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Semi Analytical Solution of Boundary-Layer Flow of a Micropolar Fluid through a Porous Channel

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Received: 24 January 2012, Revised: 12 March 2012, Accepted: 30 October 2012

Abstract

In this study, the problem of boundary-layer flow of a micropolar fluid through a porous channel is dealt with. The flow occurs due to suction and injection at the walls of the channel. The micropolar fluid fills the space inside the channel. The similarity transformations are applied to reduce governing partial differential equations (PDEs) to a set of nonlinear coupled ordinary differential equations (ODEs) in dimensionless form. An efficient mathematical technique, called the differential transform method (DTM), is used to solve the nonlinear differential equations governing the problem in the form of series with easily computable terms. Graphical results are presented to investigate the influence of the pertinent parameters on the velocity and micro-rotation. The results illustrate the reliability and the performance of the DTM in comparison with the numerical method (forth-order Runge-Kutta) in solving this problem.

Keywords: Micropolar fluid, differential transform method (DTM), boundary-layer, similarity transformation

Introduction

A micropolar fluid is the fluid with internal structures in which coupling between the spin of each particle and the macroscopic velocity field is taken into account. The classical theories of continuum mechanics are inadequate to explicit the microscopic manifestations of such complex hydrodynamic behavior. The dynamics of micropolar fluids originated from the theory of Eringen [1] and have been a popular area of research for the last few decades as this class of fluids represents mathematically many industrially important fluids that may be applied to explain the flow of colloidal suspensions [2], liquid crystals [3], human and animal blood [4] and many other situations. Interesting aspects of theory and applications of micropolar fluids are dealt with in the books by Eringen [5] and Lukaszewicz [6]. The boundary-layer flow of a Newtonian fluid through a porous channel is studied by Berman [7]. He obtained a self similar exact solution of the problem. This solution of Newtonian fluid was further generalized in the references [8-11]. Vafai and Tien [12] studied the boundary and inertia effects in a porous media. Later, mixed heat transfer in porous media was studied by Tien and Vafai [13]. The suction flow in a channel for an upper-convected Maxwell fluid is presented by Choi *et al.* [14]. Ahmadi [15] presented solutions for the flow of a micropolar fluid past a semi-infinite plate taking into account micro-inertia effects using a Runge-Kutta shooting method with Newtonian iteration. Rees and Pop [16] studied free convection boundary-layer flow of micropolar fluids from a vertical flat plate. Char and Chang [17] have studied laminar free convection flow of micropolar fluids from a curved surface using a cubic spline collocation method. Free convection boundary layer flows past vertical and horizontal surfaces embedded in a porous medium, in which the flow is generated by Newtonian heating were considered by Lesnic *et al.* [18]. Rees and Pop [19] studied the effect of adopting a two-temperature model of microscopic heat transfer on the vertical free convection boundary-layer flow in a porous medium. There are several attempts available in literature by various authors who successfully describe the micropolar fluid [20-24]. Recently, Sajid *et al.* [25] studied the boundary-layer flow of a micropolar fluid through a porous channel with suction and injection at the walls of the channel.

Nomenclature

DTM	differential transform method
f	velocity fields
g	micro-rotation
H	width of porous channel
j	microinertial per unit mass
k	vortex viscosity, permeability
N	micro-rotation or angular velocity in the xy plane, series size
Re	Reynolds number
u	$x-$ component velocity
v	$y-$ component velocity
V	uniform velocity at the wall
x	horizontal coordinate
y	vertical coordinate

Greek symbols

γ	spin gradient viscosity
μ	dynamic viscosity
ν	kinematic viscosity
ρ	density

In this paper, the basic idea of the DTM is introduced and then ordinary non-linear differential equations are solved with the DTM. The method gives rapidly convergent series with specific significant features for each scheme.

Mathematical formulation

Let us consider an incompressible micropolar fluid through a porous channel of width *H*. The *x*-axis is along the centerline of the channel, parallel to the channel surfaces and the *y*-axis is perpendicular to it. The porous surfaces are at $y = \pm H/2$. The flow is symmetric about both the *x* and *y* axes. In the permeable walls, the proper relation between the velocity, *v*, and the actual permeability, *k*, is given by Darcy's law

$$v = -\frac{k}{\mu}\frac{dp}{dx}, \tag{1}$$

where *p* is the pressure drop across the permeable wall of thickness *h*. It is clear that increasing the speed of the permeable wall, causes greater pressure difference between the two sides of the permeable wall. In this work we assume pressure drop is constant and so we have constant velocity at the wall boundaries. So, the fluid is either injected into the channel or extracted out at a uniform velocity *V*/2 (the velocity $V > 0$ corresponds to the suction and $V < 0$ for injection). There is no pressure gradient. A schematic diagram of the problem is shown in **Figure 1**. The equations which govern the boundary-layer flow of a micropolar fluid are [25].

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=(\nu+\frac{k}{\rho})\frac{\partial^2 u}{\partial y^2}+\frac{k}{\rho}\frac{\partial N}{\partial y}, \quad (3) \qquad u\frac{\partial N}{\partial x}+v\frac{\partial N}{\partial y}=\frac{\gamma}{\rho j}\frac{\partial^2 N}{\partial y^2}-\frac{k}{\rho j}(2N+\frac{\partial u}{\partial y}). \quad (4)$$

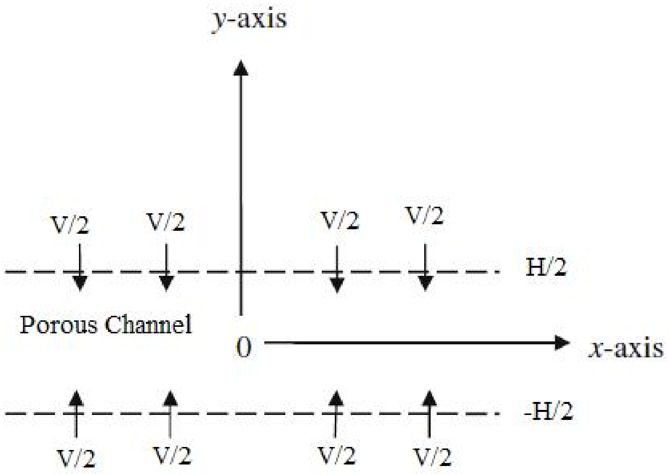

Figure 1 Geometry of the porous channel.

Here, γ is assumed to be [16]

$$\gamma=(\mu+\frac{k}{2})\,j, \quad (5)$$

in which μ is the dynamic viscosity and we take $j=H^2$ to be the reference length. As pointed out by Ahmadi [15] relation (5) is invoked to allow Eq. (2) - (4) to predict the correct behaviour in the limiting case when the microstructure effects become negligible and in this case micro-rotation reduces to the angular velocity, by assuming that the microelements close to the wall are unable to rotate [26]. In this case there is the strong concentration of microelements [27]. The appropriate boundary conditions are

$$\begin{aligned}&\frac{\partial u}{\partial y}=v=\frac{\partial N}{\partial y}=0 \quad \text{at} \quad y=0,\\ &v=\frac{V}{2}, \quad N=0 \quad \text{at} \quad y=\frac{H}{2}.\end{aligned} \quad (6)$$

Introducing [25]

$$x^*=\frac{x}{H}, \quad y^*=\frac{y}{H}, \quad u=-Vx^*f'(y^*), \quad v=Vf(y^*), \quad N=\frac{Vx^*}{H}g(y^*). \quad (7)$$

Eq. (2) is automatically satisfied and Eq. (3) - (6) give

$$(1+K)\,f'''+\mathrm{Re}(\,f'^2-ff'')-Kg'=0, \quad (8)$$

$$(1+\frac{K}{2})g''-\mathrm{Re}(fg'-g\,f')-K(2g-f'')=0, \tag{9}$$

$$f=0,\quad f''=0,\quad g'=0\quad \text{at}\quad y=0,$$
$$f=\frac{1}{2},\quad g=0\quad \text{at}\quad y=\frac{1}{2}, \tag{10}$$

where

$$\mathrm{Re}=\frac{HV}{\nu},\quad K=\frac{k}{\mu}, \tag{11}$$

and $\mathrm{Re}>0$ holds for suction and $\mathrm{Re}<0$ corresponds to injection.

The differential transform method

Most scientific problems and phenomena are modeled by non-linear ordinary or partial differential equations. Therefore, the study of the various methods used for solving the non-linear differential equations is a very important topic for the analysis of engineering practical problems. There are various techniques for solving non-linear problems, e.g. Adomian decomposition method (ADM) [28,29], homotopy perturbation method (HPM) [30-34], homotopy analysis method (HAM) [35-43] and variational iteration method (VIM) [44,45].

The DTM is a semi analytical-numerical technique that is based on Taylor series expansion. The concept of DTM was first introduced by Zhou [46] in solving linear and non-linear initial value problems in electrical circuit analysis. Chen and Ho [47] developed this method for partial differential equations and Ayaz [48] applied it to a system of differential equations. The method is very powerful [49]. The DTM obtains an analytical solution in the form of a polynomial. It is different from the traditional higher order Taylor series expansion method. The Taylor series expansion method is computationally expensive for large orders. The DTM is an alternative procedure for obtaining analytic Taylor series solution of the differential equations. In recent years, the DTM has been successfully employed to solve many types of non-linear problems [50-57]. All of these successful applications verified the validity, effectiveness and flexibility of the DTM.

The transformation of a function in one variable is as follows [58]

$$F(i)=\frac{1}{i!}[\frac{d^i f(y)}{dy^i}]_{y=y_0}, \tag{12}$$

in Eq. (12) $f(y)$ is the original function and $F(i)$ is the transformed function which is called the T-function (it is also called the spectrum of the $f(y)$ at $y=y_0$ in the Ω domain). The differential inverse transformation of F is defined as

$$f(y)=\sum_{i=0}^{\infty}F(i)\left(y-y_0\right)^i. \tag{13}$$

Eq. (13) implies that the concept of the differential transformation is derived from Taylor's series expansion, but the method does not evaluate the derivatives symbolically. However, relative derivatives are calculated by an iterative procedure that is described by the transformed equations of the original functions. In real applications, the function $f(\eta)$ in Eq. (13) is expressed by a finite series and can be written as

$$f(\eta)\cong\sum_{i=0}^{N}F(i)(y-y_0)^i. \tag{14}$$

In this paper initial conditions are at 0 so we let $y_0=0.$ Eq. (14) implies that $\sum_{i=N+1}^{\infty}F(i)(y-y_0)^i$ is negligibly small, when N is large.

The fundamental mathematical operations performed by DTM are listed in **Table 1**.

Table 1 The fundamental operations of differential transform method.

Original function	**Transformed function**
$f(y)=g(y)\pm h(y)$	$F(i)=G(i)\pm H(i)$
$f(y)=c\,g(y)$	$F(i)=c\,G(i)$
$f(y)=\dfrac{d^n g(y)}{dy^n}$	$F(i)=\dfrac{(i+n)!}{i!}G(i+n)$
$f(y)=\dfrac{d^n g(y)}{dy^n}\dfrac{d^m h(y)}{dy^m}$	$F(i)=\sum_{l=0}^{i}\dfrac{(l+n)!}{l!}G(l+n)\dfrac{(i-l+m)!}{(i-l)!}H(i-l+m)$
$f(y)=g(y)h(y)$	$F(i)=\sum_{l=0}^{i}G(l)H(i-l)$

Taking differential transformations of Eq. (8) and Eq. (9), we obtain

$$F(i+3)=\frac{-1}{(i+3)(i+2)(i+1)(1+K)}[\mathrm{Re}\sum_{r=0}^{i}(r+1)(i-r+1)F(r+1)F(i-r+1)$$
$$-\mathrm{Re}\sum_{r=0}^{i}(i-r+2)(i-r+1)F(i-r+2)F(r)-K(i+1)G(i+1)], \tag{15}$$

$$G(i+2)=\frac{1}{(i+2)(i+1)(1+\frac{K}{2})}[\mathrm{Re}\sum_{r=0}^{i}(i-r+1)F(r)G(i-r+1)$$
$$-\mathrm{Re}\sum_{r=0}^{i}(r+1)F(r+1)G(i-r)+2KG(i)-K(i+2)(i+1)F(i+2)], \tag{16}$$

where $F(i)$ and $G(i)$ are the differential transformations of $f(y^*)$ and 86 respectively. The transformation of the boundary conditions are

$$F(0)=0,\quad F(1)=\alpha,\quad F(2)=0,$$
$$G(0)=\beta,\quad G(1)=0, \tag{17}$$

where $\alpha,\ \beta$ are constants to be determined by imposing the end interval conditions stated in Eq. (10). So, for computing their value, the problem is solved with initial conditions (17) and then the boundary conditions (10) are applied

$$F(\frac{1}{2})=\frac{1}{2}\ \text{or}\ \sum_{i=0}^{N}(\frac{1}{2})^i F(i)=\frac{1}{2},$$
$$g(\frac{1}{2})=0\ \text{or}\ \sum_{i=0}^{N}(\frac{1}{2})^i G(i)=0. \tag{18}$$

Solving the two above equations simultaneously α,β are obtained.
For $N=10$ the solution of Eq. (8) and Eq. (9) (using the DTM) are as follows

$$
\begin{aligned}
f(y^*) &= \alpha y^* - \frac{\mathrm{Re}\,\alpha^2}{6(1+K)}(y^*)^3 + \frac{K(2K\beta - \mathrm{Re}\,\alpha\beta)}{24(1+\frac{K}{2})(1+K)}(y^*)^4 \\
&+ \frac{K^2\,\mathrm{Re}\,\alpha^2}{120(1+\frac{K}{2})(1+K)^2}(y^*)^5 - \frac{1}{120(1+K)}\Big(-\frac{K\,\mathrm{Re}\,\alpha(2K\beta - \mathrm{Re}\,\alpha\beta)}{6(1+\frac{K}{2})(1+K)} \\
&-(K\Big(\frac{K(2K\beta - \mathrm{Re}\,\alpha\beta)}{(1+\frac{K}{2})} - \frac{K^2(2K\beta - \mathrm{Re}\,\alpha\beta)}{2(1+\frac{K}{2})(1+K)} + \frac{\mathrm{Re}\,\alpha(2K\beta - \mathrm{Re}\,\alpha\beta)}{(1+\frac{K}{2})} \\
&- \mathrm{Re}\Big(-\frac{\mathrm{Re}\,\alpha^2\beta}{2(1+K)} + \frac{\alpha(2K\beta - \mathrm{Re}\,\alpha\beta)}{2(1+\frac{K}{2})}\Big)\Big)/3(1+\frac{K}{2})\Big)(y^*)^6 \\
&+ \cdots,
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
g(y^*) &= \beta + \frac{(2K\beta - \mathrm{Re}\,\alpha\beta)}{2(1+\frac{K}{2})}(y^*)^2 + \frac{K\,\mathrm{Re}\,\alpha^2}{6(1+\frac{K}{2})(1+K)}(y^*)^3 \\
&+ \frac{1}{12(1+\frac{K}{2})}\Big(\frac{K(2K\beta - \mathrm{Re}\,\alpha\beta)}{(1+\frac{K}{2})} - \frac{K^2(2K\beta - \mathrm{Re}\,\alpha\beta)}{2(1+\frac{K}{2})(1+K)} + \frac{\mathrm{Re}\,\alpha(2K\beta - \mathrm{Re}\,\alpha\beta)}{(1+\frac{K}{2})} \\
&- \mathrm{Re}\Big(-\frac{\mathrm{Re}\,\alpha^2\beta}{2(1+K)} + \frac{\alpha(2K\beta - \mathrm{Re}\,\alpha\beta)}{2(1+\frac{K}{2})}\Big)\Big)(y^*)^4 + \frac{1}{20(1+\frac{K}{2})}\Big(-\frac{K^3\,\mathrm{Re}\,\alpha^2}{6(1+\frac{K}{2})(1+K)^2} \\
&+ \frac{K^2\,\mathrm{Re}\,\alpha^2}{3(1+\frac{K}{2})(1+K)} + \frac{K\,\mathrm{Re}^2\,\alpha^3}{2(1+\frac{K}{2})(1+K)} - \mathrm{Re}\Big(\frac{K\,\mathrm{Re}\,\alpha^3}{6(1+\frac{K}{2})(1+K)} + \frac{K\beta(2K\beta - \mathrm{Re}\,\alpha\beta)}{6(1+\frac{K}{2})(1+K)}\Big)\Big)(y^*)^5 \\
&+ \cdots.
\end{aligned}
\tag{20}
$$

The Eq. (19) and Eq. (20) have sufficient accuracy. On the other hand, if the order of series size increases, the accuracy of the solution increases.

Results and discussion

The ordinary differential Eq. (8) and Eq. (9), with the boundary conditions (10) are solved numerically using forth-order Runge-Kutta scheme. For Reynolds number $\mathrm{Re} = -10$ and $\mathrm{Re} = 10$, the velocity fields $f(y^*)$ and micro-rotation $g(y^*)$ obtained by the different order of approximation for $K = 4$ are compared with the numerical results in **Tables 2** and **3**, respectively. We can see a very good agreement between the analytic results of the DTM and numerical results.

Table 2 The analytic results of $f(y^*)$ at different orders of approximation compared with the numerical results, when $K=4$.

Re	*y*	5th-order	10th-order	Numerical
–10	0.0	0.000000	0.000000	0.000000
	0.1	0.093136	0.093130	0.093130
	0.2	0.187999	0.187986	0.187986
	0.3	0.286308	0.286289	0.286289
	0.4	0.389758	0.389738	0.389738
	0.5	0.500000	0.500000	0.500000
10	0.0	0.000000	0.000000	0.000000
	0.1	0.109493	0.109488	0.109488
	0.2	0.216580	0.216571	0.216571
	0.3	0.318890	0.318876	0.318876
	0.4	0.414107	0.414093	0.414093
	0.5	0.500000	0.500000	0.500000

Table 3 The analytic results of $g(y^*)$ at different orders of approximation compared with the numerical results, when $K=4$.

Re	*y*	5th-order	10th-order	Numerical
–10	0.0	0.026357	0.026444	0.026444
	0.1	0.026734	0.026824	0.026824
	0.2	0.026352	0.026450	0.026450
	0.3	0.023034	0.023142	0.023142
	0.4	0.014805	0.014905	0.014905
	0.5	0.000000	0.000000	0.000000
10	0.0	–0.084492	–0.085347	–0.085347
	0.1	–0.083533	–0.084383	–0.084383
	0.2	–0.078453	–0.079291	–0.079291
	0.3	–0.065708	–0.066507	–0.066507
	0.4	–0.041202	–0.041840	–0.041840
	0.5	0.000000	0.000000	0.000000

Figures 2 and **3** show comparison between the numerical solutions obtained by the forth-order Runge–Kutta method and the DTM solution for $f(y^*)$ at Reynolds number $\mathrm{Re}=10$ (suction) and $\mathrm{Re}=-10$ (injection), respectively, over a range of K. From the graphs, it is very clear that unlike the previous solutions [25] the present solution agrees well with the obtained numerical solution over a wide range of Re. One can obtain the DTM solution for $\mathrm{Re}>10$ or $\mathrm{Re}<-10$ by increasing N. **Figure 4** illustrates the micro-rotation $g(y^*)$ for different values of K at Reynolds number $\mathrm{Re}=10$ (suction) and **Figure 5** shows the effect of parameter K on the velocity component $f(y^*)$. **Figures 6** and **7** show the analytical solutions of velocity components in the $x-$ direction and $y-$ direction at $x^*=0.5$ when $H=1$, $V=2$, $\mathrm{Re}=10$ and $K=6$, respectively.

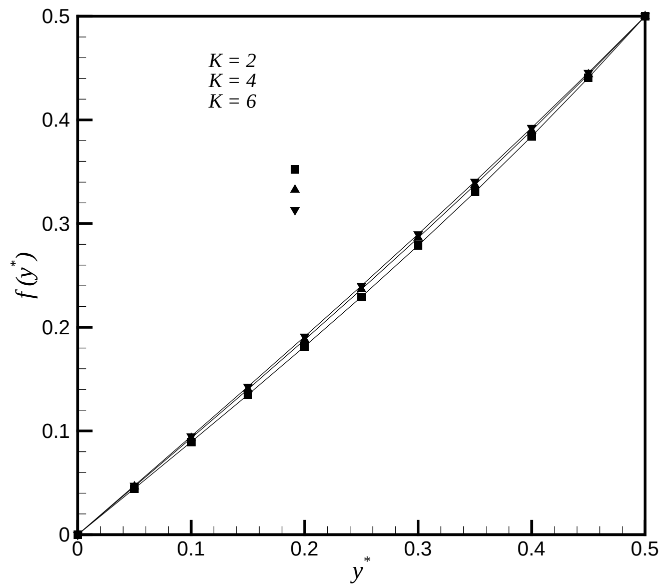

Figure 2 Comparison between the numerical and the analytical velocity fields $f(y^*)$ shown over a range of K at $\mathrm{Re}=-10$: (symbol) numerical results and (solid line) DTM.

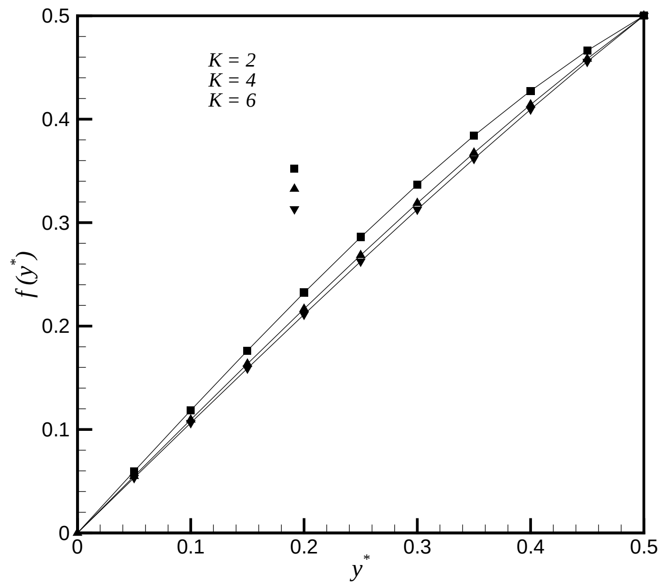

Figure 3 Comparison between the numerical and the analytical velocity fields $f(y^*)$ shown over a range of K at $\mathrm{Re}=10$: (symbol) numerical results and (solid line) DTM.

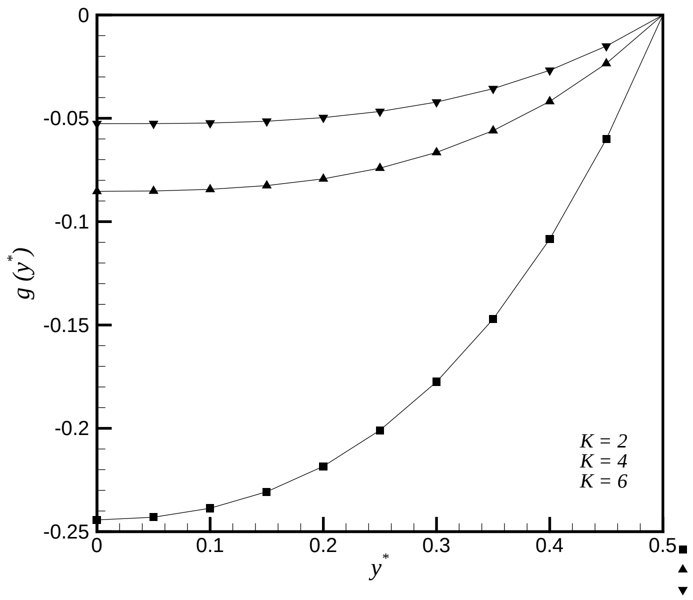

Figure 4 Comparison between the numerical and the analytical micro-rotation $g(y^*)$ shown over a range of K at $\mathrm{Re} = 10$:(symbol) numerical results and (solid line) DTM.

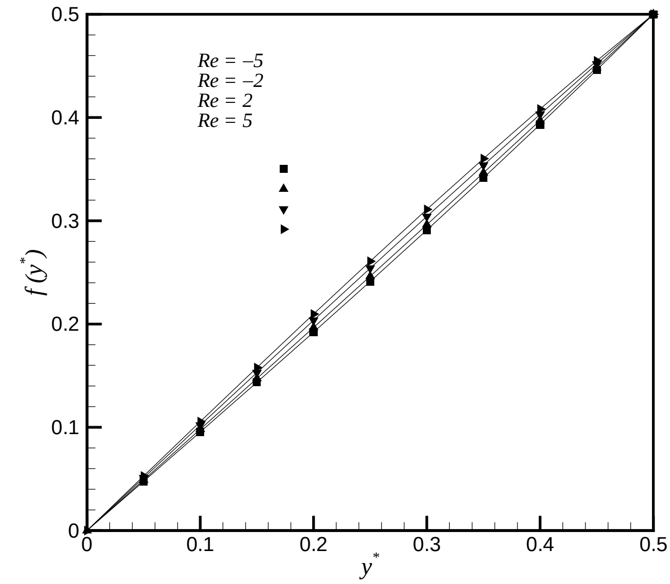

Figure 5 Comparison between the numerical and the analytical velocity fields $f(y^*)$ shown over a range of Re at $K = 3$:(symbol) numerical results and (solid line) DTM.

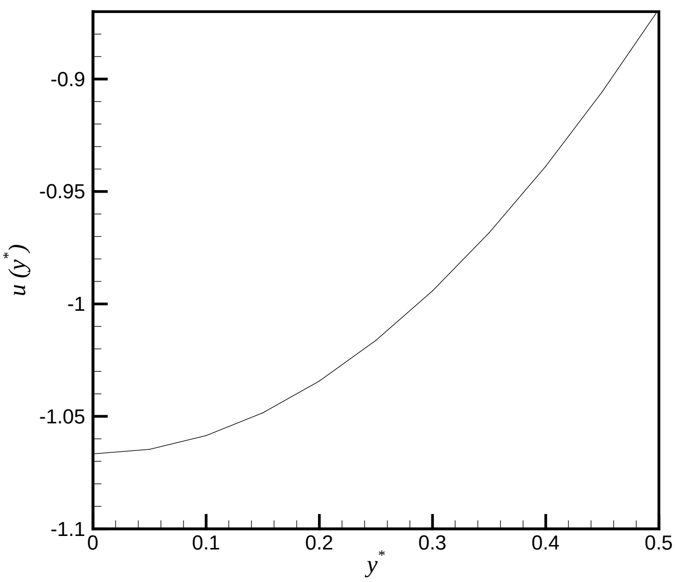

Figure 6 Analytical solution of the velocity component in the $x-$direction $u(x^*, y^*)$, when $x^* = 0.5$, $\mathrm{Re} = 10$ and $K = 6$.

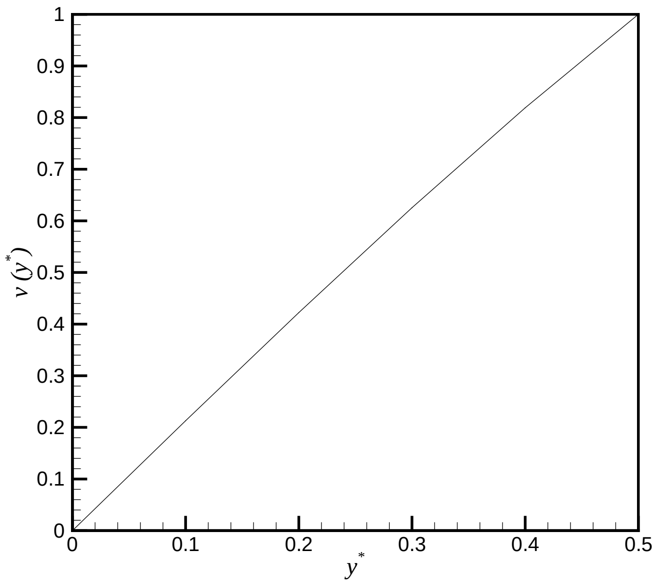

Figure 7 Analytical solution of the velocity component in the $y-$direction $v(x^*, y^*)$, when $x^* = 0.5$, $\mathrm{Re} = 10$ and $K = 6$.

Conclusions

In this paper, the DTM is employed to study the boundary-layer flow of a micropolar fluid through a porous channel. The main advantage of using a micropolar fluid model to study the boundary-layer flow in comparison with other classes of non-Newtonian fluids is that it takes care of the rotation of fluid particles by means of an independent kinematic vector called the micro-rotation vector. The reduced system of coupled ordinary differential equations is solved using DTM. The validity of our analytic solution is verified by numeric results. The results show that the DTM does not require small parameters in the equations, so the limitations of the traditional perturbation methods can be eliminated. The reliability of the method and reduction in the size of computational domain give this method a wider applicability. Therefore, this method can be applied to many nonlinear integral and differential equations without linearization, discretization or perturbation.

Acknowledgements

We express our gratitude to the anonymous referees for their constructive reviews of the manuscript and for helpful comments.

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