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Finite Element Method: An Overview

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Abstract

The finite element method (FEM) is a numerical analysis technique for obtaining approximate solutions to a wide variety of engineering problems. A finite element model of a problem gives a piecewise approximation to the governing equations. The basic premise of the FEM is that a solution region can be analytically modeled or approximated by replacing it with an assemblage of discrete elements (discretization). Since these elements can be put together in a variety of ways, they can be used to represent exceedingly complex shapes.

Keywords: FEM, discretization, numerical analysis, approximate solution

Introduction

Several approximate numerical analysis methods have evolved over the years. As an example of how a finite difference model and a finite element model might be used to represent a complex geometrical shape, consider the turbine blade cross section in **Figure 1** and plate geometry in **Figure 2**. A uniform finite difference mesh would reasonably cover the blade (the solution region), but the boundaries must be approximated by a series of horizontal and vertical lines (or "stair steps"). On the other hand, the finite element model (using the simplest two-dimensional element-the triangle) gives a better approximation of the region. Also, a better approximation to the boundary shape results because the curved boundary is represented by straight lines of any inclination. This is not intended to suggest that finite element models are decidedly better than finite difference models for all problems. The only purpose of these examples is to demonstrate that the finite element method is particularly well suited for problems with complex geometries and numerical solutions to even very complicated stress problems can now be obtained routinely using finite element analysis (FEA).

History of the method

Although the label finite element method first appeared in 1960, when it was used by Clough [1] in a paper on plane elasticity problems, the ideas of finite element analysis date back much further. The first efforts to use piecewise continuous functions defined over triangular domains appear in the applied mathematics literature with the work of Courant [2] in 1943. Courant developed the idea of the minimization of a functional using linear approximation over sub-regions, with the values being specified at discrete points which in essence become the node points of a mesh of elements.

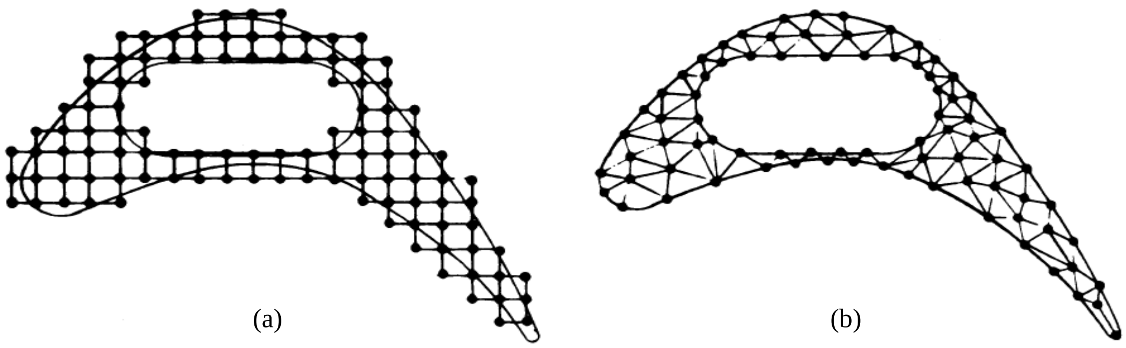

Figure 1 (a) Finite difference and (b) finite element discretizations of a turbine blade profile.

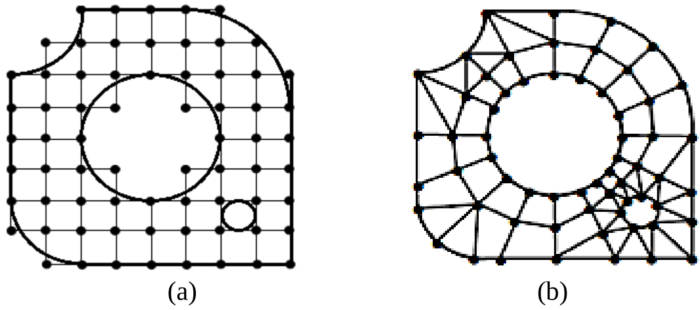

Figure 2 (a) Plate geometry finite difference model and (b) Finite element model.

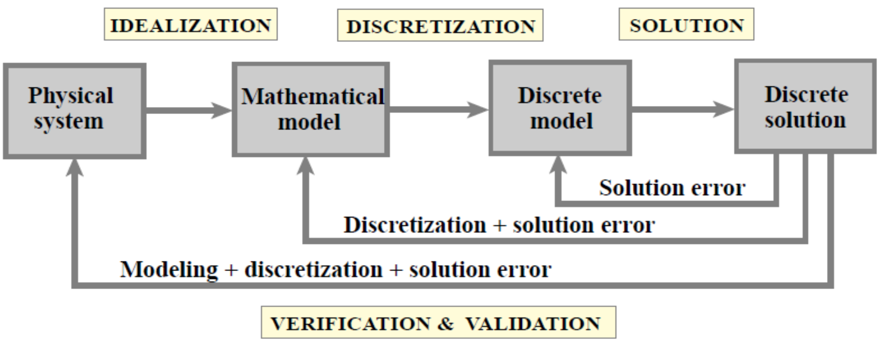

Figure 3 Flowchart of model-based simulation (MBS) by computer.

The overall schematics of a model-based simulation (MBS) by computer are shown in a flowchart in **Figure 3**. For mechanical systems such as structures the Finite Element Method (FEM) is the most widely used discretization and solution technique. Historically the ancestor of the FEM is the MSA, as illustrated in **Figure 4**. On the left "human computer" means computations under direct human control, possibly with the help of analog devices (slide rule) or digital devices (desk calculator). The FEM configuration shown on the right was settled by the mid 1960s.

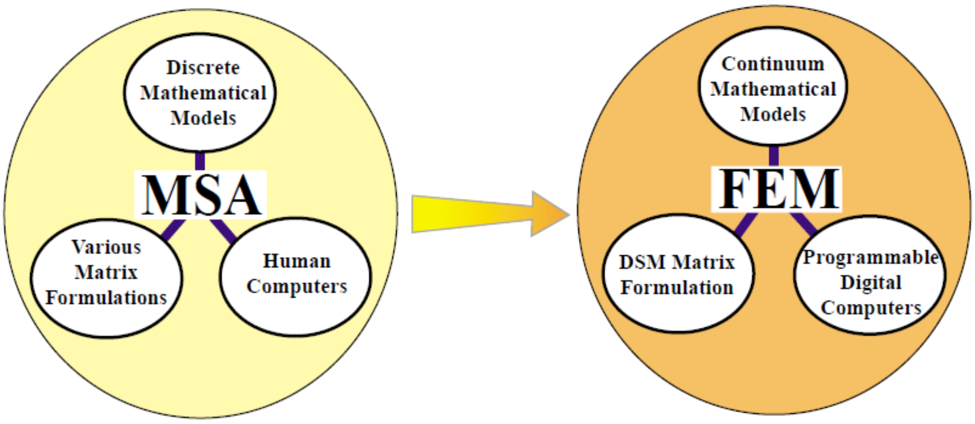

Figure 4 Morphing of the pre-computer MSA (before 1950) into the present FEM.

As the popularity of the finite element method began to grow in the engineering and physics communities, more applied mathematicians became interested in giving the method a firm mathematical foundation. As a result, a number of studies were aimed at estimating discretization error, rates of convergence, and stability for different types of finite element approximations. In the 1930s when a structural engineer encountered a truss problem, to solve for component stresses and deflections as well as the overall strength of the unit. He recognized that the truss was simply an assembly of rods whose force-deflection characteristics he knew well. Then he combined these individual characteristics according to the laws of equilibrium and solved the resulting system of equations for the unknown forces and deflections for the overall system. This procedure worked well whenever the structure had a finite number of interconnection points, but then a question arose: What can we do when we encounter an elastic continuum structure such as a plate that has an infinite number of interconnection points? For example, if a plate replaces the truss, the problem becomes considerably more difficult. Intuitively, Hrenikoff [3] reasoned that this difficulty could be overcome by assuming the continuum structure to be divided into elements or structural sections (beams) interconnected at only a finite number of node points. Under this assumption the problem reduces to that of a conventional structure, which could be handled by the old methods. Attempts to apply Hrenikoff's "framework method" were successful, and thus the seed to finite element techniques began to germinate in the engineering community.

Shortly after Hrenikoff, McHenry [4] and Newmark [5] offered further development of these discretization ideas, while Kron [6,7] studied topological properties of discrete systems. There followed a ten-year spell of inactivity, which was broken in 1954 when Argyris and his collaborators [8-12] began to publish a series of papers extensively covering linear structural analysis and efficient solution techniques well suited to automatic digital computation. The actual solution of plane stress problems by means of triangular elements whose properties were determined from the equations of elasticity theory was first given in 1956 paper of Turner, Clough, Martin, and Topp [13]. These investigators were the first to introduce what is now known as the direct stiffness method for determining finite element properties. Their studies, along with the advent of the digital computer at that time, opened the way to the solution of complex plane elasticity problems. After further treatment of the plane elasticity problem by Clough [1] in 1960, engineers began to recognize the importance of the finite element method. The time line of developments in the field of finite element method is given in **Table 1**.

Table 1 A time line of developments in finite elements.

Year	Analysis technology
1930	Collar and Duncan formulated discrete aeroelasticity in matrix form
1941	Framework method
1943	Courant studies of St. Venant torsion problem
1954 & 1955	Argyris publishes efficient solution techniques in classic paper solving plane stress
1956	Argyris presented a formal unification of Force and Displacement Methods using dual energy theorems
1959	Greenstadt's discretization approach
1959	Turner proposed the direct stiffness method
1960	Phrase *finite element* coined
1964	First commercial offering of finite element software

In 1965 the finite element method received an even broader interpretation when Zienkiewicz and Cheung [14] reported that it was applicable to all field problems that can be cast into variational form. During the late 1960s and early 1970s (while mathematicians were working on establishing errors, bounds, and convergence criteria for finite element approximations) engineers and other practitioners of the finite element method were also studying similar concepts for various problems in the area of solid mechanics. In the years since 1960 the finite element method has received widespread acceptance in engineering. Thousands of papers, hundreds of conferences, and many books have appeared on the subject.

How the finite element method works

The finite element discretization procedure reduces the problem by dividing a continuum to be a body of matter (solid, liquid, or gas) or simply a region of space into elements and by expressing the unknown field variable in terms of assumed approximating functions within each element. The approximating functions (sometimes called interpolation functions) are defined in terms of the values of the field variables at specified points called nodes or nodal points. Nodes usually lie on the element boundaries where adjacent elements are connected. In addition to boundary nodes, an element may also have a few interior nodes. The nodal values of the field variable and the interpolation functions for the elements completely define the behaviour of the field variable within the elements.

For the finite element representation of a problem the nodal values of the field variable become the unknowns. Once these unknowns are found, the interpolation functions define the field variable throughout the assemblage of elements. Clearly, the nature of the solution and the degree of approximation depend not only on the size and number of the elements used but also on the interpolation functions selected. As one would expect, we cannot choose functions arbitrarily, because certain compatibility conditions should be satisfied. Often functions are chosen so that the field variable or its derivatives are continuous across adjoining element boundaries.

An important feature of the finite element method that sets it apart from other numerical methods is the ability to formulate solutions for individual elements before putting them together to represent the entire problem. This means if we are treating a problem in stress analysis, we find the force–displacement or stiffness characteristics of each individual element and then assemble the elements to find the stiffness of the whole structure. In essence, a complex problem reduces to a series of greatly simplified problems. Another advantage of the finite element method is the variety of ways in which one can formulate the properties of individual elements. There are basically three different approaches.

The first approach to obtaining element properties is called the direct approach because its origin is traceable to the direct stiffness method of structural analysis. Although the direct approach can be used only for relatively simple problems, it is the easiest to understand when meeting the finite element method for the first time. The direct approach suggests the need for matrix algebra in dealing with the finite element equations. Element properties obtained by the direct approach can also be determined by the variational approach. The variational approach relies on the calculus of variations. For problems in solid mechanics the functional turns out to be the potential energy, the complementary energy, or some variant of these, such as the Reissner variational principle. Knowledge of the variational approach is necessary to work beyond the introductory level and to extend the finite element method to a wide variety of engineering problems. Whereas the direct approach can be used to formulate element properties for only the simplest element shapes, the variational approach can be employed for both simple and sophisticated element shapes.

A third and even more versatile approach to deriving element properties has its basis in mathematics and is known as the weighted residuals approach. The weighted residuals approach begins with the governing equations of the problem and proceeds without relying on a variational statement. This approach is advantageous because it thereby becomes possible to extend the finite element method to problems where no functional is available. The method of weighted residuals is widely used to derive element properties for nonstructural applications such as heat transfer and fluid mechanics.

Regardless of the approach used to find the element properties, the solution of a continuum problem by the finite element method always follows an orderly step-by-step process. To summarize in general terms how the finite element method works these are the steps.

Discretize the continuum

The first step is to divide the continuum or solution region into elements. In the example of **Figure 1** the turbine blade has been divided into triangular elements that might be used to find the temperature distribution or stress distribution in the blade. A variety of element shapes may be used, and different element shapes may be employed in the same solution region. Indeed, when analyzing an elastic structure that has different types of components such as plates and beams, it is not only desirable but also necessary to use different elements in the same solution. Although the number and type of elements in a given problem are matters of engineering judgment, the analyst can rely on the experience of others for guidelines.

Select interpolation functions

The next step is to assign nodes to each element and then choose the interpolation function to represent the variation of the field variable over the element. The field variable may be a scalar, a vector, or a higher-order tensor. Often, polynomials are selected as interpolation functions for the field variable because they are easy to integrate and differentiate. The degree of the polynomial chosen depends on the number of nodes assigned to the element, the nature and number of unknowns at each node, and certain continuity requirements imposed at the nodes and along the element boundaries. The magnitude of the field variable as well as the magnitude of its derivatives may be the unknowns at the nodes.

Find the element properties

Once the finite element model has been established (that is, once the elements and their interpolation functions have been selected), we are ready to determine the matrix equations expressing the properties of the individual elements. For this task we may use one of the three approaches just mentioned: the direct approach, the variational approach, or the weighted residuals approach.

Assemble the element properties to obtain the system equations

To find the properties of the overall system modelled by the network of elements we must "assemble" all the element properties. In other words, we combine the matrix equations expressing the behavior of the elements and form the matrix equations expressing the behavior of the entire system. The matrix equations for the system have the same form as the equations for an individual element except that they contain many more terms because they include all nodes. The basis for the assembly procedure stems from the fact that at a node, where elements are interconnected, the value of the field variable is the same for each element sharing that node. A unique feature of the finite element method is that the system equations are generated by assembly of the individual element equations. In contrast, in the finite difference method the system equations are generated by writing nodal equations.

Impose the boundary conditions

Before the system equations are ready for solution they must be modified to account for the boundary conditions of the problem. At this stage we impose known nodal values of the dependent variables or nodal loads.

Solve the system equations

The assembly process gives a set of simultaneous equations that we solve to obtain the unknown nodal values of the problem. If the problem describes steady or equilibrium behavior, then we must solve a set of linear or nonlinear algebraic equations. If the problem is unsteady, the nodal unknowns are a function of time, and we must solve a set of linear or nonlinear ordinary differential equations.

Make additional computations if desired

Many times we use the solution of the system equations to calculate other important parameters. For example, in a structural problem the nodal unknowns are displacement components. From these displacements we calculate element strains and stresses. Similarly, in a heat-conduction problem the nodal unknowns are temperatures, and from these we calculate element heat fluxes.

Range of applications

Applications of the finite element method divide into three categories, depending on the nature of the problem to be solved. In the first category are the problems known as equilibrium problems or time-independent problems. The majority of applications of the finite element method fall into this category, for the solution of equilibrium problems in the solid mechanics area, we need to find the displacement distribution and the stress distribution for a given mechanical or thermal loading. Similarly, for the solution of equilibrium problems in fluid mechanics, we need to find pressure, velocity, temperature, and density distributions under steady-state conditions.

In the second category are the so-called eigen value problems of solid and fluid mechanics. These are steady-state problems whose solution often requires the determination of natural frequencies and modes of vibration of solids and fluids. Examples of eigen value problems involving both solid and fluid mechanics appear in civil engineering when the interaction of lakes and dams is considered and in aerospace engineering when the sloshing of liquid fuels in flexible tanks is involved. Another class of eigen value problems includes the stability of structures and the stability of laminar flows.

The third category is the multitude of time-dependent or propagation problems of continuum mechanics. This category is composed of the problems that result when the time dimension is added to the problems of the first two categories. Just about every branch of engineering is a potential user of the finite element method. But the mere fact that this method can be used to solve a particular problem does not mean that it is the most practical solution technique. Often several are attractive but civil, mechanical, and aerospace engineers are the most frequent users of the method. In addition to structural analysis other areas of applications include heat transfer, fluid mechanics, electromagnetism, biomechanics, geomechanics, and acoustics. The method finds acceptance in multidisciplinary problems where there is a coupling between two or more of the disciplines. Examples include thermal structures where there is a natural coupling between heat transfer and displacements, as well as aeroelasticity where there is a strong coupling between external flow and the distortion of the wing. Techniques are available to solve a given problem. Each technique has its relative merits, and no technique enjoys the lofty distinction of being "the best" for all problems, the range of possible applications of the finite element method extends to all engineering disciplines.

Commercial finite element software

The first commercial finite element software made its appearance in 1964. The Control Data Corporation sold it in a time-sharing environment. No pre-processors (mesh generators) were available, so engineers had to prepare data element by element and node by node. A keypunched IBM (Hollerith) card represented each element and each node. Batch-mode line plots were used to check geometry and to post-process results. Only linear problems could be addressed. Nevertheless it represented a breakthrough in the complexity of the problem that could be handled in a practical time frame. Later, finite element software could be purchased or leased to run on corporate computers. Typically the corporate computer had been purchased to process financial data, so that computer availability to the engineer was restricted, perhaps to nights and weekends. The introduction of workstations circa 1980 brought several breakthrough advantages. Interactive graphics were practical and availability of computer power to solve problems on a dedicated basis was achieved. Finally, the introduction of personal computers (PCs) powerful enough to run finite element software provides extremely cost effective problem solving.

Today we have hundreds of commercial software packages to choose from. A small number of these dominate the market. It is difficult to make comparisons purely on a finite element basis, because the software houses are often diversified. Data from Daratech suggest that the companies listed in **Table 2** are dominant providers of general-purpose finite element software. Choice among these, or other providers, involves a complex set of criteria, usually including: analysis versatility, ease of use, efficiency, cost, technical support, training, and even the labor pool locally available to use particular software.

In contrast to the early days, we can now use computer-aided design (CAD) software or solid modelers to generate complex geometries, at either the component or assembly level. We can (with some restrictions) automatically generate elements and nodes, by merely indicating the desired nodal density. Software is available that works in conjunction with finite elements to generate structures of optimum topology, shape, or size. Nonlinear analyses including contact, large deflection, and nonlinear material behaviour are routinely addressed.

Table 2 Leading commercial finite element software companies.

Company name	Product name	Web site
Hibbitt, Karlsson & Sorensen	ABAQUS	http://www.hks.com
Ansys, Incorporated	ANSYS	http://www.ansys.com
Structural Data Research Corp.	SDRC-Ideas	http://www.sdrc.com
Parametric Technology, Inc.	RASNA	http://www.ptc.com
MSC Software Corp.	MSC/NASTRAN	http://www.mscsoftware.com

Conclusion

Our brief look at the history of the finite element method shows us that its early development was sporadic. The applied mathematicians, physicists, and engineers all dabbled with finite element concepts, but they did not recognize at first the diversity and the multitude of potential applications. After 1960 this situation changed and the tempo of development increased. By 1972 the finite element method had become the most active field of interest in the numerical solution of continuum problems. It remains the dominant method today. Part of its strength is that it can be used in conjunction with other methods. Software components such as solvers can be used in a modular fashion, so that improvements in diverse areas can be rapidly assimilated. Certainly, improved iterative solvers, mesh less formulations, better error indicators, and special-purpose elements are on the list of things to come. Although the finite element method can be used to solve a very large number of complex problems, there are still some practical engineering problems that are difficult to address because we lack an adequate theory of failure, or because we lack appropriate material data.

The mechanical and thermal properties of many nonmetallic materials are difficult to acquire, especially over a range of temperatures. Fatigue data is often lacking. Fatigue failure theory often lags our ability to calculate changing complex stress states. Data on friction is often difficult to obtain. Calculations based on the assumption of Coulomb friction are often unrealistic. There is a general paucity of thermal data, especially regarding absorbvity and emissivity needed for radiation calculations. The World Wide Web should offer a means of placing material properties into accessible databases. From a practitioner's viewpoint, the finite element method, like any other numerical analysis techniques, can always be made more efficient and easier to use. As the method is applied to larger and more complex problems, it becomes increasingly important that the solution process remains economical.

The rapid growth in engineering usage of computer technology will undoubtedly continue to have a significant effect on the advancement of the finite element method. Improved efficiency achieved by computer technology advancements such as parallel processing will surely occur. Since the mid 1970s interactive finite element programs on small but powerful personal computers and workstations have played a major role in the remarkable growth of computer-aided design. With continuing economic pressures to improve engineering productivity, this decade will see an accelerated role of the finite element method in the design process. This methodology is still exciting and an important part of an engineer's tool kit.

Acknowledgement

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