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On the Convergence of the Homotopy Analysis Method for Solving Fredholm Integral Equations

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Abstract

The aim of this paper is to study the convergence of the Homotopy analysis method (HAM) for solving Fredholm integral equations. A sufficient condition for convergence of the method is illustrated. The validity of the presented condition for convergence of the HAM is studied for two examples. The comparison of the obtained results by the method with an exact solution shows that the method is reliable and capable of providing analytic treatment for solving such equations.

Keywords: Homotopy analysis method, study of convergence, Fredholm integral equations

Introduction

The Homotopy analysis method (HAM) [1] has been proved to be one of the most useful techniques to solve numerous linear and non-linear functional equations. As mentioned in [2,3], unlike all previous analytic techniques [4-10], the HAM provides great freedom to express solutions of a given non-linear problem by means of different base functions. Furthermore, this method provides us with a way to adjust and control the convergence region and rate of convergence of solution series by introducing the auxiliary parameter, $\hbar$. Finally, the HAM is independent of any small or large parameters. So, the method can be applied no matter if governing equations and boundary or initial conditions of a given non-linear problem depend on small or large quantities or not. By properly choosing the base functions, initial approximations, auxiliary linear operators, auxiliary functions, and auxiliary parameter, $\hbar$, HAM gives rapidly convergent successive approximations of the exact solution.

We aim in this work to study convergence of the HAM, for solving integral equations of the Fredholm type, of the form;

$$u(x) = f(x) + \mu \int_a^b k(x,t)\big(u(t)\big)^p \, dt, \qquad a \le x \le b,\ p \in N, \quad (1)$$

subject to the initial condition;

$$u(0) = \alpha, \quad (2)$$

where μ is a real number, the kernel $k(x,t)$ is a continuous function over $[a,b]\times[a,b]$, and $f(x)$ is a given continuous function defined over $[a,b]$.

Since the integral equations appear frequently in modeling of physical phenomena, they have a major role in the fields of science and engineering and a considerable amount of research work has been investigated [11-15].

In Section 2, we illustrate the main idea of the Homotopy perturbation method. In Section 3, the convergence analysis of the method is discussed. Then 2 numerical examples are presented in Section 4. Finally, conclusions are stated in the last section.

Basic ideas of HAM

Let us consider a non-linear equation in the following form;

$$N[u(x)] = 0, \tag{3}$$

where N is a non-linear operator, $u(x)$ is an unknown function of the independent variable x.
Let $u_0(x)$ denote an initial guess of the solution $u(x)$, $\hbar \neq 0$ is an auxiliary parameter, $H(0) \neq 0$ is an auxiliary function, and L is an auxiliary linear operator with the property;

$$L[f(x)] = 0 \text{ when } f(x) = 0. \tag{4}$$

Then, using $q \in [0,1]$ as an embedding parameter, the following homotopy can be constructed.

$$\Omega[\Phi(x;q); u_0(x), H(x), \hbar, q] = (1-q)\{L[\Phi(x;q) - u_0(x)]\} - q\hbar H(x) N[\Phi(x;q)]. \tag{5}$$

It should be emphasized that we have great freedom to choose the initial guess $u_0(x)$, auxiliary linear operator L, non-zero auxiliary parameter $\hbar$, and auxiliary function $H(x)$.
Enforcing the homotopy (5) to be zero, i.e.;

$$\Omega[\Phi(x;q); u_0(x), H(x), \hbar; q] = 0.$$

We have the so-called zero-order deformation equation.

$$(1-q)\{L[\Phi(x;q) - u_0(r,t)]\} = q\hbar H(x) N[\Phi(x;q)], \tag{6}$$

where $\Phi(x;q)$ is the solution which depends upon not only on the initial guess $u_0(x)$, auxiliary linear operator L, auxiliary function $H(r,t)$ and auxiliary parameter $\hbar$, but also the embedding parameter q.
When $q = 0$, the zero-order deformation Eq. (6), turns into;

$$L[\Phi(x;0) - u_0(x)] = 0. \tag{7}$$

Property (4), leads to;

$$\Phi(x;0) = u_0(x).$$

When $q = 1$, since $\hbar \neq 0$ and $H(x) \neq 0$, the zero-order deformation Eq. (6) is equivalent to

$$N[\Phi(x;1)] = 0,$$

which is exactly the same as the original Eq. (3), provided [2];

$$\Phi(x;1) = u(x). \tag{8}$$

Thus, according to (7) and (8), as the embedding parameter q increases from 0 to 1, $\Phi(x;q)$ varies continuously from the initial approximation $u_0(x)$ to the exact solution $u(x)$ of the original Eq. (3).
Under the assumption that the Taylor series of $\Phi(x;q)$ with respect to q;

$$\Phi(x;q) = \phi_0(x) + \sum_{m=1}^{+\infty} \phi_m(x) q^m, \tag{9}$$

be convergent at $q=1$, the solution series will be presented as;

$$u(x) = \Phi(x;1) = \phi_0(x) + \sum_{m=1}^{+\infty} \phi_m(x). \tag{10}$$

This expression provides us with a relationship between the exact solution $u(x)$ and the initial approximation $u_0(x)$ by means of the terms $\phi_m(x)$ that are determined as follows.

Differentiating the zero-order deformation Eq. (6) $m(m>1)$ times with respective to the embedding parameter q and then dividing it by $m!$ and finally setting $q=0$, we have the so-called mth-order deformation equation.

$$L[\phi_m(x) - \chi_m \phi_{m-1}(x)] = \hbar H(x) R_m(\vec{\phi}_{m-1}, x), \tag{11}$$

subject to the initial condition;

$$\phi_m(0) = 0, \tag{12}$$

where χ_m is defined by;

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & Otherwise. \end{cases} \tag{13}$$

Also;

$$\vec{\phi}_m = \{\phi_0(x), \phi_1(x), \phi_2(x), \ldots, \phi_m(x)\}. \tag{14}$$

and

$$R_m(\vec{\phi}_{m-1}, x) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\Phi(x;q)]}{\partial q^{m-1}} \Big|_{q=0}. \tag{15}$$

Substituting (10) into (15), gives;

$$R_m(\vec{\phi}_{m-1}, x) = \frac{1}{(m-1)!} \left\{ \frac{\partial^{m-1}}{\partial q^{m-1}} N\left[\sum_{n=0}^{+\infty} \phi_n(x) q^n \right] \right\} \Big|_{q=0}. \tag{16}$$

According to Eq. (1), we consider the non-linear operator.

$$N[\Phi] = \Phi(x) - f(x) - \mu \int_a^b k(x,t)(\Phi(x))^p \, dt. \tag{17}$$

Also, we can choose the initial guess $u_0(x)$, in such a way that it satisfies the initial condition (2), i.e.;

$$u_0(0) = u(0) = \alpha. \tag{18}$$

Using Eqs. (16) and (17), we have;

$$R_m\left[\vec{\phi}_{m-1}(x)\right]=\phi_{m-1}(x)-(1-\chi_m)f(x)$$
$$-\mu\int_a^b k(x,t)\left[\sum_{r_1=0}^{m-1}\phi_{m-1-r_1}(x)\sum_{r_2=0}^{r_1}\phi_{r_1-r_2}(x)\sum_{r_3=0}^{r_2}\phi_{r_2-r_3}(x)\ldots\sum_{r_{p-2}=0}^{r_{p-3}}\phi_{r_{p-3}-r_{p-2}}(x)\sum_{r_{p-1}=0}^{r_{p-2}}\phi_{r_{p-2}-r_{p-1}}(x)\phi_{r_{p-1}}(x)\right]dt. \tag{19}$$

From (15), it should be noted that the right-hand side of Eq. (11) is only dependent upon $\vec{\phi}_{m-1}$. Thus, we recursively gain $\phi_0(r,t),\phi_1(r,t),\phi_2(r,t),\ldots$ by means of solving the linear high-order deformation Eq. (11) subject to (12). The m th-order approximation of $u(x)$ is given by;

$$u_m(x)=\sum_{k=0}^{m}\phi_k(x).$$

Convergence analysis

In this section, some conditions of convergence of the Homotopy analysis method are stated and addressed briefly.

Theorem 1. The following series;

$$\phi_0(x)+\sum_{m=1}^{+\infty}\phi_m(x)$$

where $\phi_m(x)$'s result from Eqs. (11), (12) and (19), is an exact solution of the Eq. (1) and (2).

Proof. The series is convergent, i.e.;

$$s(x)=\sum_{m=0}^{+\infty}\phi_m(x).$$

So, by necessary condition for the convergence of the Series, it holds;

$$\lim_{m\to\infty}\phi_m(x)=0. \tag{20}$$

Using (11) and (20), we have;

$$\begin{aligned}\hbar H(x)\sum_{m=1}^{+\infty}R_m\left[\vec{\phi}_{m-1}(x)\right]&=\lim_{n\to\infty}\sum_{m=1}^{n}L\left[\phi_m(x)-\chi_m\phi_{m-1}(x)\right]\\&=L\left\{\lim_{n\to\infty}\sum_{m=1}^{n}\left[\phi_m(x)-\chi_m\phi_{m-1}(x)\right]\right\}\\&=L\left\{\lim_{n\to\infty}\phi_n(x)\right\}\\&=0.\end{aligned}$$

Since $\hbar\neq0$ and $H(x)\neq0$, we have;

$$\sum_{m=1}^{+\infty}R_m\left[\vec{\phi}_{m-1}(x)\right]=0. \tag{21}$$

On the other hand, we have;

$$\begin{aligned}
&\sum_{m=1}^{+\infty} R_m\left[\vec{\phi}_{m-1}(x)\right] \\
&=\sum_{m=1}^{n}\left[\phi_{m-1}(x)-(1-\chi_m)f(x)-\int_a^b k(x,t)\left[\sum_{r_1=0}^{m-1}\phi_{m-1-r_1}(x)\sum_{r_2=0}^{r_1}\phi_{r_1-r_2}(x)\sum_{r_3=0}^{r_2}\phi_{r_2-r_3}(x)\ldots\sum_{r_{p-2}=0}^{r_{p-3}}\phi_{r_{p-3}-r_{p-2}}(x)\sum_{r_{p-1}=0}^{r_{p-2}}\phi_{r_{p-2}-r_{p-1}}(x)\phi_{r_{p-1}}(x)\right]dt\right] \\
&=\sum_{m=0}^{\infty}\phi_m(x)-f(x)-\mu\sum_{m=1}^{\infty}\left\{\int_a^b k(x,t)\left[\sum_{r_1=0}^{m-1}\phi_{m-1-r_1}(x)\sum_{r_2=0}^{r_1}\phi_{r_1-r_2}(x)\sum_{r_3=0}^{r_2}\phi_{r_2-r_3}(x)\ldots\sum_{r_{p-2}=0}^{r_{p-3}}\phi_{r_{p-3}-r_{p-2}}(x)\sum_{r_{p-1}=0}^{r_{p-2}}\phi_{r_{p-2}-r_{p-1}}(x)\phi_{r_{p-1}}(x)\right]dt\right\} \\
&=\sum_{m=0}^{\infty}\phi_m(x)-f(x)-\mu\int_a^b k(x,t)\left[\sum_{r_{p-1}=0}^{\infty}\phi_{r_{p-1}}(x)\sum_{r_{p-2}=r_{p-1}}^{\infty}\phi_{r_{p-2}-r_{p-1}}(x)\sum_{r_{p-3}=r_{p-2}}^{\infty}\phi_{r_{p-3}-r_{p-2}}(x)\ldots\sum_{r_2=r_3}^{\infty}\phi_{r_2-r_3}(x)\sum_{r_1=r_2}^{\infty}\phi_{r_1-r_2}(x)\sum_{m=r_1}^{\infty}\phi_{m-r_1}(x)\right]dt \\
&=\sum_{m=0}^{\infty}\phi_m(x)-f(x)-\mu\int_a^b k(x,t)\left[\sum_{i_1=0}^{\infty}\phi_{i_1}(x)\sum_{i_2=0}^{\infty}\phi_{i_2}(x)\sum_{i_3=0}^{\infty}\phi_{i_3}(x)\ldots\sum_{i_{p-2}=0}^{\infty}\phi_{i_{p-2}}(x)\sum_{i_{p-1}=0}^{\infty}\phi_{i_{p-1}}(x)\sum_{i_p=0}^{\infty}\phi_{i_p}(x)\right]dt \\
&=s(x)-f(x)-\mu\int_a^b k(x,t)\left(s(t)\right)^p dt.
\end{aligned}$$

So, from Eq. (21), we obtain;

$$s(x)-f(x)-\mu\int_a^b k(x,t)\left(s(t)\right)^p dt=0. \tag{22}$$

From the initial conditions (12) and (18), the following holds;

$$s(0)=\sum_{i=0}^{\infty}\phi_i(0)=\phi_0(0)=u_0(0)=\alpha, \tag{23}$$

Since, $s(x)$ satisfies Eqs. (22) and (23), so it is an exact solution of Eq. (1) with the initial condition (2). This ends the proof.

Theorem 2. Suppose that $\Im\subset R$ be a Banach space with a suitable norm, say $\|.\|_\infty$, over which the sequence $\phi_m(x)$ of (9) is defined for a prescribed value of $\hbar$. Assume also that the initial approximation $\phi_0(x)$ remains inside the ball of the solution $u(x)$. Taking $r\in R$ as a constant, the following statements hold.
(i) If there exists some $r\in[0,1]$, such that for all $k \in R$ we have $\|\phi_{k+1}(x)\|\geq r\|\phi_k(x)\|$, then the series solution $u(x)=\sum_{k=0}^{\infty}\phi_k(x)q^k$, converges absolutely to (10), at $q=1$, over the domain of definition of x,
(ii) If there exists some $r>1$, such that for all $k \in R$ we have $\|\phi_{k+1}(x)\|\leq r\|\phi_k(x)\|$, then the series solution $u(x)=\sum_{k=0}^{\infty}\phi_k(x)q^k$ diverges, at $q=1$, over the domain of definition of x.

Proof. Indeed, this is a special case of the Banach fixed-point theorem, which can be found in standard texts on real analysis such as in [16].

Theorems 1 and 2 state that the homotopy series solution $\sum_{k=0}^{\infty}\phi_k(x)$, of the non-linear problem (1), converges to an exact solution $u(x)$, under the condition that $\exists\gamma, 0<\gamma<1$ such that

$\forall k \geq k_0, \|\phi_{k+1}(x)\| \leq \gamma \|\phi_k(x)\|$, for some $k_0 \in R$. In other words, if for every $i \geq k_0$, we consider the parameters β_i's, as defined in [17].

$$\beta_{i+1} = \begin{cases} \dfrac{\|\phi_{i+1}\|}{\|\phi_i\|}; & \|\phi_i\| \neq 0, \\ 0; & \|\phi_i\| = 0. \end{cases} \tag{24}$$

For $\in R \cup \{0\}$, then the series solution $\sum_{k=0}^{\infty} \phi_k(x)$ of problem (1) converges to an exact solution $u(x)$, when $\forall i \geq k_0, 0 \leq \beta_i < 1$.

Numerical examples

In this section, 2 examples are presented to illustrate the convergence study.

Example 1. Consider the following linear Fredholm integral equation [15];

$$\begin{cases} u(x) = e^x - 2\sin(x) + \int_{-1}^{1} e^{-t} \sin(x) u(t) dt, \\ u(0) = 1. \end{cases}$$

with the exact solution $u(x) = e^x$.
According to HAM, the following homotopy is constructed.

$$(1-q)L[\Phi(x) - \phi_0(x)] = q\hbar H(x)\left(\Phi(x) - e^x + 2\sin(x) - \int_{-1}^{1} e^{-t} \sin(x)\Phi(t)dt\right).$$

By taking $H = 1, L\phi = \phi$, and substituting (9) into the above homotopy, the following deformation equations are obtained.

$$\begin{aligned} \phi_1 &= \hbar\left(\phi_0(x) - e^x + 2\sin(x) - \int_{-1}^{1} e^{-t} \sin(x)\phi_0(t)dt\right) + C_1, \\ \phi_{k+1} &= \phi_k + \hbar\left(\phi_k(x) - \int_{-1}^{1} e^{-t} \sin(x)\phi_k(t)dt\right) + C_{k+1}, \quad k \geq 1 \end{aligned} \tag{25}$$

where the constants $C_k, k \geq 1,$ are determined from the condition (12). Starting with $\phi_0 = 1$ in (25), we recursively obtain the approximations. In this example;

$$\begin{aligned} \phi_1(x) &= \hbar\left(-e^x + e^{-1}\sin(x) + 1\right), \\ \phi_2(x) &= \frac{1}{2}\hbar\left((-2e^x + 2)(\hbar + 1) + \sin(x)\left[\hbar e^{-2}\sin(1) + \hbar e^{-2}\cos(1) + 2e^{-1} + 3\hbar e^{-1}\right]\right). \end{aligned}$$

Table 1 shows some values of β_i's, defined as in (24), for the truncated series solution $u_7(x) = \sum_{i=0}^{7} \phi_i(x)$, which was obtained from the iterative scheme (25) by using different values of $\hbar$. From **Table 1**, since $\beta_i < 1$ for $\hbar = -1.1$, $\hbar = -1$ and $\hbar = -0.9$, we can conclude that the HAM approach converges to the exact solution of problem for $\hbar=-1.1$, $\hbar=-1$ and $\hbar=-0.9$. Also, we observe that β_i's are not less than one for $\hbar=1$. So, the HAM approach may be divergent when $\hbar=1$.

Table 1 Numerical values of β_i's for different values of $\hbar$ in Example 1.

	$\hbar=-1.1$	$\hbar=-1$	$\hbar=-0.9$	$\hbar=1$
β_1	0.1064777183	0.0273588205	1.9470005312	2.005888808
β_2	0.1225203824	0.1394710424	0.07293168915	2.004620076
β_3	0.1665256283	0.1772552582	0.1048370531	2.003192583
β_4	0.1887994255	0.0728215595	0.2171489807	2.001605553
β_5	0.2262985828	0.1434132848	0.2128619930	1.999857309
β_6	0.1143905693	0.1543150838	0.1933299028	1.997944868

In **Table 2**, relative errors δ_n of the n terms approximation of HAM, defined as;

$$\delta_n(x_j)=\left|\frac{u(x_j)-u_n(x_j)}{u(x_j)}\right|, \tag{26}$$

for different values of $\hbar$ at different x_i's are presented. It is evident that the auxiliary parameter $\hbar$ can also be effectively implemented to adjust and control the rate of convergence of series solutions by HAM.

Table 2 Comparison of relative errors δ_n for Example 1.

	$\hbar=-1.1$	$\hbar=-1$	$\hbar=-0.9$	$\hbar=1$
$x_1=0.1$	9.41E-7	1.65E-7	3.55E-7	2.597236983
$x_2=0.2$	2.60E-6	7.28E-7	7.02E-7	9.554876418
$x_3=0.3$	7.65E-7	1.67E-7	4.49E-7	33.19152096
$x_4=0.4$	6.15E-5	9.03E-7	2.67E-6	68.22084816
$x_5=0.5$	6.12E-7	4.21E-6	7.00E-8	114.6188038
$x_6=0.6$	7.16E-6	9.72E-7	1.74E-7	172.3762215
$x_7=0.7$	3.44E-7	6.14E-6	2.20E-6	241.4889172
$x_8=0.8$	1.12E-6	4.17E-6	1.84E-6	321.9545818
$x_9=0.9$	2.52E-6	7.27E-7	1.31E-6	413.7719315
$x_{10}=1.0$	9.14E-6	9.91E-6	4.83E-6	516.9401309

Example 2. In this example, we consider the following non-linear Fredholm integral equation with the exact solution $x\ln(1+x)$ [14].

$$\begin{cases} u(x)=x\ln(x+1)-\dfrac{53}{108}+\dfrac{1}{3}\ln 2\left[\dfrac{8}{3}x+2-x\ln 2\right]-\dfrac{241}{576}+\dfrac{1}{2}\int_0^1 (x-t)\big(u(x)\big)^2\,dt,\ x\in[0,1], \\ u(0)=0. \end{cases} \tag{27}$$

To obtain the approximate solution of (27), by taking $H=1,\ L\phi=\phi$ in (6), we have;

$$(1-q)L\left[\Phi(x)-\phi_0(x)\right]=q\hbar H(x)\left(\Phi(x)-x\ln(x+1)+\frac{53}{108}-\frac{1}{3}\ln 2\left(\frac{8}{3}x+2-x\ln 2\right)+\frac{241}{576}-\frac{1}{2}\int_0^1(x-t)\left(\Phi(x)\right)^2dt\right). \tag{28}$$

Substituting (9) into (28) and starting with the initial guess $\phi_0(x)=x\ln(x+1)+x\left(-\frac{55}{108}+\frac{8}{9}\ln 2-\frac{1}{3}(\ln 2)^2\right)x,$ the following deformation equations are obtained.

$$\phi_1=\hbar\left(\phi_0(x)-x\ln(x)+\frac{53}{108}-\frac{1}{3}\ln\left(\frac{8}{3}x+2-x\ln 2\right)+\frac{241}{576}-\frac{1}{2}\int_0^1(x-t)\left(\phi_0(x)\right)^2dt\right)+C_1,$$

$$\phi_{k+1}=\phi_k+\hbar\left(\phi_k(x)-\frac{1}{2}\int_0^1(x-t)\left[\sum_{i=0}^{m}\phi_i(x)\phi_{m-i}(x)\right]dt\right)+C_{k+1},k\geq 1$$

where the constants $C_k,\ k\geq 1,$ can be determined by the initial condition (12).

Table 3 shows the values of β_i's for different values of $\hbar$. Moreover, **Table 4** shows the relative error (26), of the truncated series u_7, for different values of $\hbar$. Clearly, one can observe that the approximate solution for $\hbar=-1.25$, is more accurate than the approximate solutions obtained when $\hbar=-1.1$, $\hbar=-1$ and $\hbar=-0.9$. So, one can claim that the auxiliary parameter $\hbar$ plays an important role in adjusting and controlling the convergence of the series solution. It seems that the more accurate approximations will be obtained for smaller values of β_i.

Table 3 Numerical values of β_i's for different values of $\hbar$ in Example 2.

	$\hbar=-1.25$	$\hbar=-1.1$	$\hbar=-1$	$\hbar=-0.9$
β_1	0.41799351	0.08321659	0.16656051	0.249904473
β_2	0.31550415	0.18968116	0.21052058	0.27363689
β_3	0.11432698	0.17663097	0.23611584	0.27363689
β_4	0.25497653	0.20555668	0.25311880	0.30883061
β_5	0.16748886	0.20770612	0.26565744	0.32097967
β_6	0.18793950	0.22965814	0.27066721	0.33114110

Table 4 Comparison of relative errors δ_n for Example 2.

	$\hbar=-1.25$	$\hbar=-1.1$	$\hbar=-1.0$	$\hbar=-0.9$
$x_1=0.1$	2.09E-8	9.59E-7	4.71E-6	1.94E-5
$x_2=0.2$	1.39E-7	1.75E-7	6.74E-6	3.28E-5
$x_3=0.3$	1.60E-7	3.46E-6	9.49E-6	4.62E-5
$x_4=0.4$	7.20E-8	3.13E-6	1.22E-5	5.96E-5
$x_5=0.5$	2.99E-7	3.86E-6	1.49E-5	7.29E-5
$x_6=0.6$	4.86E-7	4.61E-6	1.77E-5	8.63E-5
$x_7=0.7$	1.96E-8	5.37E-6	2.04E-5	9.96E-5
$x_8=0.8$	1.37E-7	6.07E-6	2.32E-5	1.12E-4
$x_9=0.9$	1.23E-7	6.92E-6	2.58E-5	1.26E-4
$x_{10}=1.0$	2.67E-7	7.50E-6	2.86E-5	1.39E-4

Conclusions

In this study, the problem of convergence of the Homotopy analysis method, when it used for solving a special form of the Fredholm integral equation has been studied. The sufficient condition for convergence of the method has been illustrated, and verified for 2 examples. The obtained approximations of the solutions confirm the power and ability of the HAM as a reliable device for computing the solutions to the Fredholm integral equation. This is mainly due to the fact that the method provides a way to ensure the convergence of series solutions. The study of convergence conditions in applying HAM for other equations and systems of differential equations, integral equations and integro-differential equations, are also under investigation by our research team.

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