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A General Family of Fifth-Order Methods for Finding Simple Roots of Nonlinear Equations

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Abstract

In this paper, a new fifth-order family of methods free from second derivative is obtained. The new iterative family contains the King's family, which is one of the most well-known family of methods for solving nonlinear equations, and some other known methods as its particular case. To illustrate the efficiency and performance of the proposed family, several numerical examples are presented. Numerical results illustrate better efficiency and performance of the presented methods in comparison with the other compared fifth-order methods. Therefore, the proposed family can be effectively used for solving nonlinear equations.

Keywords: Iterative methods, Simple-root of nonlinear equations, Newton's method

Introduction

Nonlinear equations arise in a wide variety of forms in all branches of science, engineering, and technology. In recent years, a large number of methods of different order have been proposed and analyzed for solving nonlinear equations. For example, we refer the readers to [1-8] and the references therein.

In this paper, we propose an iterative method to obtain a simple root α of a nonlinear equation $f(x)=0$. i.e., $f(\alpha)=0$ and $f'(\alpha)\neq 0$.

It's well-known that the Newton's method is the most widely used (second-order) method for solving such equations, given by

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}. \tag{1}$$

Here, our approach is based on the fifth-order method defined in [5], as

$$x_{n+1}=y_n-\frac{f'(y_n)+3f'(x_n)}{5f'(y_n)-f'(x_n)}.\frac{f(y_n)}{f'(x_n)}, \tag{2}$$

where

$$y_n=x_n-\frac{f(x_n)}{f'(x_n)}. \tag{3}$$

Throughout the rest of this section, y_n is defined by Eq. (3).

This paper is organized as follows: In section 2, we consider a general iterative scheme, analyze it to present a family of fifth-order methods then several known special cases of this family are listed. Section 3 is devoted to numerical comparisons between the results obtained in this work and some known iterative methods. Finally, conclusions are drawn in the last section.

Development of method and convergence analysis

To obtain a fifth-order family of methods, we introduce and analyze the following iterative scheme:

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}-H(u_n)\frac{f(y_n)}{f'(x_n)} \tag{4}$$

where $H(u)$ is a function to be determined such that the iterative method defined by Eq. (4) has the order of convergence five, and

$$u_n = \frac{f'(y_n)}{f'(x_n)}. \tag{5}$$

It follows that for $H(u) = \frac{3+u}{-1+5u}$, the iterative scheme Eq. (4), reduces to the method of Kou and Li as given by Eq. (2). For the proposed family of methods Eq. (4) we have the following analysis of convergence.

Theorem 1. Let $\alpha \in I$ be a simple root of a sufficiently differentiable function $f : I \to \Re$ on an open interval which contains x_0 as a close initial approximation to α. If $H(u)$ satisfies the conditions

$$H(1) = 1,\ H'(1) = -1,\ H''(1) = 5/2 \tag{6}$$

then the method defined by Eq. (2) and Eq. (3) is of order at least five.

Proof. Let α be a simple zero of *f*. Consider the iteration function F defined by

$$F(x) = x - \frac{f(x)}{f'(x)} - H(u(x))\frac{f(y(x))}{f'(x)}$$

where $y(x) = x - \frac{f(x)}{f'(x)}$, and $u(x) = \frac{f'(y(x))}{f'(x)}$

In view of an elementary, tedious evaluation of derivatives of *F*, we employ the symbolic computation of the Maple package to compute the Taylor expansion of $F(x_n)$ around $x = \alpha$. We find after simplifying that

$$x_{n+1} = F(x_n) = \alpha + K_2 e_n^2 + K_3 e_n^3 + K_4 e_n^4 + O(e_n^5).$$

where

$$K_2 = (1 - H(1))c_2,$$
$$K_3 = (2 - 2H(1))c_3 + (4H(1) + 2H'(1) - 2)c_2^2,$$
$$K_4 = (3 - 3H(1))c_4 + (4 - 13H(1) - 14H'(1) - 2H''(1))c_2^3 + (14H(1) + 7H'(1) - 7)c_2c_3,$$

It can be easily verified that K_2, K_3 and K_4 vanish when

$$H(1) = 1,\ H'(1) = -1,\ H''(1) = 5/2$$

That is the method defined by Eq. (2) and Eq. (3) is of order at least five. This completes the proof.

From Theorem 1, we can see that the order of Newton's method can be improved by three units with additional evaluations of the one function and one derivative. So the order of convergence and computational efficiency of the method are greatly improved.

The iterative scheme Eq. (4) with some special choices for the function of $H(u)$ leads to some known fifth-order methods, as follows:

Case1: For the function H defined by

$$H(u) = \frac{3+u}{-1+5u}, \tag{7}$$

we obtain the fifth-order scheme Eq. (2).

Case2: For the function H giving by

$$H(u) = \frac{5+3u^2}{1+7u^2}, \tag{8}$$

in Eq. (4), a fifth-order method is obtained which has been introduced Fang *et al.* [6].

Case3: For the function H given by

$$H(u) = \frac{5-2u+u^2}{4u}, \tag{9}$$
$$H(u) = \frac{3+u^2}{2+2u^2},$$
$$H(u) = \frac{-4}{1-6u+u^2},$$
$$H(u) = \frac{5+3u^2}{1+7u},$$
$$H(u) = \frac{6+u+u^2}{-1+7u+2u^2},$$

respectively, which all have been introduced by Fang *et al.* in [7]. It is worth mentioning the functions defined in [8] as

$$H(u)=\frac{1+u}{-1+3u},$$
$$H(u)=\frac{2u^2}{1-4u+5u^2},$$
$$H(u)=-\frac{2}{1-4u+u^2},$$

do not satisfy the conditions in Eq. (6). So, against what is claimed in [8] the methods resulting aren't methods of order five. This fact has also been mentioned in [7].

To obtain a general family, let's consider the function H, as

$$H(u)=\frac{A+Bu+Cu^2}{D+Eu+Fu^2}. \tag{10}$$

It can easily be shown that $H(u)$ satisfies the conditions of Theorem 1, when

$$A=\frac{13D+5E+F}{4}, \tag{11}$$
$$B=-\frac{7D+E-F}{2},$$
$$C=\frac{5D+E+F}{4},$$

where D,E,F are real parameters that can be freely chosen.

It can easily be verified that Eq. (10) and Eq. (11) covers all the functions defined in Eq. (7) - Eq. (9). If we take $D=1, E=0$ and $F=0$ in Eq. (10) and Eq. (11), we obtain the following fifth-order method

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}-\left[\frac{13}{4}-\frac{7}{2}\frac{f'(y_n)}{f'(x_n)}+\frac{5}{4}\frac{f'(y_n)^2}{f'(x_n)^2}\right]\frac{f(y_n)}{f'(x_n)}. \tag{12}$$

If we take $D=0, E=0$ and $F=1$ in (10) and (11), we obtain the following fifth-order method

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}-\left[\frac{1}{4}+\frac{1}{2}\frac{f'(x_n)}{f'(y_n)}+\frac{1}{4}\frac{f'(x_n)^2}{f'(y_n)^2}\right]\frac{f(y_n)}{f'(x_n)}. \tag{13}$$

Each iteration in the method defined by Eq. (4) requires two functions and two first derivative evaluations. If we consider the definition of the efficiency index [11] as $\sqrt[r]{p}$, where p is the order of the method and r is the number of functional evaluations per iteration required by the method, we have an iteration formula defined by Eq. (10) and Eq. (11) has the efficiency index equal to $\sqrt[4]{5}\approx 1.5874$, which is better than that of Newton's method $\sqrt{2}\approx 1.4142$.

Numerical examples

In this section, some numerical tests of some various root-finding methods as well as our new methods and Newton's method are presented. Methods used for comparison were Newton's method Eq. (1) (NM), Chun's method Eq. (2) (CM), the Grau and Diaz-Barrero method (GM) [9] defined by

$$x_{n+1}=x_n-\left[1+\frac{f''(x_n)(f(x_n)+f(z_n))}{2f'(y_n)^2}\right]\frac{f(x_n)+f(z_n)}{f'(x_n)},$$

where $z_n=x_n-\left[1+\frac{1}{2}\frac{f''(x_n)f(x_n)}{f'(x_n)^2}\right]\frac{f(x_n)}{f'(x_n)}$, the method of Kou and Li (KM) [10] defined by

$$x_{n+1}=z_n-\left[1+\frac{M(x_n)}{1+M(x_n)}\right]\frac{f(z_n)}{f'(x_n)},$$

where $z_n=x_n-\left[1+\frac{1}{2}\frac{t(x_n)}{1-t(x_n)}\right]\frac{f(x_n)}{f'(x_n)}$, and

$M(x_n)=\dfrac{f''(x_n)(f(x_n)-f(z_n))}{f'(x_n)^2}$, with the new presented methods given by Eq. (12) (BGM1) and Eq. (13) (BGM2), introduced in this contribution. All computations were done using Maple with 128 digit floating point arithmetics (Digits = 128). Displayed in **Table 1** are the number of iterations and functional evaluations required such that $|f(x_n)|<10^{-15}$. The following functions (which are taken from [5-7]), are used for the comparison and we display the approximate zeros α found, up to the 28th decimal place.

$f_1(x)=x^3+4x^2-10,\ \alpha=1.3652300134140968457608068290,$

$f_2(x)=x^2-e^x-3x+2,\ \alpha=0.2575302854398607604553673049\text{4},$

$f_3(x)=x\,e^{x^2}-\sin^2(x)+3\cos(x)+5,\ \alpha=-1.2076478271309189270094167584,$

$f_4(x)=\sin(x)e^x+\ln(x^2+1),\ \ \alpha=0,$

$f_5(x)=(x-1)^3-2,\ \ \alpha=2.2599210498948731647672106073,$

$f_6(x)=(x+2)e^x-1,\ \alpha=-0.4428544010023885831413280000\text{0},$

$f_7(x)=\sin^2(x)-x^2+1,\alpha=1.4044916482153412260350868178.$

The results presented in **Table 1** show that for the functions we tested, the new methods introduced in this contribution reduce the number of iterations and needed functional evaluations showing that this family can be competitive to the known fifth-order methods and Newton's method and converge more quickly than the other compared methods.

Table 1 Comparison of number of iterations of various fifth-order convergent iterative methods.

	NM	CM	GM	KM	BGM1	BGM2
$f_1, x_0=-0.3$	55	11	27	24	8	7
$f_1, x_0=1$	6	3	4	4	3	3
$f_2, x_0=0$	5	3	3	3	2	2
$f_2, x_0=1$	5	3	4	4	3	3
$f_3, x_0=-1$	6	4	4	4	3	4
$f_3, x_0=-2$	9	5	5	Div	6	5
$f_4, x_0=2$	6	4	4	4	4	4
$f_4, x_0=-5$	8	4	6	4	5	5
$f_5, x_0=3$	7	4	4	4	3	3
$f_5, x_0=4$	8	4	5	4	4	4
$f_6, x_0=2$	9	5	5	Div	5	5
$f_6, x_0=3.5$	11	5	6	Div	6	6
$f_7, x_0=1$	7	4	4	5	6	4
$f_7, x_0=2$	6	4	2	4	3	3

Conclusion

In this paper, we have constructed a new general fifth-order iterative family of methods for solving nonlinear equations. This proposed iterative family contains several well-known methods as a special case. It is noteworthy that the presented methods show better performance and faster convergence than the King's method and some recent variants. Further research to find the optimal values of the function to achieve faster convergence is required.

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