

## On System of Time-Fractional Partial Differential Equations

Umer HAYAT<sup>1</sup>, Abid KAMRAN<sup>1</sup>, Bibi AMBREEN<sup>1</sup>, Ahmet YILDIRIM<sup>2</sup> and Syed Tauseef MOHYUD-DIN<sup>1,\*</sup>

<sup>1</sup>Department of Mathematics, HITEC University, Taxila Cantt, Pakistan

<sup>2</sup>Zeytinolani Mah, Urla-Izmir, Turkey

(\*Corresponding author's e-mail: syedtauseefs@hotmail.com)

Received: 23 April 2012, Revised: 15 May 2012, Accepted: 25 June 2013

### Abstract

In this paper, we apply Homotopy Perturbation Transformation Method (HPM) using the Laplace transformation to tackle time-fractional systems of Partial Differential equations. The proposed technique is fully compatible with the complexity of these problems and obtained results are highly encouraging. Numerical results coupled with graphical representations explicitly reveal the complete reliability and efficiency of the suggested algorithm.

**Keywords:** Fractional partial differential equations, homotopy perturbation method, Laplace transform, system of PDEs, HPTM

### Introduction

Nonlinear partial differential equations [1-21] are of extreme importance in applied and engineering sciences. The thorough study of the literature reveals that most of the physical phenomena are nonlinear in nature and hence there is a dire need to find their appropriate solutions, see [1-21] and the references therein. It is to be highlighted that the expansion idea was also used to obtain exact solutions around an integrable ODE [22], and exact solutions of traveling wave type can be generated through the transformed rational function method, see [23]. Furthermore, multiple wave solutions can be computed by using the multiple exp-function method [24] and a new kind of exact solution with generalized separation of variables can be recognized through the invariant subspace method [25]. On the other hand, the linear superposition [22,23] principle has been used to solve Hirota bilinear differential equations. Recently, scientists have observed that the number of real time problems is modeled by fractional nonlinear differential equations, which are very hard to tackle. We apply the Homotopy Perturbation Transformation Method (HPTM) to solve a time-fractional system of partial differential equations.

$$\begin{aligned} D_t^\alpha u + R_1(u, v, w) + N_1(u, v, w) &= g_1, \\ D_t^\alpha v + R_2(u, v, w) + N_2(u, v, w) &= g_2, \\ D_t^\alpha w + R_3(u, v, w) + N_3(u, v, w) &= g_3, \end{aligned} \quad (1)$$

with initial conditions;

$$\begin{aligned} u(x, 0) &= f_1(x), \\ v(x, 0) &= f_2(x), \\ w(x, 0) &= f_3(x). \end{aligned} \quad (2)$$

$D_t^\alpha$  is the time-fractional derivative with  $0 < \alpha \leq 1$ ,  $R_j, 1 \leq j \leq 3$  and  $N_j, 1 \leq j \leq 3$  are the linear and non-linear operators and  $g_1, g_2$  and  $g_3$  are source terms. The fractional derivative is considered in the Caputo sense. It is to be highlighted that such equations arise frequently in applied, physical and

engineering sciences. The proposed algorithm is fully synchronized with the complexity of fractional differential equations. Numerical results coupled with graphical representations explicitly reveal the complete reliability and efficiency of the proposed algorithm.

### Definitions: [13-17]

**Definition 1** A real function  $f(x)$ ,  $x > 0$ , is said to be in the space  $C_\mu$ ,  $\mu \in \mathbb{R}$  if there exists a real number  $p(> \mu)$ , such that  $f(x) = x^p f_1(x)$ , where  $f_1(x) \in C[0, \infty)$ , and it is said to be in the space  $C_\mu^\infty$  if and only if  $f^{(m)} \in C_\mu$ ,  $\mu \geq 1$ ,  $m \in \mathbb{N}$ .

**Definition 2** The Riemann-Liouville fractional integral operator of order  $\alpha \geq 0$ , of a function  $f \in C_\mu$ ,  $\mu \geq -1$ , is defined as;

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \quad \alpha > 0, x > 0,$$

$$J^0 f(x) = f(x).$$

Properties of the operator  $J^\alpha$  can be found in [13-17], we mention only the following.

For  $f \in C_\mu$ ,  $\mu \geq -1$ ,  $\alpha, \beta \geq 0$  and  $\gamma > -1$ ;

1.  $J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t),$
2.  $J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t),$
3.  $J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}.$

**Definition 3** The fractional derivative of  $f(x)$  in the Caputo sense is defined as;

$$D_*^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt,$$

for  $m-1 < \alpha \leq m$ ,  $m \in \mathbb{Z}$ ,  $x > 0$ ,  $f \in C_{-1}^m$ .

also, we need here two of its basic properties.

**Lemma 1** if  $m-1 < \alpha \leq m$ ,  $m \in \mathbb{N}$  and  $f \in C_\mu^m$ ,  $\mu \geq -1$ , then

$$D_*^\alpha J^\alpha f(x) = f(x),$$

and

$$J^\alpha D_*^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0^+) x^k}{k!}, \quad x > 0.$$

### Analysis of Modified HPTM [13-17]

To illustrate the basic idea of this method, we consider a general fractional nonlinear non-homogeneous partial differential equation with initial conditions of the form.

$$D_t^\alpha u(x, t) + Ru(x, t) + Nu(x, t) = g(x, t), \quad (3)$$

$$u(x, 0) = h(x), \quad u_t(x, 0) = f(x). \quad (4)$$

where  $g(x, t)$  is the source term,  $N$  represents the general non-linear differential operator and  $R$  is the linear differential operator,  $D_t^\alpha u(x, t)$  is the Caputo fractional derivative of function  $u(x, t)$  which is defined as;

where  $\Gamma(.)$  denotes the Gamma function. The properties of the fractional derivative can be found in [1,2,4,6]. Laplace transform (denoted throughout this paper by  $L$ ) of the Caputo operator is an important property which will be used in this paper.

Taking the Laplace transform on both sides;

Using the property of the Laplace transform, we have;

Operating with the Laplace inverse on both sides;

where  $G(x, t)$  represents the term arising from the source term and the prescribed initial conditions. Then we apply the homotopy perturbation method, the basic assumption is that the solutions can be written as a power series in  $p$ .

and the nonlinear term can be decomposed as;

where  $p \in [0, 1]$  is an embedding parameter.  $H_n(u)$  is He's polynomials which can be generated by;

Substituting Eqs. (11) and (12) in Eq. (9) we get;

Equating the terms with identical powers in  $p$ , we obtain the following approximations;

•  
•  
•

The best approximations for the solution are;

$$u(x, t) = u_0 + u_1 + u_2 + u_3 + \dots \quad (17)$$

This method does not resort to linearization or assumptions of weak nonlinearity. The solution generated in the form of a general solution and it is more realistic compared to the method of simplifying the physical problems.

### Numerical examples

**Example 1** We first consider the homogenous linear system;

$$\begin{cases} D_t^\alpha u - v_x + (u + v) = 0, \\ D_t^\alpha v - u_x + (u + v) = 0, \end{cases} \quad (18)$$

with initial conditions;

$$u(x, 0) = \sinh(x), \quad v(x, 0) = \cosh(x). \quad (19)$$

Taking the Laplace transform on both sides;

$$\begin{cases} L[D_t^\alpha u] = L[v_x - (u + v)], \\ L[D_t^\alpha v] = L[u_x - (u + v)], \end{cases}$$

Using the property of Laplace transform, we have;

$$\begin{cases} L[u(x, t)] = \frac{\sinh(x)}{s} + \frac{1}{s^\alpha} L[v_x - (u + v)], \\ L[v(x, t)] = \frac{\cosh(x)}{s} + \frac{1}{s^\alpha} L[u_x - (u + v)], \end{cases}$$

Operating with the Laplace inverse on both sides;

$$\begin{cases} u(x, t) = \sinh(x) + L^{-1}\left[\frac{1}{s^\alpha} L[v_x - (u + v)]\right], \\ v(x, t) = \cosh(x) + L^{-1}\left[\frac{1}{s^\alpha} L[u_x - (u + v)]\right], \end{cases}$$

Then, we apply the homotopy perturbation method.

$$\begin{aligned} \sum_{n=0}^{\infty} p^n u_n(x, t) &= \sinh(x) + pL^{-1}\left[\frac{1}{s^\alpha} L\left(\sum_{n=0}^{\infty} p^n v_{nx}(x, t) - \left(\sum_{n=0}^{\infty} p^n u_n(x, t) + \sum_{n=0}^{\infty} p^n v_n(x, t)\right)\right)\right], \\ \sum_{n=0}^{\infty} p^n v_n(x, t) &= \cosh(x) + pL^{-1}\left[\frac{1}{s^\alpha} L\left(\sum_{n=0}^{\infty} p^n u_{nx}(x, t) - \left(\sum_{n=0}^{\infty} p^n u_n(x, t) + \sum_{n=0}^{\infty} p^n v_n(x, t)\right)\right)\right], \end{aligned}$$

Comparing the coefficient of like power of  $p$ , we have;

$$\begin{aligned} p^0: & \begin{cases} u_0(x, t) = \sinh(x), \\ v_0(x, t) = \cosh(x), \end{cases} \\ p^1: & \begin{cases} u_1(x, t) = -\cosh(x) \frac{t^\alpha}{\Gamma(\alpha+1)}, \\ v_1(x, t) = -\sinh(x) \frac{t^\alpha}{\Gamma(\alpha+1)}, \end{cases} \end{aligned}$$

$$p^2: \begin{cases} u_2(x, t) = \sinh(x) \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ v_2(x, t) = \cosh(x) \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \end{cases}$$

$$p^3: \begin{cases} u_3(x, t) = -\sinh(x) \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \\ v_3(x, t) = -\sinh(x) \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \end{cases}$$

$$\vdots$$

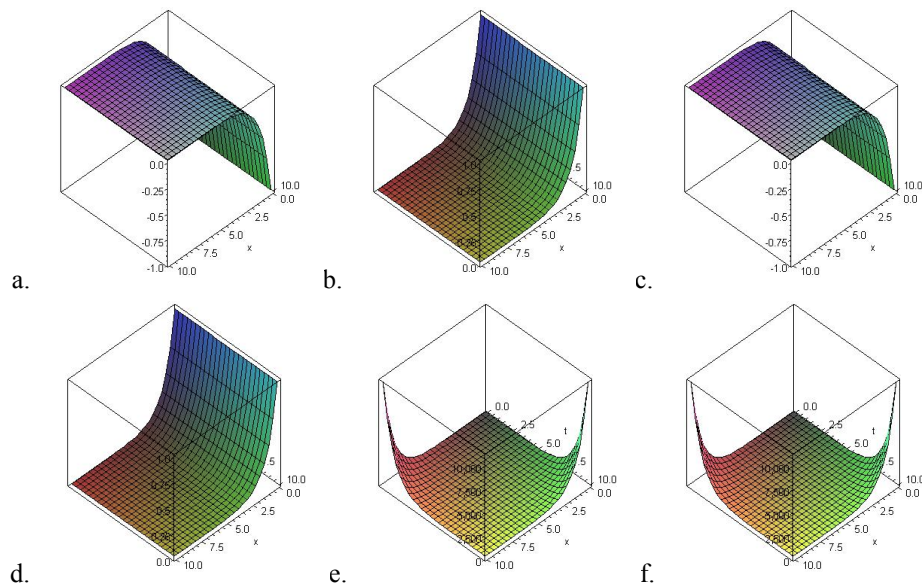
The solution in the series form is given by;

$$\begin{aligned} u(x, t) &= \sum_{n=0}^{\infty} p^n u_n(x, t) = u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \dots, \\ v(x, t) &= \sum_{n=0}^{\infty} p^n v_n(x, t) = v_0 + p v_1 + p^2 v_2 + p^3 v_3 + \dots, \\ \begin{cases} u(x, t) = \sinh(x) \left[ 1 + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \dots \right] - \cosh(x) \left[ t + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots \right], \\ v(x, t) = \cosh(x) \left[ 1 + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \dots \right] - \sinh(x) \left[ t + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots \right], \end{cases} \end{aligned} \quad (20)$$

For the special case  $\alpha = 1$ , we obtain the from;

$$\begin{cases} u(x, t) = \sinh(x - t). \\ v(x, t) = \cosh(x - t). \end{cases} \quad (21)$$

which are the exact solutions. The results for the exact solution Eq. (21) and the approximate solution Eq. (20) are obtained using HPTM, for  $\alpha = 0.50$  and  $1$ , are shown in **Figure 1**.



**Figure 1** The surface shows solutions  $u(x, t)$  and  $v(x, t)$  for the Eq. (4.3) (a)  $u(x, t)$  when  $\alpha = 0.5$ , (b)  $v(x, t)$  when  $\alpha = 0.5$ , (c)  $u(x, t)$  when  $\alpha = 1$ , (d)  $v(x, t)$  when  $\alpha = 1$ , (e) exact solution  $u(x, t)$ , (f) exact solution  $v(x, t)$ .

**Example 2** Consider the following inhomogeneous linear system;

$$\begin{cases} D_t^\alpha u - v_x - (u - v) = -2, \\ D_t^\alpha v - u_x - (u - v) = -2, \end{cases} \quad (22)$$

with initial conditions;

$$u(x, 0) = 1 + e^x, \quad v(x, 0) = -1 + e^x.$$

Taking the Laplace transform on both sides;

$$\begin{cases} L[D_t^\alpha u] = L[-2 + v_x + (u - v)], \\ L[D_t^\alpha v] = L[-2 + u_x - (u - v)], \end{cases}$$

Using the property of Laplace transform, we have;

$$\begin{cases} L[u(x, t)] = \frac{1 + e^x}{s} + \frac{1}{s^\alpha} L[-2 + v_x - (u - v)], \\ L[v(x, t)] = \frac{-1 + e^x}{s} + \frac{1}{s^\alpha} L[-2 + u_x - (u - v)], \end{cases}$$

Operating with the Laplace inverse on both sides;

$$\begin{cases} u(x, t) = 1 + e^x + L^{-1}\left[\frac{1}{s^\alpha} L[-2 + v_x - (u - v)]\right], \\ v(x, t) = -1 + e^x + L^{-1}\left[\frac{1}{s^\alpha} L[-2 + u_x - (u - v)]\right], \end{cases}$$

Then, we apply the homotopy perturbation method;

$$\begin{aligned} \sum_{n=0}^{\infty} p^n u_n(x, t) &= 1 + e^x + pL^{-1}\left[\frac{1}{s^\alpha} L[-2 + (\sum_{n=0}^{\infty} p^n v_{nx}(x, t)) - (\sum_{n=0}^{\infty} p^n u_n(x, t)) + (\sum_{n=0}^{\infty} p^n v_n(x, t))]\right], \\ \sum_{n=0}^{\infty} p^n v_n(x, t) &= -1 + e^x + pL^{-1}\left[\frac{1}{s^\alpha} L[-2 + (\sum_{n=0}^{\infty} p^n v_{nx}(x, t)) - (\sum_{n=0}^{\infty} p^n u_n(x, t)) + (\sum_{n=0}^{\infty} p^n v_n(x, t))]\right], \end{aligned}$$

Comparing the coefficient of like power of  $p$ , we have;

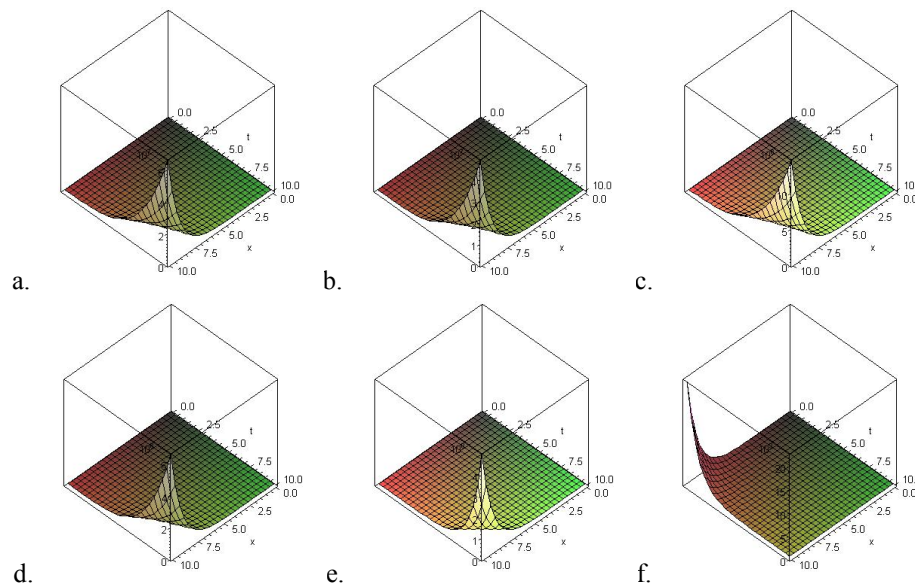
$$\begin{aligned} p^0: & \begin{cases} u_0(x, t) = 1 + e^x, \\ v_0(x, t) = -1 + e^x, \end{cases} \\ p^1: & \begin{cases} u_1(x, t) = e^x \frac{t^\alpha}{\Gamma(\alpha+1)}, \\ v_1(x, t) = e^x \frac{t^\alpha}{\Gamma(\alpha+1)}, \end{cases} \\ p^2: & \begin{cases} u_2(x, t) = e^x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ v_2(x, t) = e^x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \end{cases} \\ p^3: & \begin{cases} u_3(x, t) = e^x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \\ v_3(x, t) = -e^x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \end{cases} \\ & \vdots \end{aligned}$$

The solution in the series form is given by;

$$\begin{cases} u(x, t) = -1 + e^x \left[ 1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \dots \right], \\ v(x, t) = 1 + e^x \left[ 1 - \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \dots \right], \end{cases} \quad (23)$$

$$\begin{cases} u(x, t) = -1 + e^{x+t}, \\ v(x, t) = 1 + e^{x-t}, \end{cases} \quad (24)$$

which is the exact solution of the system. The results for the exact solution Eq. (24) and the approximate solution Eq. (23) obtained using HPTM, for  $\alpha = 0.5$  and  $1$ , are shown in **Figure 2**.



**Figure 2** The surface shows solutions  $u(x, t)$  and  $v(x, t)$  for the Eq. (4.6) (a)  $u(x, t)$  when  $\alpha = 0.5$ , (b)  $v(x, t)$  when  $\alpha = 0.5$ , (c)  $u(x, t)$  when  $\alpha = 1$ , (d)  $v(x, t)$  when  $\alpha = 1$ , (e) exact solution  $u(x, t)$ , (f) exact solution  $v(x, t)$ .

**Example 3** Consider the following homogenous nonlinear system;

$$\begin{cases} D_t^\alpha u - u_{xx} - 2uu_x + (uv)_x = 0, \\ D_t^\alpha v - v_{xx} - 2vv_x + (uv)_x = 0, \end{cases} \quad (25)$$

with initial conditions;

$$u(x, 0) = \sin(x), \quad v(x, 0) = \sin(x). \quad (26)$$

Applying the method defined above, we get;

$$\begin{aligned} p^0: & \begin{cases} u_0(x, t) = \sin(x), \\ v_0(x, t) = \sin(x), \end{cases} \\ p^1: & \begin{cases} u_1(x, t) = -\sin(x) \frac{t^\alpha}{\Gamma(\alpha+1)}, \\ v_1(x, t) = -\sin(x) \frac{t^\alpha}{\Gamma(\alpha+1)}, \end{cases} \\ p^2: & \begin{cases} u_2(x, t) = \sin(x) \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ v_2(x, t) = \sin(x) \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \end{cases} \\ p^3: & \begin{cases} u_3(x, t) = -\sin(x) \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \\ v_3(x, t) = -\sin(x) \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \end{cases} \\ & \vdots \end{aligned}$$

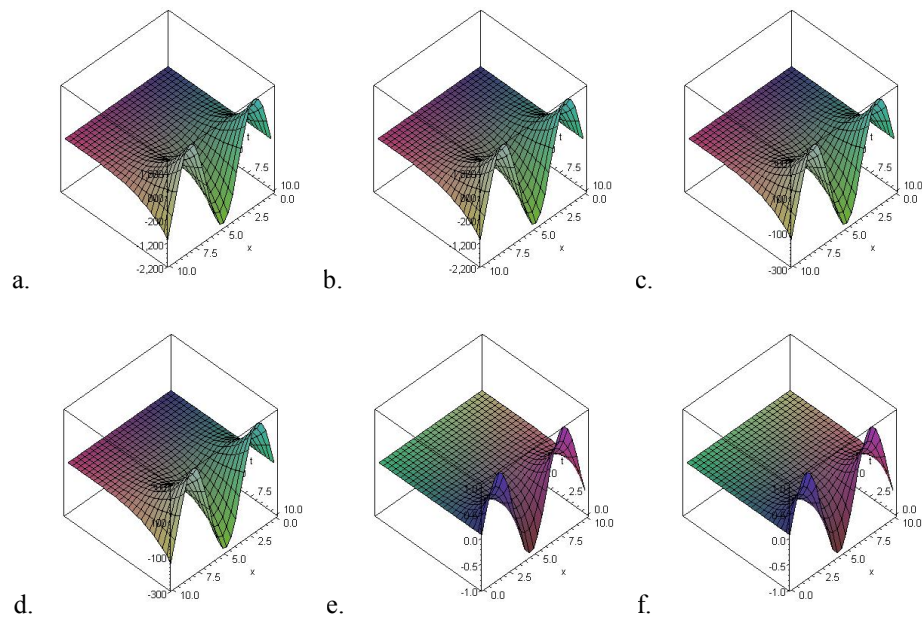
The solution in the series form is given by;

$$\begin{aligned} u(x, t) &= \sum_{n=0}^{\infty} p^n u_n(x, t) = u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \dots, \\ v(x, t) &= \sum_{n=0}^{\infty} p^n v_n(x, t) = v_0 + p v_1 + p^2 v_2 + p^3 v_3 + \dots, \\ \begin{cases} u(x, t) = \sin(x) \left[ 1 - \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} - \dots \right], \\ v(x, t) = \sin(x) \left[ 1 - \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} - \dots \right], \end{cases} \end{aligned} \quad (27)$$

For the special case  $\alpha = 1$ , we obtain the form;

$$\begin{cases} u(x, t) = \sin(x)e^{-t}. \\ v(x, t) = \sin(x)e^{-t}. \end{cases} \quad (28)$$

which is the exact solution of the system. The results for the exact solution Eq. (28) and the approximate solution Eq. (27) obtained using HPTM, for  $\alpha = 0.50$  and 1, are shown in **Figure 3**.



**Figure 3** The surface shows solutions  $u(x, t)$  and  $v(x, t)$  for the Eq. (4.11) (a)  $u(x, t)$  when  $\alpha = 0.5$ , (b)  $v(x, t)$  when  $\alpha = 0.5$ , (c)  $u(x, t)$  when  $\alpha = 1$ , (d)  $v(x, t)$  when  $\alpha = 1$ , (e) exact solution  $u(x, t)$ , (f) exact solution  $v(x, t)$ .

**Example 4** Consider the following homogenous nonlinear time-fractional system;

$$\begin{cases} D_t^\alpha u + u_x v_x + u_y v_y + u = 0, \\ D_t^\alpha v + v_x w_x - v_y w_y - v = 0, \\ D_t^\alpha w + w_x u_x + w_y u_y - w = 0, \end{cases} \quad (29)$$

with initial conditions;

$$u(x, y, 0) = e^{x+y}, \quad v(x, y, 0) = e^{x-y}, \quad w(x, y, 0) = e^{-x+y}. \quad (30)$$

applying the method defined above (3) - (17), we get;

$$\begin{aligned} p^0: & \begin{cases} u_0(x, y, t) = e^{x+y}, \\ v_0(x, y, t) = e^{x-y}, \\ u_0(x, t) = e^{-x+y}, \end{cases} \\ p^1: & \begin{cases} u_1(x, t) = e^{x+y} \frac{t^\alpha}{\Gamma(\alpha+1)}, \\ v_1(x, t) = e^{x-y} \frac{t^\alpha}{\Gamma(\alpha+1)}, \\ w_1(x, t) = e^{-x+y} \frac{t^\alpha}{\Gamma(\alpha+1)}, \end{cases} \end{aligned}$$

$$p^2: \begin{cases} u_2(x, t) = e^{x+y} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ v_2(x, t) = e^{x-y} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ w_2(x, t) = e^{-x+y} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \end{cases}$$

$$p^3: \begin{cases} u_3(x, t) = -e^{x+y} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \\ v_3(x, t) = e^{x-y} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \\ w_3(x, t) = e^{-x+y} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \end{cases}$$

$\vdots$

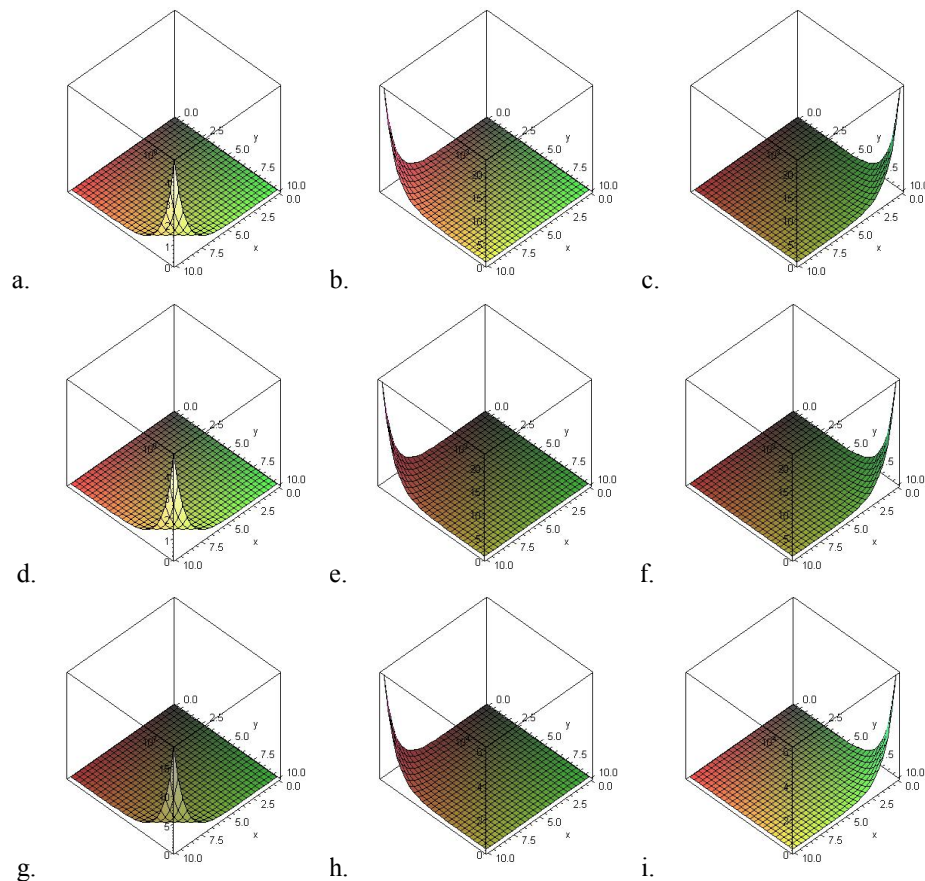
The solution in the series form is given by;

$$\begin{cases} u(x, y, t) = e^{x+y} \left[ 1 - \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \dots \right], \\ v(x, y, t) = e^{x-y} \left[ 1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \dots \right], \\ w(x, y, t) = e^{-x+y} \left[ 1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \dots \right], \end{cases} \quad (31)$$

For the special case  $\alpha = 1$ , we obtain the form (32).

$$\begin{cases} u(x, t) = e^{x+y-t}, \\ v(x, t) = e^{x-y+t}, \\ w(x, t) = e^{-x+y+t}. \end{cases} \quad (32)$$

which is the exact solution of the system. The results for the exact solution Eq. (32) and the approximate solution Eq. (31) obtained using HPTM, for  $\alpha = 0.50$  and  $1$ , are shown in **Figure 4**.



**Figure 4** The surface shows solutions  $u(x,y,t)$ ,  $v(x,y,t)$  and  $w(x,y,t)$  for the Eq. (4.14) (a)  $u(x,y,t)$  when  $\alpha = 0.5$ , (b)  $v(x,y,t)$  when  $\alpha = 0.5$ , (c)  $w(x,y,t)$  when  $\alpha = 1$ , (d)  $u(x,y,t)$  when  $\alpha = 1$ , (e)  $v(x,y,t)$  when  $\alpha = 1$ , (f)  $w(x,y,t)$  when  $\alpha = 1$  (g) exact solution  $u(x,y,t)$ , (h) exact solution  $v(x,y,t)$ , (i) exact solution  $w(x,y,t)$ .

## Conclusions

HPTM has been implemented to find appropriate solutions of time-fractional linear and non-linear system of partial differential equations. Numerical results coupled with graphical representations explicitly reveal the complete reliability and efficiency of the proposed algorithm.

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