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Affine Transformations of Itô Diffusions and their Transition Densities

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Abstract

For a given Itô diffusion, we derive the forward Kolmogorov equation (FKE) associated with the adjoint operator of the infinitesimal generator of an affine transformation of the given Itô diffusion. The fundamental solution obtained by solving the FKE is, in fact, the transition density of the transformed diffusion. Moreover, we prove that the transition density can be represented in terms of a product of two functions, a Jacobian term and a composition of the transition density of the given Itô diffusion and the inverse of the transformation. Finally, we present an application of our results in parameter estimation in commodity markets in which the commodity prices are assumed to follow an extended Black-Scholes model.

Keywords: Itô diffusions, forward Kolmogorov equation, transition densities

Introduction

Itô stochastic differential equations (Itô SDE) are a natural choice to model the time evolution of dynamic systems which are subject to random influences [1-2]. For example, in physics the dynamics of ions in super-ionic conductors are modeled via Langevin equations [3], and in engineering the dynamics of mechanical devices are described by differential equations under the influence of process noise as errors of measurement [4]. Other applications are in biology [5], medicine [6], econometrics [7], finance [8], geophysics [9], and oceanography [10]. Nevertheless, the models are less used if they contain unknown parameters that are without estimation.

It is natural that a model contains unknown parameters. We consider the model as the Itô SDE

$$dX_t = b(t, X_t;\theta)dt + a(t, X_t;\theta)dW_t,\ t \geq 0,\ X_0 = \zeta \quad (1)$$

where $\{W_t, t \geq 0\}$ is a one-dimensional Wiener process. The functions

$$b : [0,T] \times \mathbb{R} \times \Theta \to \mathbb{R} \text{ and}$$

$$a : [0,T] \times \mathbb{R} \times \Theta \to \mathbb{R} \setminus \{0\}$$

are known as the *drift coefficient* and the *diffusion coefficient*, respectively. These functions are known except the unknown parameter vector θ which is assumed to belong in a particular parameter space $\Theta \subseteq \mathbb{R}^r$ for some positive integer r and $E[\zeta^2] < \infty$. Under Lipschitz and the linear growth conditions on the coefficients b and a, there exists a unique non-exploding strong solution of the above Itô SDE, called the *Itô diffusion process* or simply the *Itô diffusion*, which is a continuous strong Markov semi-martingale. The drift and the diffusion coefficients are respectively the instantaneous mean and instantaneous standard deviation of the process. It should be noted that the diffusion coefficient is almost surely determined by the process, i.e. it can be estimated without any error if observed continuously throughout a time interval (see [11,12]). However, continuous observation of an Itô diffusion is a mathematical idealization because the path of the Itô diffusion is very jagged and no measuring device can follow an Itô diffusion trajectory continuously. Hence, the observation is always discrete in practice. Consequently, research on discretely observed Itô diffusions is growing recently with a powerful theory of statistical inference for Itô diffusions.

Statistical inference for Itô diffusions based on discrete-time observations is based on the likelihood function (see for example in [13-15]). The literature has mainly concentrated on the maximum likelihood (ML) approach. Suppose that an Itô diffusion $X = \{X_t, t \geq 0\}$ is observed at times t_i's with $0 = t_0 < t_1 < t_2 < ... < t_n = T$. Since X is Markov, hence if the *transition probability density function* or simply the *transition density* of X is known and denoted by $p_X \equiv p_X(t, x, s, x_0; \theta)$, one can use the log likelihood function

$$l_n(\theta) = \sum_{i=1}^{n} \ln p_X(t_i, X_{t_i}, t_{i-1}, X_{t_{i-1}}; \theta) \tag{2}$$

to estimate θ. Under the regularity conditions, the corresponding maximum likelihood estimate $\hat{\theta}_n$ is known to have usual good properties such as the consistency and asymptotic normality property (see for example in [16-19]).

According to the theory of Markov processes, under a set of conditions on the coefficients b and a and for a fixed parameter vector θ, p_X satisfies the *forward Kolmogorov equation* (FKE), for every fixed $(s, x_0) \in [0, T) \times D_X$:

$$\frac{\partial p_X}{\partial t}(t, x, s, x_0; \theta) = \mathcal{A}_t^* p_X(t, x, s, x_0; \theta)$$

for all $(t, x) \in (s, T] \times D_X$, (3)

subject to the condition

$$\lim_{t \downarrow s} \int_{D_X} p_X(t, x, s, x_0; \theta) f(x) dx = f(x_0)$$

for all $f \in C^0(D_X)$, (4)

where $C^0(D)$ denotes the set of bounded continuous functions on $D \subseteq \mathbb{R}$. The operator $\mathcal{A}_t^*$ on the RHS of (3), known as the adjoint operator of the infinitesimal generator of X starting at $X_t = x \in D_X$, is given by

$$\mathcal{A}_t^* p(x) = \frac{1}{2} \frac{\partial^2}{\partial x^2}\left(a^2(t, x) p(x)\right) - \frac{\partial}{\partial x}\left(b(t, x) p(x)\right)$$

for all $p \in C^2(\mathbb{R})$, (5)

where for a probability space $(\Omega, \mathcal{F}, \mathbb{P}_\theta)$,

$$D_X = \{x \in \mathbb{R} \mid \exists t \in [0, T], \mathbb{P}_\theta(X_t = x) = 1\}$$

denotes the domain of X. We assume for simplicity that D_X is a bounded domain, i.e. an open connected bounded subset of $\mathbb{R}$. In view of the theory of partial differential equations of a parabolic type, p_X which satisfies the FKE (3) and the condition (4) is, in fact, a *fundamental solution* of the FKE (3) (see Chapter 1 of Friedman [20]). In view of the theory of Markov processes, we have

$$\mathbb{P}_\theta(X_t \in B \mid X_s = x_0) = \int_B p_X(t, x, s, x_0; \theta) dx$$

for every Borel set $B \subseteq \mathbb{R}$ (see Theorem 5.4 of Chapter 6 of Friedman [21]). One can see from (2) that, in order to follow the method of ML, the FKE has to be solved to obtain a closed form of p_X or at least the approximates of p_X at the observed points (t_i, X_{t_i}), for $i = 0, 1, ..., n$.

In terms of analytical methods for solving the FKE (3) subject to the condition (4), for several special cases of the drift and diffusion coefficients, the corresponding transition densities can be derived in closed form. For instance,

$$a(t, x) = \sigma, \text{ and } b(t, x) = \mu$$

where $\sigma > 0$, μ are constants (the corresponding Itô diffusion is called a *Brownian motion with drift* μ), its transition density is

$$p_X(t, x, s, x_0; \theta) = \frac{1}{\sqrt{2\pi(t-s)}\,\sigma} \exp\left(-\frac{\left(x - (x_0 + \mu(t-s))\right)^2}{2(t-s)\sigma^2}\right), \tag{6}$$

where $\theta = (\mu, \sigma)$. One can verify that p_X as expressed in (6) satisfies the FKE (3) and the condition (4), namely, p_X is the fundamental solution of the FKE (3) for which

$$\mathcal{A}_t^* \equiv \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} - \mu \frac{\partial}{\partial x}.$$

In many cases, however, a set of data which relates to the transformation of X, i.e.

$$\{F(t_i, X_{t_i}; \theta), i = 0, 1, 2, ..., n\},$$

is observed instead of the data of X. Although, for each time t_i, the transformation is one-to-one in the space variable x, the data of X cannot be retrieved since the transformation depends on unknown parameters. Fortunately, the method of ML still works by replacing the transition density p_X in (2) by the transition density of the transformed process $Y_t = F(t, X_t; \theta)$.

The aim of this paper is to derive the transition density of a transformed process $Y = \{Y_t, t \geq 0\}$ which is an affine transformation of X, defined as follows: for a fixed parameter vector θ,

$$Y_t = m(t;\theta)X_t + c(t;\theta) \tag{7}$$

for all $t \in [0,T]$, where $m(t;\theta)$ and $c(t;\theta)$ are deterministic functions depending on the time t and parameter θ. We provide a set of sufficient conditions for proving that the transition density of Y, denoted by p_Y, can be represented in terms of a product of two functions, a Jacobian term and a composition of p_X and the inverse of the transformation.

Affine transformations of Itô diffusions and their transition densities

In the present section, we provide sufficient conditions to guarantee the existence and uniqueness of the transition density of Y. Note that from this point forward, the parameter vector θ is considered to be fixed and we omit writing θ as an argument of the relevant functions.

We begin with the sufficient conditions to guarantee the existence and uniqueness of the transition density of X.

Assumption 1: The coefficients $a(t,x)$ and $b(t,x)$ are Lipschitz continuous in both the time variable t and space variable x on $[0,T] \times D_X$.

Assumption 2: The coefficients $a(t,x)$ and $b(t,x)$ satisfy the linear growth condition in the space variable x, i.e.

$$|a(t,x)|^2 + |b(t,x)|^2 \leq K^2(1 + |x|^2)$$

for all $(t,x) \in [0,T] \times D_X$, where K is a positive constant.

Assumption 3: There is $\lambda > 0$ such that $a(t,x) \geq \lambda$ for all $(t,x) \in [0,T] \times D_X$. In other words, $a(t,x)$ is uniformly elliptic on $\mathbb{R}$ for all $(t,x) \in [0,T] \times D_X$.

Assumption 4: The partial derivatives

$$\frac{\partial^2 a}{\partial x^2}, \ \frac{\partial a}{\partial x}, \text{ and } \frac{\partial b}{\partial x}$$

are Lipschitz continuous in both the time variable t and space variable x on $[0,T] \times D_X$.

The following theorem is an important implication of Friedman's work in [20] and [21]. He established a connection between fundamental solutions of parabolic partial differential equations and transition densities of Itô diffusions and investigated some of their properties. We state the theorem below without proof, because the proof is quite intricate and needs several pages.

Theorem 1: *Under Assumptions 1 - 4, there is a unique fundamental solution of the FKE (3). Moreover, if p_X is the fundamental solution then it is nonnegative and its partial derivatives*

$$\frac{\partial p_X}{\partial t}, \ \frac{\partial p_X}{\partial x}, \textit{and } \frac{\partial^2 p_X}{\partial x^2}$$

are continuous in the variables $t, x, s,$ and x_0 on

$$E_X = \{(t,x,s,x_0) \mid 0 \leq s < t \leq T, x, x_0 \in D_X\}.$$

In addition,

$$\int_{D_X} p_X(t,x,s,x_0)dx = 1$$

for all $0 \leq s < t \leq T$ and all $x_0 \in D_X$.

Remark 1: We recommend Chapters 1 and 9 of Friedman [20] and Chapter 6 of Friedman [21] for the proofs of the obtained results in the theorem.

Remark 2: The condition in Assumption 3 can be relaxed somewhat by making it the local requirement (the condition A3b′ in [22]) as follows:

Assumption 3′: There is a sequence $(D_n)_{n=1}^{\infty}$ of bounded domains with $\overline{D_n} \subseteq D_X$ such that $D_X = \bigcup_{i=1}^{\infty} D_n$ and for each n there is $\lambda_n > 0$ such that $a(t,x) \geq \lambda_n$ for all $(t,x) \in D_n$.

Replacing Assumption 3 by Assumption 3′, the conclusions of Theorem 1 still hold. The following assumption is required for preserving Y as a random process and for applying Itô's lemma.

Assumption 5: The real-valued functions $m(t)$ and $c(t)$ belong to $C^2([0,T])$ and $m(t)$ is either negative or positive on $[0,T]$.

Under Assumption 5, one can easily see that Y is a random process because $m(t)$ never reaches zero for all $t \in [0,T]$. Furthermore, $m(t)$ and $c(t)$ are smooth on $[0,T]$, and hence, we can apply Itô's lemma to Y as defined in (7). Then, we have Y that is also an Itô diffusion and satisfies the Itô SDE

$$dY_t = \hat{b}(t,Y_t)dt + \hat{a}(t,Y_t)dW_t,\ t \geq 0,$$
$$Y_0 = m(0)X_0 + c(0), \tag{8}$$

where the coefficient functions in (8) are given by, for any $(t,y) \in [0,T] \times D_Y$,

$$\hat{a}(t,y) = m(t)a\left(t, \frac{y - c(t)}{m(t)}\right), \tag{9}$$

$$\hat{b}(t,y) = m'(t)\left(\frac{y - c(t)}{m(t)}\right) + c'(t) + m(t)b\left(t, \frac{y - c(t)}{m(t)}\right). \tag{10}$$

Here, D_Y denotes the domain of Y and $m'(t)$ and $c'(t)$ signify the derivatives of $m(t)$ and $c(t)$ with respect to t, respectively. It should be noted that, under Assumption 5 and the boundedness of D_X as previously assumed, D_Y is also a bounded domain in $\mathbb{R}$.

From the previous section, if the transition density of Y exists then it must satisfy the FKE, for every fixed $(s,y_0) \in [0,T) \times D_Y$:

$$\frac{\partial p_Y}{\partial t}(t,y,s,y_0) = \hat{\mathcal{A}}_t^* p_Y(t,y,s,y_0)$$

for all $(t,y) \in (s,T] \times D_Y$, (11)

subject to the condition

$$\lim_{t \downarrow s} \int_{D_Y} p_Y(t,y,s,y_0)f(y)dy = f(y_0)$$

for all $f \in C^0(\mathbb{R})$. (12)

The operator $\hat{\mathcal{A}}_t^*$ denotes the adjoint operator of the infinitesimal generator of Y starting at $Y_t = y \in D_Y$,

$$\hat{\mathcal{A}}_t^* p(y) = \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\hat{a}^2(t,y)p(y)\right) - \frac{\partial}{\partial y}\left(\hat{b}(t,y)p(y)\right)$$

for all $p \in C^2(\mathbb{R})$. (13)

Corollary 1: *Under Assumptions 1-5, there is a unique fundamental solution of the FKE (11). Moreover, if p_Y is the fundamental solution then it is nonnegative and its partial derivatives*

$$\frac{\partial p_Y}{\partial t},\ \frac{\partial p_Y}{\partial y},\ \text{and}\ \frac{\partial^2 p_Y}{\partial y^2}$$

are continuous in the variables $t, y, s,$ and y_0 on

$$E_Y = \{(t,y,s,y_0) \mid 0 \leq s < t \leq T, y, y_0 \in D_Y\}.$$

In addition,

$$\int_{D_Y} p_Y(t,y,s,y_0)dy = 1$$

for all $0 \leq s < t \leq T$ and all $y_0 \in D_Y$.

Proof. Suppose that Assumptions 1-5 are fulfilled. It is easy to show that the coefficient functions $\hat{a}(t,y)$ and $\hat{b}(t,y)$ of the Itô diffusion Y as written in (9) and (10) satisfy the conditions in Assumptions 1-4 with respect to the variables t and y. It should be pointed that $\hat{b}(t,y)$ is

Lipschitz continuous since $m(t)$ and $c(t)$ belong to $C^2([0,T])$ which implies that $m'(t)$ and $c'(t)$ are Lipschitz continuous. Hence, by Theorem 1, the conclusions of the corollary are obtained.

Main results

Theorem 2: *Let X be a random variable for which the probability density function (p.d.f.) is f_X and $\Pr(a < X < b) = 1$. Let $Y = g(X)$, and suppose that $g(x)$ is continuous and either strictly increasing or strictly decreasing on $[a,b]$. Suppose also that $a < X < b$ if and only if $\alpha < Y < \beta$, and let $X = h(Y)$ be the inverse function for $\alpha < Y < \beta$. Then the p.d.f. of Y is specified by the relation*

$$f_Y(y) = \begin{cases} \left|\dfrac{dh(y)}{dy}\right| f_X(h(y)) & \text{for } \alpha < y < \beta \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if $Y = mX + c$, for some constants $m \neq 0$ and $c \in \mathbb{R}$ then the p.d.f. of Y is given by

$$f_Y(y) = \begin{cases} \dfrac{1}{|m|} f_X\left(\dfrac{y-c}{m}\right) & \text{for } \min(\alpha,\beta) < y < \max(\alpha,\beta) \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha = ma + c$ and $\beta = mb + c$.

Proof. The proof of the theorem is provided in Chapter 2 of [23].

We now return to the transformation (7). Theorem 2 guides us to construct a nonnegative function which is claimed to coincide with the transition density of Y as follows. For a fixed $(s, y_0) \in [0,T) \times D_Y$, we define

$$\overline{p}_Y(t,y,s,y_0) = \frac{1}{|m(t)|} p_X\left(t, \frac{y-c(t)}{m(t)}, s, \frac{y_0 - c(s)}{m(s)}\right)$$

for $(t,y) \in (s,T] \times D_Y$. (14)

It should be noted that the first term on the RHS of (14) is the absolute value of the Jacobian of the inverse of the transformation (7) and the second term is the composition of p_X and the inverse of the transformation (7).

Under Assumptions 1-5, we have the following two lemmas.

Lemma 1: $\overline{p}_Y$ *satisfies the FKE (11) on* E_Y.

Lemma 2: $\overline{p}_Y$ *satisfies the condition (12).*

The proofs of Lemmas 1 and 2 are provided in Appendices A and B, respectively.

Now, we state the main theorem.

Theorem 3: Under Assumptions 1-5,

$$\overline{p}_Y = p_Y \text{ on } E_Y.$$

Proof. By Lemma 1 and 2, $\overline{p}_Y$ satisfies the FKE (11) and the condition (12). From the existence and uniqueness of the fundamental solution in Corollary 1, $\overline{p}_Y$ must coincide with p_Y on E_Y.

Application in parameter estimation in commodity markets

In this section, we consider parameter estimation of an extended Black-Scholes model described by the Itô SDE as follows:

$$dS_t = (r - \delta(t))S_t\,dt + \sigma S_t\,dW_t,\ t \geq 0, \qquad (15)$$

$$\delta(t) = \alpha_0 + \sum_{k=1}^{K} \alpha_k \sin(2\pi k t), \qquad (16)$$

where $S = \{S_t, t \geq 0\}$ is a price process, $\delta(t)$ is a deterministic function of time t, r is a risk-free interest rate, and $\sigma > 0$, α_i, for $i = 0,1,...,K$ for some integer $K \geq 1$, are unknown parameters. One can verify that the drift and diffusion coefficients of S sufficiently satisfy the conditions in Section 2, i.e. Assumptions 1, 2, 3′, 4, and 5. The models (15) and (16) describe, respectively, dynamics of commodity prices and seasonal variation of *convenience yields* (see Chapter 2 in [24] for the definition of convenience yields). From the Introduction, we need the transition density of S for constructing the log-likelihood function, and hence, for calculating the maximum likelihood estimates of the unknown parameters. The associated FKE of S is

$$\frac{\partial p_S}{\partial t} = \frac{1}{2}\sigma^2 y^2 \frac{\partial^2 p_S}{\partial y^2} - (r - \delta(t))\frac{\partial p_S}{\partial y}, \qquad (17)$$

for every fixed $(s, y_0) \in [0,T) \times D_S$, where $p_S \equiv p_S(t, y, s, y_0)$ and D_S denote, the transition density of S and the domain of S, respectively. To avoid solving the FKE (17), we notice that S is, in fact, an affine transformation of a log-normal diffusion:

$$S_t = m(t)X_t,\ \ t \geq 0 \tag{18}$$

where

$$m(t) = \exp\left(-\int_0^t \delta(\eta)d\eta\right),$$

and $X = \{X_t, t \geq 0\}$ is a log-normal diffusion (the Black-Scholes model), i.e.

$$dX_t = rX_t dt + \sigma X_t dW_t.$$

Applying Theorem 3 to the transformation (18), for discretely observed data $\{(t_i, S_{t_i}), i = 0,1,...,n\}$, we obtain

$$p_S\left(t_i, S_{t_i}, t_{i-1}, S_{t_{i-1}}; \theta\right) = \frac{1}{\exp\left(-\int_0^{t_i} \delta(\eta)d\eta\right)} \times p_X\left(t_i, \frac{S_{t_i}}{\exp\left(-\int_0^{t_i} \delta(\eta)d\eta\right)}, t_{i-1}, \frac{S_{t_{i-1}}}{\exp\left(-\int_0^{t_{i-1}} \delta(\eta)d\eta\right)}; \theta\right),$$

where p_X is the transition density of X which is known in closed form as

$$p_X(t, x, s, x_0; \theta) = \frac{1}{\sqrt{2\pi(t-s)}\,\sigma x} \exp\left(-\frac{(\ln(x / x_0) - (r - \frac{1}{2}\sigma^2)(t-s))^2}{2\sigma^2(t-s)}\right),$$

where $\theta = (\sigma, \alpha_0, \alpha_1, ..., \alpha_K)$.

Conclusions

This paper provides a set of sufficient conditions for the existence and uniqueness of the transition densities of affine transformations of a given Itô diffusion. Moreover, the paper has established a general formula for the transition densities of the transformed Itô diffusions, providing the transition density of the given Itô diffusion. An application to parameter estimation in commodity markets has been demonstrated. Finally, an extension of these results to affine transformations of multivariate Itô diffusions will be investigated in future work.

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Appendix A (Proof of Lemma 1)

Proof. Assume that $m(t)$ is positive on $[0,T]$. From Theorem 1, the continuities of the partial derivatives of p_X allow us to apply the chain rule to the formula of $\overline{p}_Y$. The results obtained are the following relations:

$$\frac{\partial \overline{p}_Y}{\partial t} = \frac{d}{dt}\left(\frac{1}{m(t)}\right) p_X + \frac{1}{m(t)}\left(\frac{\partial p_X}{\partial t} + \frac{\partial p_X}{\partial x}\frac{dx}{dt}\right)$$

$$= -\frac{m'(t)}{m^2(t)} p_X + \frac{1}{m(t)}\frac{\partial p_X}{\partial t} + \left(\frac{c(t)m'(t) - c'(t)m(t)}{m^3(t)} - \frac{m'(t)}{m^3(t)}(m(t)x + c(t))\right)\frac{\partial p_X}{\partial x},$$

$\frac{\partial \overline{p}_Y}{\partial y} = \frac{1}{m^2(t)}\frac{\partial p_X}{\partial x}$, and $\frac{\partial^2 \overline{p}_Y}{\partial y^2} = \frac{1}{m^3(t)}\frac{\partial^2 p_X}{\partial x^2}$, where for any $(t,y) \in [s,T]\times D_Y$,

$$\overline{p}_Y \equiv \overline{p}_Y(t,y,s,y_0),\ p_X \equiv p_X(t,x(t),s,x(s)), \text{ and } x \equiv x(t) = \frac{y - c(t)}{m(t)}.$$

We use the above relations to compute the partial derivatives contained in $\hat{\mathcal{A}}_t^* \overline{p}_Y$. Thus we have

$$\frac{\partial}{\partial y}\left\{\hat{b}(t,y)\overline{p}_Y\right\} = \left\{\frac{m'(t)}{m(t)} + \frac{\partial b}{\partial x}\left(t, \frac{y - c(t)}{m(t)}\right)\right\}\overline{p}_Y$$

$$+\left\{b\left(t,\frac{y-c(t)}{m(t)}\right)m(t) + \left\{\frac{c'(t)m(t) - c(t)m'(t)}{m(t)}\right\} + \frac{m'(t)}{m(t)}y\right\}\frac{\partial \overline{p}_Y}{\partial y}$$

$$= \frac{m'(t)}{m^2(t)}p_X - \left\{\frac{c(t)m'(t) - c'(t)m(t)}{m^3(t)} - \frac{m'(t)}{m^3(t)}(m(t)x + c(t))\right\}\frac{\partial p_X}{\partial x}$$

$$+\frac{1}{m(t)}\frac{\partial}{\partial x}\left\{b(t,x)p_X\right\}$$

$$\frac{\partial^2}{\partial y^2}\left\{\hat{a}^2(t,y)\overline{p}_Y\right\} = \left\{\frac{2}{m^2(t)}\left\{\frac{\partial a}{\partial x}\left(t,\frac{y-c(t)}{m(t)}\right)\right\}^2 + a\left(t,\frac{y-c(t)}{m(t)}\right)\frac{\partial^2 a}{\partial x^2}\left(t,\frac{y-c(t)}{m(t)}\right)\right\}\overline{p}_Y$$

$$+\left\{\frac{4}{m(t)}a\left(t,\frac{y-c(t)}{m(t)}\right)\frac{\partial a}{\partial x}\left(t,\frac{y-c(t)}{m(t)}\right)\right\}\frac{\partial \overline{p}_Y}{\partial y} + a^2\left(t,\frac{y-c(t)}{m(t)}\right)\frac{\partial^2 \overline{p}_Y}{\partial y^2}$$

$$= \frac{1}{m(t)}\left\{2\left\{\frac{\partial a}{\partial x}(t,x)\right\}^2 + 2a(t,x)\frac{\partial^2 a}{\partial x^2}(t,x)\right\}p_X + \frac{4}{m(t)}a(t,x)\frac{\partial a}{\partial x}(t,x)\frac{\partial p_X}{\partial x}$$

$$+\frac{1}{m(t)}a^2(t,x)\frac{\partial^2 p_X}{\partial x^2} = \frac{1}{m(t)}\frac{\partial^2}{\partial x^2}\left\{a^2(t,x)p_X\right\}.$$

Replacing the partial derivatives contained in the following FKE with the above relations gives us

$$\frac{\partial \overline{p}_Y}{\partial t} - \hat{\mathcal{A}}_t^* \overline{p}_Y = \frac{\partial \overline{p}_Y}{\partial t} + \frac{\partial}{\partial y}\left\{\hat{b}(t,y)\overline{p}_Y\right\} - \frac{1}{2}\frac{\partial^2}{\partial y^2}\left\{\hat{a}^2(t,y)\overline{p}_Y\right\}$$

$$= -\frac{m'(t)}{m^2(t)} p_X + \frac{1}{m(t)}\frac{\partial p_X}{\partial t} + \left\{\frac{c(t)m'(t) - c'(t)m(t)}{m^3(t)} - \frac{m'(t)}{m^3(t)}(m(t)x + c(t))\right\}\frac{\partial p_X}{\partial x}$$

$$+\frac{m'(t)}{m^2(t)} p_X - \left\{\frac{c(t)m'(t) - c'(t)m(t)}{m^3(t)} - \frac{m'(t)}{m^3(t)}(m(t)x + c(t))\right\}\frac{\partial p_X}{\partial x} + \frac{1}{m(t)}\frac{\partial}{\partial x}\left\{b(t,x)p_X\right\}$$

$$-\frac{1}{2}\left\{\frac{1}{m(t)}\frac{\partial^2}{\partial x^2}\left\{a^2(t,x)p_X\right\}\right\}$$

$$= \frac{1}{m(t)}\frac{\partial p_X}{\partial t} + \frac{1}{m(t)}\frac{\partial}{\partial x}\left\{b(t,x)p_X\right\} - \frac{1}{2m(t)}\frac{\partial^2}{\partial x^2}\left\{a^2(t,x)p_X\right\}$$

$$= \frac{1}{m(t)}\left\{\frac{\partial p_X}{\partial t} - \mathcal{A}_t^* p_X\right\} = 0.$$

In the case $m(t)$ being negative on $[0,T]$, one can show by following the above steps that $\overline{p}_Y$ satisfies the FKE (11) on $(s,T]\times D_Y$. This completes the proof.

Appendix B (Proof of Lemma 2)

Proof. Let $f \in C^0(D_Y)$. From Theorem 1, $p_X(t,x,s,x_0)$ is continuous in both *t* and *x* for which $(t,x)\in(s,T]\times D_X$ and satisfies the condition (4). Moreover, $m(t)$ and $c(t)$ are continuous on $[0,T]$. Thus we can compute the following limit:

$$\lim_{t\downarrow s}\int_{D_Y}\overline{p}_Y(t,y,s,y_0)f(y)dy = \lim_{t\downarrow s}\int_{D_Y}\frac{1}{|\,m(t)\,|}p_X\left(t,\frac{y-c(t)}{m(t)},s,\frac{y_0-c(s)}{m(s)}\right)f(y)\,dy$$

$$= \lim_{t\downarrow s}\left\{\lim_{\tau\downarrow s}\int_{D_Y}\frac{1}{|\,m(\tau)\,|}p_X\left(t,\frac{y-c(\tau)}{m(\tau)},s,\frac{y_0-c(s)}{m(s)}\right)f(y)\,dy\right\}$$

$$= \lim_{t\downarrow s}\int_{D_Y}\lim_{\tau\downarrow s}\left\{\frac{1}{|\,m(\tau)\,|}p_X\left(t,\frac{y-c(\tau)}{m(\tau)},s,\frac{y_0-c(s)}{m(s)}\right)\right\}f(y)\,dy$$

$$= \lim_{t\downarrow s}\int_{D_Y}\frac{1}{|\,m(s)\,|}p_X\left(t,\lim_{\tau\downarrow s}\left(\frac{y-c(\tau)}{m(\tau)}\right),s,\frac{y_0-c(s)}{m(s)}\right)f(y)\,dy$$

$$= \lim_{t\downarrow s}\int_{D_Y}\frac{1}{|\,m(s)\,|}p_X\left(t,\frac{y-c(s)}{m(s)},s,\frac{y_0-c(s)}{m(s)}\right)f(y)\,dy$$

$$= \lim_{t\downarrow s}\int_{D_X}\frac{1}{|\,m(s)\,|}p_X\left(t,x,s,x_0\right)f\left(m(s)x+c(s)\right)\left|J_X\right|dx$$

$$= \lim_{t\downarrow s}\int_{D_X}p_X\left(t,x,s,x_0\right)f\left(m(s)x+c(s)\right)dx$$

$$= f\left(m(s)x_0 + c(s)\right)$$
$$= f(y_0),$$

where J_X denotes the Jacobian of the transformation $x \mapsto y = m(s)x + c(s)$ and $x_0 = \dfrac{y_0 - c(s)}{m(s)}$.

The proof is now complete.

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