xx} u_j^{m+1}] + \delta_t^{1-\alpha} u_j^m - f(x_j, t_m) = T_j^m, \tag{17}$$

with $\lambda \in [0,1]$ being the weight factor. T_j^m is the resulting truncation error. The standard difference formula is given by;

$$\delta_t u_j^m - [\lambda \delta_t^{1-\alpha} \delta_{xx} u_j^m + (1-\lambda) \delta_t^{1-\alpha} \delta_{xx} u_j^{m+1}] + \delta_t^{1-\alpha} u_j^m - f(x_j, t_m) = 0. \tag{18}$$

Now, by substituting from the difference operators given by (10), (11) and (13) in Eq.(18), the following scheme can be obtained.

$$-\phi U_{j-1}^{m+1} + (1+2\phi) U_j^{m+1} - \phi U_{j+1}^{m+1} = R, \tag{19}$$

where

$$\phi = (1-\lambda)\bar{S}, \quad \bar{S} = \frac{(\Delta t)^{\alpha}}{(\Delta x)^2}, \quad S = (\Delta t)^{\alpha}, \tag{20}$$

and

$$\begin{aligned} R = U_j^m + \bar{S}\sum_{r=0}^{m}[\lambda w_r^{(1-\alpha)} + (1-\lambda)w_{r+1}^{(1-\alpha)}][U_{j-1}^{m-r} - 2U_j^{m-r} + U_{j+1}^{m-r}] \\ - S\sum_{r=0}^{m} w_r^{(1-\alpha)} U_j^{m-r} - (\Delta t) f(x_j, t_m), \quad j = 1,2,...,N-1. \end{aligned} \tag{21}$$

Eq. (19) is the fractional weighted average finite difference scheme considered in this paper. Fortunately, Eq. (19) is a tridiagonal system that can be solved using the Thomas algorithm [29]. In the case of $\lambda = 1$ and $\lambda = \frac{1}{2}$ the backward Euler fractional quadrature method and the Crank-Nicholson fractional quadrature methods are available, respectively, which have been studied, e.g., in ([30], [31]), but at $\lambda = 0$ the scheme is called fully implicit.
Now, to study the solvability of the proposed finite difference method, let;

$U^0 = [w(x_1), w(x_2),..., w(x_{N-1})]^T$, and $U^m = [U_1^m, U_2^m,..., U_{N-1}^m]^T, \quad m = 0,1,..., M$,
respectively. Therefore, the explicit difference approximation scheme (19)-(21) can be written in matrix form:

$$AU^{m+1} = b^m, \tag{22}$$

where

$$A = \begin{bmatrix} (1+2\phi) & -\phi & & & & 0 \\ -\phi & (1+2\phi) & -\phi & & & \\ & \ddots & \ddots & \ddots & & \\ & & -\phi & (1+2\phi) & -\phi & \\ & & & -\phi & (1+2\phi) & -\phi \\ & 0 & & & & \end{bmatrix}$$

and $b^m = R$.

Theorem 1 *The difference Eq. (22) is uniquely solvable.*

Proof Because $\phi > 0$, then the coefficient matrix of the difference equation (22) is a strictly diagonally dominant matrix. Therefore A is a nonsingular matrix; this proves Theorem 1.

Lemma 1 *The coefficients* $w_k^{1-\alpha}$, *(* $k = 0,1,...$ *) satisfy;*

(1) $w_0^{1-\alpha} = 1$; $w_1^{1-\alpha} = \alpha - 1$; $w_k^{1-\alpha} < 0$, $k = 2,3,...$;

(2) $\sum_{k=0}^{\infty} w_k^{1-\alpha} = 1;\ \forall n \in N^+,\ -\sum_{k=1}^{n} w_k^{1-\alpha} < 1.$

Proof See [2].

Lemma 2

$$\tau^{\alpha-1}\sum_{k=0}^{m} w_k^{1-\alpha} = \frac{1}{\Gamma(\alpha)} + O(\tau).$$

Proof See [2].

Stability analysis

In this section, the John von Neumann method is used to study the stability analysis of the weighted average scheme (19). In this study the source term (i.e., $f(x,t)=0$) is neglected.

Proposition 1 *Assuming that* $U_j^m = \xi_m e^{\mathrm{i}qj\Delta x}$, *then*

$$\begin{aligned}&[1+4(1-\lambda)\bar{S}\sin^2(\frac{q\Delta x}{2})]\xi_{m+1} + S\sum_{r=0}^{m} w_r^{(1-\alpha)}\xi_{m-r} - \xi_m \\ &\quad + 4\bar{S}\sin^2(\frac{q\Delta x}{2})\sum_{r=0}^{m}[\lambda w_r^{(1-\alpha)} + (1-\lambda)w_{r+1}^{(1-\alpha)}]\xi_{m-r} = 0.\end{aligned} \tag{23}$$

Proof By using (21), Eq. (19) can be written in the following form

$$\begin{aligned}&-\phi U_{j-1}^{m+1} + (1+2\phi)U_j^{m+1} - \phi U_{j+1}^{m+1} = U_j^m - S\sum_{r=0}^{m} w_r^{(1-\alpha)}U_j^{m-r} \\ &+ \bar{S}\sum_{r=0}^{m}[\lambda w_r^{(1-\alpha)} + (1-\lambda)w_{r+1}^{(1-\alpha)}][U_{j-1}^{m-r} - 2U_j^{m-r} + U_{j+1}^{m-r}].\end{aligned} \tag{24}$$

In the fractional John von Neumann stability procedure, the stability of the fractional WAM is decided by putting $U_j^m = \xi_m e^{\mathrm{i}qj\Delta x}$. Inserting this expression into the weighted average difference scheme (24), the following is obtained.

$$\begin{aligned}&-\phi\xi_{m+1}e^{\mathrm{i}q(j-1)\Delta x} + (1+2\phi)\xi_{m+1}e^{\mathrm{i}qj\Delta x} - \phi\xi_{m+1}e^{\mathrm{i}q(j+1)\Delta x} - \xi_m e^{\mathrm{i}qj\Delta x} = \bar{S}\sum_{r=0}^{m}[\lambda w_r^{(1-\alpha)} \\ &+ (1-\lambda)w_{r+1}^{(1-\alpha)}][e^{\mathrm{i}q(j-1)\Delta x} - 2e^{\mathrm{i}qj\Delta x} + e^{\mathrm{i}q(j+1)\Delta x}]\xi_{m-r} - S\sum_{r=0}^{m} w_r^{(1-\alpha)}\xi_{m-r}e^{\mathrm{i}qj\Delta x},\end{aligned} \tag{25}$$

substituting by $\phi = (1-\lambda)\bar{S}$ and dividing (25) by $e^{\mathrm{i}qj\Delta x}$ results in;

$$\begin{aligned}&-(1-\lambda)\bar{S}\xi_{m+1}e^{-\mathrm{i}q\Delta x} + (1+2(1-\lambda)\bar{S})\xi_{m+1} - (1-\lambda)\bar{S}\xi_{m+1}e^{\mathrm{i}q\Delta x} - \xi_m \\ &- \bar{S}\sum_{r=0}^{m}[\lambda w_r^{(1-\alpha)} + (1-\lambda)w_{r+1}^{(1-\alpha)}][e^{-\mathrm{i}q\Delta x} - 2 + e^{\mathrm{i}q\Delta x}]\xi_{m-r} + S\sum_{r=0}^{m} w_r^{(1-\alpha)}\xi_{m-r} = 0.\end{aligned} \tag{26}$$

Using the known Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ in Eq.(26) we obtain;

$$[1+2(1-\lambda)\bar{S} - 2(1-\lambda)\bar{S}\cos(q\Delta x)]\xi_{m+1} - \xi_m - \bar{S}\sum_{r=0}^{m}[\lambda w_r^{(1-\alpha)} + (1-\lambda)w_{r+1}^{(1-\alpha)}][-2+2\cos(q\Delta x)]\xi_{m-r} + S\sum_{r=0}^{m} w_r^{(1-\alpha)}\xi_{m-r} = 0. \tag{27}$$

Under some simplifications, the above equation (27) can be written in the required form (23). This completes the proof of the proposition.

Proposition 2 *Assuming in proposition 1 that* $\xi_{m+1} = \eta\xi_m$*, then the scheme will be stable as long as:*

$$-1,\leq \frac{1-4\bar{S}\sin^2(\frac{q\Delta x}{2})\sum_{r=0}^{m}[\lambda w_r^{(1-\alpha)} + (1-\lambda)w_{r+1}^{(1-\alpha)}]\eta^{-r} - S\sum_{r=0}^{m} w_r^{(1-\alpha)}\eta^{-r}}{1+4(1-\lambda)\bar{S}\sin^2(\frac{q\Delta x}{2})} \leq 1,. \tag{28}$$

Proof The stability of the scheme (23) is determined by the behavior of ξ_m. In the John von Neumann method, the stability analysis is carried out using the amplification factor η defined by

$$\xi_{m+1} = \eta\xi_m. \tag{29}$$

Of course, η depends on m. But, for the moment, assume that, as in [30], η is independent of time. Then, inserting this expression (29) into Eq. (23), one gets;

$$[1+4(1-\lambda)\bar{S}\sin^2(\frac{q\Delta x}{2})]\eta\xi_m + S\sum_{r=0}^{m} w_r^{(1-\alpha)}\eta^{-r}\xi_m - \xi_m + 4\bar{S}\sin^2(\frac{q\Delta x}{2})\sum_{r=0}^{m}[\lambda w_r^{(1-\alpha)} + (1-\lambda)w_{r+1}^{(1-\alpha)}]\eta^{-r}\xi_m = 0, \tag{30}$$

dividing Eq. (30) by ξ_m to obtain the following formula of η;

$$\eta = \frac{1-4\bar{S}\sin^2(\frac{q\Delta x}{2})\sum_{r=0}^{m}[\lambda w_r^{(1-\alpha)} + (1-\lambda)w_{r+1}^{(1-\alpha)}]\eta^{-r} - S\sum_{r=0}^{m} w_r^{(1-\alpha)}\eta^{-r}}{1+4(1-\lambda)\bar{S}\sin^2(\frac{q\Delta x}{2})}. \tag{31}$$

The scheme will be stable as long as $|\eta| \leq 1$ using Eq. (31), which completes the proof of proposition 2 and (28).

Proposition 3 *Assuming in proposition 2 that* $\psi = \bar{S}\sin^2(\frac{q\Delta x}{2})$ *and that;*

$$L_m = \frac{2 - S\sum_{r=0}^{m} w_r^{(1-\alpha)}(-1)^{-r}}{4\{(2\lambda-1)[1-\sum_{r=1}^{m}(-1)^{r-1}w_r^{(1-\alpha)}] + (-1)^m(1-\lambda)w_{m+1}^{(1-\alpha)}\}}, \tag{32}$$

then the scheme will be stable when

$$\psi \le L_m. \tag{33}$$

Proof. By considering the time-independent limit value $\eta = -1$ and since;

$1 + 4(1-\lambda)\bar{S}\sin^2(\frac{q\Delta x}{2}) > 0$, then from Eq. (28) we have;

$$\begin{aligned} -1-4(1-\lambda)\bar{S}\sin^2(\frac{q\Delta x}{2}) \le 1 - 4\bar{S}\sin^2(\frac{q\Delta x}{2})\sum_{r=0}^{m}(-1)^{-r}[\lambda w_r^{(1-\alpha)} + (1-\lambda)w_{r+1}^{(1-\alpha)}] \\ - S\sum_{r=0}^{m} w_r^{(1-\alpha)}(-1)^{-r}. \end{aligned} \tag{34}$$

From the above inequality (34),

$$\begin{aligned} -2-4(1-\lambda)\bar{S}\sin^2(\frac{q\Delta x}{2}) + S\sum_{r=0}^{m} w_r^{(1-\alpha)}(-1)^{-r} + 4\bar{S}\sin^2(\frac{q\Delta x}{2})\sum_{r=0}^{m}[\lambda w_r^{(1-\alpha)} \\ + (1-\lambda)w_{r+1}^{(1-\alpha)}](-1)^{-r} \le 0. \end{aligned} \tag{35}$$

Using the assumption that $\psi = \bar{S}\sin^2(\frac{q\Delta x}{2})$, from (35) it is found that;

$$-2-4(1-\lambda)\psi + S\sum_{r=0}^{m} w_r^{(1-\alpha)}(-1)^{-r} + 4\psi\sum_{r=0}^{m}[\lambda w_r^{(1-\alpha)} + (1-\lambda)w_{r+1}^{(1-\alpha)}](-1)^{-r} \le 0. \tag{36}$$

We can write Eq. (36) in the following form;

$$-2-4(1-\lambda)\psi + S\sum_{r=0}^{m} w_r^{(1-\alpha)}(-1)^{-r} + 4\psi[(1-2\lambda)\sum_{r=1}^{m}(-1)^{r-1}w_r^{(1-\alpha)} + \lambda + (-1)^m(1-\lambda)w_{m+1}^{(1-\alpha)}] \le 0. \tag{37}$$

Using the assumption from Eqs. (32) - (33) in (37), one finds that the mode is stable when $\psi \le L_m$. This ends the proof of the proposition.

Theorem 2 *The fractional weighted average finite difference scheme derived in (19) is stable at* $0 \le \lambda \le \frac{1}{2}$ *under the following stability criterion;*

$$\frac{1}{\bar{S}} \geq \frac{4(2\lambda-1)2^{-\alpha}}{1-S2^{-\alpha}}. \tag{38}$$

Proof Since L_m depends on m, L_m tends towards its limit value;

$$L = \lim_{m\to\infty} L_m. \tag{39}$$

In this limit (39), the stability condition is;

$$\psi \leq \frac{2-S\sum_{r=0}^{\infty} w_r^{(1-\alpha)}(-1)^{-r}}{4\{(2\lambda-1)[1-\sum_{r=1}^{\infty}(-1)^{r-1}w_r^{(1-\alpha)}]+\lim_{m\to\infty}(-1)^m(1-\lambda)w_{m+1}^{(1-\alpha)}\}}, \tag{40}$$

but from Eq. (14) with $z=-1$ one sees that $\sum_{r=0}^{\infty}(-1)^r w_r^{(1-\alpha)} = 2^{1-\alpha}$, so that;

$$L = \frac{2-S2^{1-\alpha}}{4\{(2\lambda-1)[2-2^{1-\alpha}]+\lim_{m\to\infty}(-1)^m(1-\lambda)w_{m+1}^{(1-\alpha)}\}}, \tag{41}$$

and by replacing $\sin^2(\frac{q\Delta x}{2})$ by its highest value, so $\psi \to \bar{S}$ as $\sin^2(\frac{q\Delta x}{2}) \to 1$ and $\lim_{m\to\infty}(-1)^m(1-\lambda)w_{m+1}^{(1-\alpha)} = 0$, then from (40)-(41), it is found that the sufficient condition for the presented method is stable, and this completes the proof of Eq.(38) and theorem 2.

Remark For $\frac{1}{2} < \lambda \leq 1$ the stability condition (19) can be satisfied under specific values of $\bar{S} = \frac{(\Delta t)^\alpha}{(\Delta x)^2}$.

Numerical results

In this section, the proposed method is tested by considering the following numerical examples.

Example 1 Consider the initial-boundary value problem of fractional reaction-subdiffusion equation type with a non-homogeneous term;

$$\frac{\partial}{\partial t}u(x,t) = D_t^{1-\alpha}[\frac{\partial^2}{\partial x^2}u(x,t)-u(x,t)]+(1+\alpha)e^x t^\alpha, \quad 0<x<1, \quad 0\leq t\leq T, \tag{42}$$

with the following boundary conditions $u(0,t)=t^{\alpha+1}, \quad u(1,t)=et^{\alpha+1}, \quad 0\le t\le T$, and the initial condition $u(x,0)=0, \quad 0\le x\le 1$.

The exact solution of this problem is $u(x,t)=e^{x}t^{\alpha+1}$.

The behavior of the analytical solution and the numerical solution of the proposed fractional reaction-subdiffusion Eq. (42) by means of the weighted average FDM with different values of λ, α, Δt, Δx and final time T are presented in **Figures 1 - 5**.

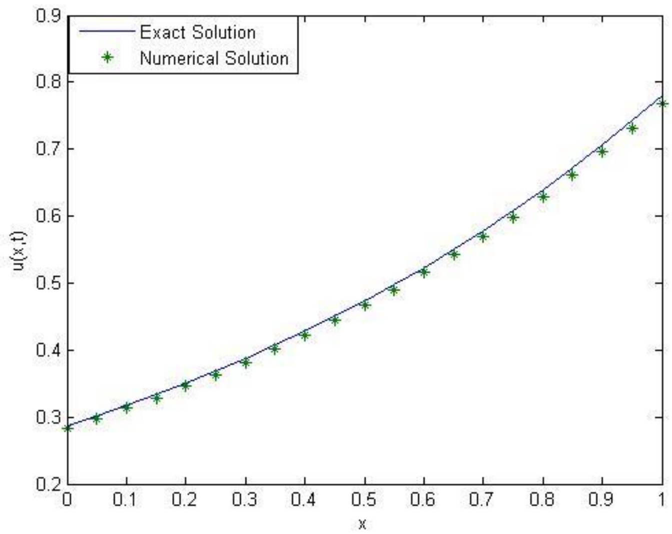

Figure 1 The behavior of the exact solution and the numerical solution of (42) by means of the proposed method at $\lambda=0$ for $\alpha=0.8$, $\Delta x=\dfrac{1}{40}$, $\Delta t=\dfrac{1}{20}$, and $T=0.5$.

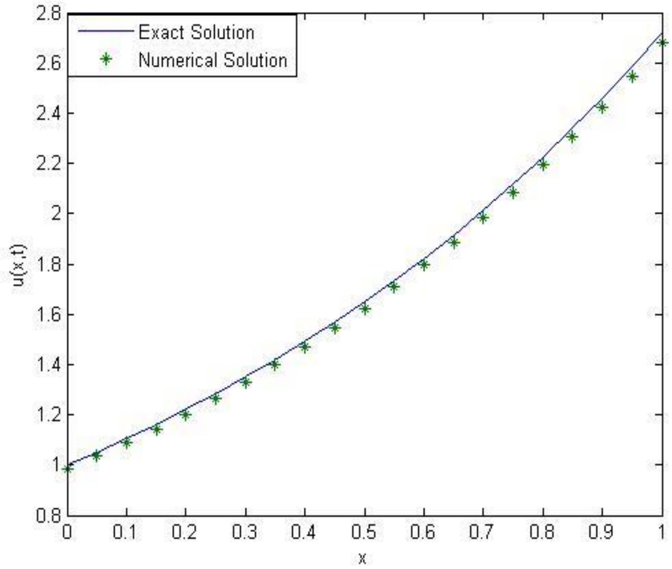

Figure 2 The behavior of the exact solution and the numerical solution of (42) by means of the proposed method at $\lambda = 0.5$ for $\alpha = 0.3$, $\Delta x = \frac{1}{60}$, $\Delta t = \frac{1}{70}$, and $T = 1$.

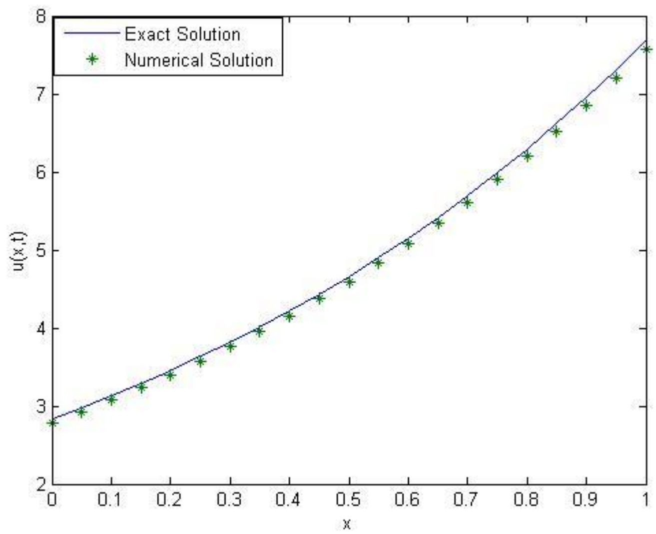

Figure 3 The behavior of the exact solution and the numerical solution of (42) by means of the proposed method at $\lambda = 1$ for $\alpha = 0.5$, $\Delta x = \frac{1}{60}$, $\Delta t = \frac{1}{20}$, and $T = 2$.

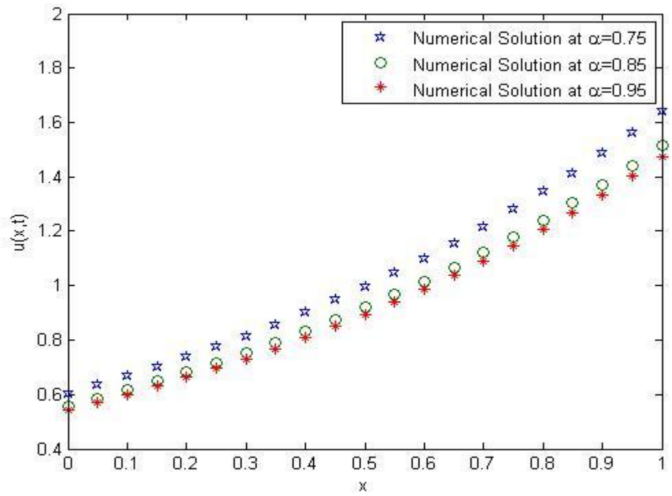

Figure 4 The behavior of the numerical solution of (42) by means of the proposed method at $\lambda = 0.5$, $\Delta x = \dfrac{1}{20}$, $\Delta t = \dfrac{1}{20}$, and $T = 0.5$, with different values of α.

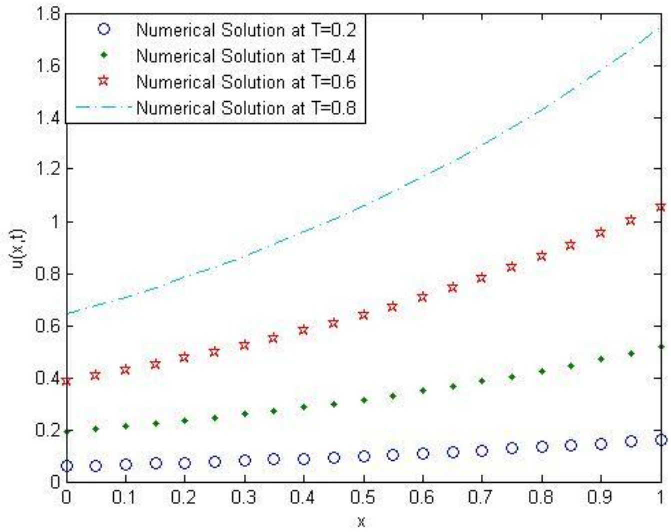

Figure 5 The behavior of the numerical solution of (42) by means of the proposed method at $\lambda = 0.5$, $\Delta x = \dfrac{1}{40}$, $\Delta t = \dfrac{1}{40}$, $\alpha = 0.7$, with different values of T.

Example 2 Consider the following initial-boundary problem of the fractional reaction-subdiffusion equation:

$$u_t(x,t) = D_t^{1-\alpha}[u_{xx}(x,t) - u(x,t)] + f(x,t), \tag{43}$$

on a finite domain $0 < x < 1$, with $0 \le t \le T$, $0 < \alpha < 1$ and the following source term;

$$f(x,t) = 2\left(\frac{t(\Gamma(2+\alpha)) + (\pi^2+1)t^{\alpha+1}}{\Gamma(2+\alpha)}\right)\sin(\pi x)$$

under the boundary conditions $u(0,t) = u(1,t) = 0$, and the initial condition $u(x,0) = 0$.
The exact solution of Eq. (43) in this case is $u(x,t) = t^2\sin(\pi x)$.

The behavior of the exact solution and the numerical solution of the proposed fractional reaction-subdiffusion Eq. (43) by means of the fractional weighted average FDM with different values of λ, α, Δt, Δx, and final time T are presented in **Figures 6 - 9**.

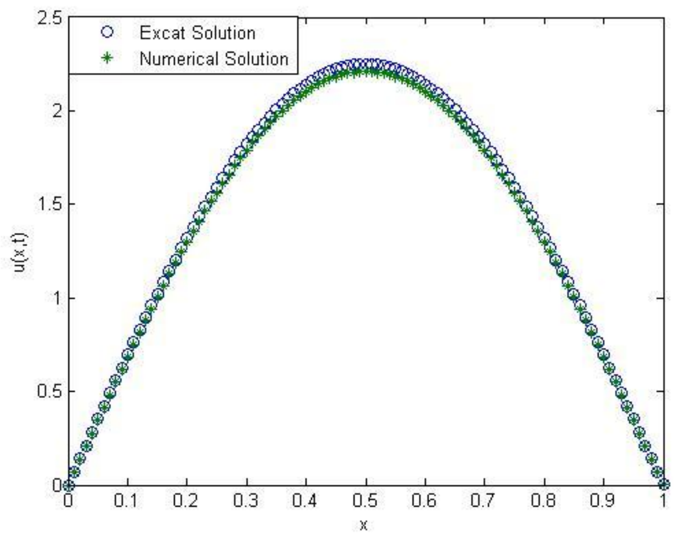

Figure 6 The behavior of the exact solution and the numerical solution of (43) by means of the proposed method at $\lambda = 0$ for $\alpha = 0.5$, $\Delta x = \dfrac{1}{70}$, $\Delta t = \dfrac{1}{60}$, and $T = 2$.

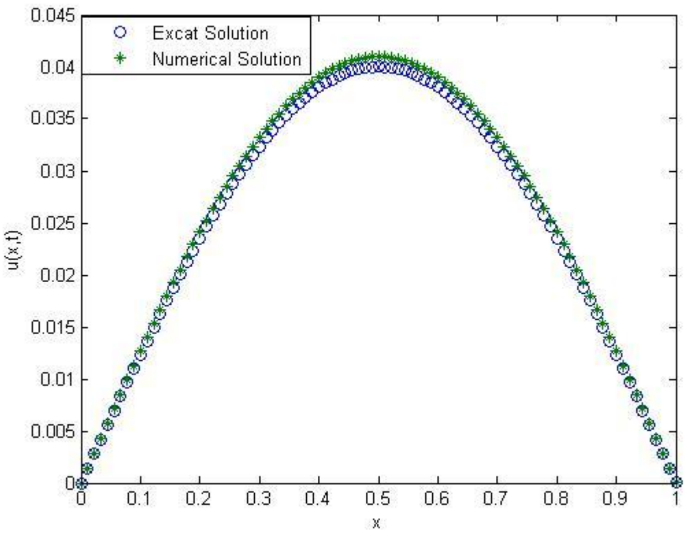

Figure 7 The behavior of the exact solution and the numerical solution of (43) by means of the proposed method at $\lambda = 0.5$ for $\alpha = 0.2$, $\Delta x = \frac{1}{60}$, $\Delta t = \frac{1}{20}$, and $T = 0.2$.

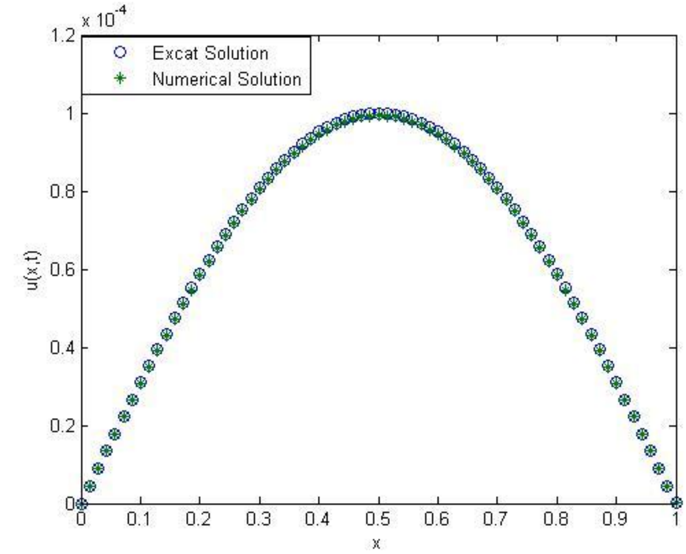

Figure 8 The behavior of the exact solution and the numerical solution of (43) by means of the proposed method at $\lambda = 1$ for $\alpha = 0.9$, $\Delta x = \frac{1}{50}$, $\Delta t = \frac{1}{150}$, and $T = 0.01$.

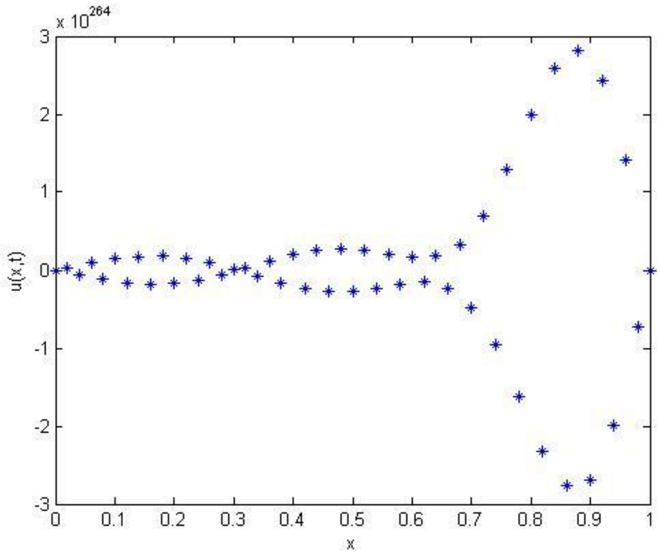

Figure 9 The behavior of the numerical solution of the proposed problem (43) by means of the proposed method for $\lambda = 1$, $\alpha = 0.4$, $\Delta x = \frac{1}{40}$, $\Delta t = \frac{1}{100}$, and $T = 1$.

From the previous figure, it can be seen that the numerical solution is unstable, since the stability condition (38) is not satisfied.

Tables 1 and **2** show the magnitude of the maximum error between the numerical solution and the exact solution obtained by using the fractional weighted average FDM discussed above at $\lambda = 0$ and $\lambda = 0.5$ respectively, with different values of Δx, Δt and the final time T.

Table 1 The maximum error with different values of Δx, Δt at $\lambda = 0$, $\alpha = 0.5$ and $T = 0.2$.

Δx	Δt	**Maximum error**
$\frac{1}{10}$	$\frac{1}{20}$	0.00687
$\frac{1}{20}$	$\frac{1}{40}$	0.00411
$\frac{1}{50}$	$\frac{1}{40}$	0.00404
$\frac{1}{50}$	$\frac{1}{70}$	0.00385
$\frac{1}{100}$	$\frac{1}{70}$	0.00384

Table 2 The maximum error with different values of Δx, Δt at $\lambda = 0.5$, $\alpha = 0.3$ and $T = 0.1$.

Δx	Δt	**Maximum error**
$\frac{1}{20}$	$\frac{1}{10}$	0.02385
$\frac{1}{20}$	$\frac{1}{30}$	0.00808
$\frac{1}{40}$	$\frac{1}{30}$	0.00805
$\frac{1}{50}$	$\frac{1}{70}$	0.00071
$\frac{1}{80}$	$\frac{1}{60}$	0.00021

Conclusions

This paper presents a class of numerical methods for solving fractional reaction-subdiffusion equations. This class of method is very close to the weighted average finite difference method. Special attention is given to study the stability of the fractional weighted average FDM. To execute this aim John von Neumann stability analysis is used. From the theoretical study it can be concluded that this procedure is suitable for the fractional finite weighted average FDM, and leads to very good predictions for the stability bounds. The stability of the fractional finite weighted average FDM presented depends strongly on the value of the weighting parameter λ. Numerical solutions and exact solutions of the proposed problem are compared, and the derived stability condition is checked numerically. From this comparison, it can be concluded that the numerical solutions are in excellent agreement with the exact solutions. All computations in this paper are run using Matlab programming.

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