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Short Communication

Parameter Estimation of the Extended Vasiček Model

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Abstract

In this paper, an estimate of the drift and diffusion parameters of the extended Vasiček model is presented. The estimate is based on the method of maximum likelihood. We derive a closed-form expansion for the transition (probability) density of the extended Vasiček process and use the expansion to construct an approximate log-likelihood function of a discretely sampled data of the process. Approximate maximum likelihood estimators (AMLEs) of the parameters are obtained by maximizing the approximate log-likelihood function. The convergence of the AMLEs to the true maximum likelihood estimators is obtained by increasing the number of terms in the expansions with a small time step size.

Keywords: The extended Vasiček model, transition density, maximum likelihood estimation

Introduction

A classical stochastic model for short rate dynamics is to represent the short rate at time t, denoted by r_t, as the solution of the stochastic differential equation:

$$dr_t = \alpha(\beta - r_t)dt + \sigma dW_t, \tag{1}$$

for all $t > 0$ and $r_0 = r^* > 0$, where W_t is the standard one-dimensional Brownian motion, *a*, *b*, and *s* are positive constants designated as the model parameters. This model was first introduced by Vasiček [1] and has been widely used for bond and option pricing under the equilibrium market model. The advantage of the Vasiček model is analytical expressions of bonds and bond options can be possibly expressed by using the dynamics of the short rates. Moreover, the model can be used to evaluate all interest rate contingent claims in a consistent way. However, the classic problem with the Vasiček model is its endogenous nature. If we have the initial zero coupon bond curve from a market, we wish our model to incorporate this curve. This means we need to force the model parameters to produce a model curve as close as possible to the market curve. This is not suitable for the Vasiček model due to the fact that the parameters are assumed to be constants. To improve this situation, exogenous term structures are introduced which are built by modifying the endogenous ones. The basic strategy is to transform an endogenous model to an exogenous model by using time-varying parameters. Hence, the Vasiček model is extended to

$$dr_t = \alpha(t;\theta)(\beta(t;\theta) - r_t)dt + \sigma(t;\theta)dW_t, \tag{2}$$

for all $t > 0$ and $r_0 = r^* > 0$, where $\alpha(t;\theta)$, $\beta(t;\theta)$, and $\sigma(t;\theta)$ are nonrandom functions of time and θ. Here, $\theta \in \mathbb{R}^m$ is a vector of m exogenous parameters. The model just introduced is called "*the extended Vasiček model*" and a rigorous study of this model is given in [2].

One problem arising from the use of the extended Vasiček model is estimation of model parameters based on the method of maximum likelihood. This is not the case for the Vasiček model because the transition density of the process and the log-likelihood function have explicit formulas [3]. In contrast to the Vasiček model, for given functions $\alpha(t;\theta)$, $\beta(t;\theta)$, and $\sigma(t;\theta)$, the transition density of the extended Vasiček process, in general, is unknown. Under regular conditions [4], the transition density of the extended Vasiček process, denoted by $p(r,t \mid r_0,s;\theta)$, satisfies the fundamental partial differential equation:

$$\frac{\partial p}{\partial t}-\frac{\sigma^2(t;\theta)}{2}\frac{\partial^2 p}{\partial r^2}+\frac{\partial}{\partial r}\Big[\alpha(t;\theta)\big(\beta(t;\theta)-r\big)p\Big]=0, \quad (3)$$

for all $r\in\mathbb{R}$ and for all $t>s$, and the boundary condition:

$$\lim_{t\to s} p(r,t \mid r_0,s;\theta)=\delta(r-r_0), \quad (4)$$

for all $r\in\mathbb{R}$, for a fixed initial time s, a fixed initial short rate r_0, and a fixed parameter vector θ. The function $\delta(x)$ is known as the one-dimensional Dirac delta function.

Suppose that a discrete sample data of r_t, i.e. $\{r_{t_0}, r_{t_1},\dots,r_{t_N}\}$, is observed at sample dates $\{t_n = n\Delta t \mid n=0,1,\dots,N\}$ for some positive integer N and for some equidistant time step size $\Delta t>0$. Applying Bayes's rule to the Markov process r_t, we simply get the log-likelihood function of the data:

$$L_N(\theta):=\sum_{n=1}^{N}\ln p(r_{t_n},t_n \mid r_{t_{n-1}},t_{n-1};\theta). \quad (5)$$

Maximizing $L_N(\theta)$ with respect to θ over a particular parameter space Θ, one can get a maximum likelihood estimator θ_N^{MLE} which is a solution of the optimization problem:

$$\theta_N^{MLE}=\arg\max_{\theta\in\Theta} L_N(\theta). \quad (6)$$

In this research, we suppose that the logarithmic of the transition density of the extended Vasiček process can be expressed in a closed-form expansion as follows:

$$\ln p(r,t \mid r_0,s;\theta)=-\hat{D}(r,t;\theta)-\frac{1}{2}\ln(2\pi\Delta t) +\frac{\hat{C}^{(-1)}(r,t \mid r_0,s;\theta)}{\Delta t}+\sum_{k=0}^{\infty}\frac{\hat{C}^{(k)}(r,t \mid r_0,s;\theta)(\Delta t)^k}{k!}.$$

(7)

Using the method proposed by Choi [5], the coefficient functions $\hat{D}$ and $\hat{C}^{(k)}$, $k=-1,0,1,\dots,K$, for some positive integer K, are computed and so is an approximate log-likelihood function which from hereon is denoted by $L_N^{(K,\Delta t)}(\theta)$. Replacing $L_N(\theta)$ in (6) by $L_N^{(K,\Delta t)}(\theta)$ and solving the problem, we can get an approximate maximum likelihood estimator $\hat{\theta}_N^{(K,\Delta t)}$. Under the regular conditions imposed in [6], $\hat{\theta}_N^{(K,\Delta t)}$ is close to θ_N^{MLE} (in a probability sense) when K is large and Δt is small.

The rest of this paper is organized as follows. In the next section, we set up the model and assumptions. Choi's method for univariate time-inhomogeneous diffusions is presented in the third section. In the fourth section, a closed-form expansion for the transition density of the extended Vasiček process is derived; that is, the coefficient functions in (7) are obtained. The convergence of the approximate maximum likelihood estimators (AMLEs) to the true maximum likelihood estimators (MLEs) is discussed within the section. The last section is the conclusion.

Set up and assumptions

A formal statement of the parameter estimation problem to be addressed in this paper is as follows. Given a univariate time-inhomogeneous diffusion:

$$dX_t=\mu_X(X_t,t;\theta)dt+\sigma_X(X_t,t;\theta)dW_t, \quad (8)$$

where W_t is the standard one-dimensional Brownian motion, and $\theta\in\mathbb{R}^m$ is a vector of m parameters that need to be estimated. The drift and diffusion coefficients in (8), i.e. $\mu_X(\cdot\,;\theta)$ and $\sigma_X(\cdot\,;\theta)$ are assumed to be prescribed functions of state X_t and time t. We easily see that, by setting $\mu_X\equiv\alpha(t;\theta)(\beta(t;\theta)-X_t)$ and $\sigma_X\equiv\sigma(t;\theta)$, the diffusion process (8) belongs to the class of the extended Vasiček model.

Let Θ be a compact subset of $\mathbb{R}^m$ and $D_X\subseteq\mathbb{R}$ be the domain of the diffusion process X. We modify the regularity conditions imposed by Egorov *et al* [6] to suit the extended Vasiček process. The following modified conditions guarantee the existence and uniqueness of the transition density of the extended Vasiček process.

(A1) Functions $\alpha(t;\theta)$, $\beta(t;\theta)$, and $\sigma(t;\theta)$ belong to $C^{\infty,2}\left([0,\infty)\times\Theta\right)$.

(A2) The diffusion coefficient $\sigma_X(x,t;\theta)=\sigma(t;\theta)$ is non-degenerate [6].

Materials and methods

In the present section, we shortly review Choi's method [5] for derivation of a closed-form expansion for the transition density of the extended Vasiček process. Firstly, we transform the diffusion process (8) to a unit diffusion process (10) by using the following transformation from X to Y,

$$y := \gamma(x,t;\theta) = \int^x \frac{1}{\sigma_X(\omega,t;\theta)}\,d\omega. \tag{9}$$

Applying the Itô formula to (9), one can get

$$dY_t = \mu_Y(Y_t,t;\theta)dt + dW_t, \tag{10}$$

where

$$\mu_Y(y,t;\theta) = \frac{\mu_X(\gamma^{-1}(y,t;\theta),t;\theta)}{\sigma_X(\gamma^{-1}(y,t;\theta),t;\theta)} + \frac{\partial\gamma}{\partial t}(\gamma^{-1}(y,t;\theta),t;\theta) - \frac{1}{2}\frac{\partial\sigma_X}{\partial x}(\gamma^{-1}(y,t;\theta),t;\theta). \tag{11}$$

Let p_X be the transition density of X. Suppose that Assumptions (A1) and (A2) are fulfilled. The following theorem proved in [5] ensures that $l_X \equiv \ln p_X$ has a closed-form expansion.

Theorem 1 (Application of Theorem 3.2 in [5]) *Let $\Delta t = t - t_0$ and $\theta\in\Theta$ be fixed. The closed-form expansion of l_X is given by*

$$l_X(x,t\mid x_0,t_0;\theta) = -D_v(x,t;\theta) - \frac{1}{2}\ln(2\pi\Delta t) + \frac{C_Y^{(-1)}(\gamma(x,t;\theta),t\mid\gamma(x_0,t_0;\theta),t_0;\theta)}{\Delta t} + \sum_{k=0}^{\infty} C_Y^{(k)}(\gamma(x,t;\theta),t\mid\gamma(x_0,t_0;\theta),t_0;\theta)\frac{(\Delta t)^k}{k!}, \tag{12}$$

for all $(x,t)\in D_X\times(s,\infty)$. The coefficient functions in the expansion (12) satisfy the following equations:

$$D_v(x,t;\theta) = \frac{1}{2}\ln(\sigma_X^2(x,t;\theta)) = \ln\sigma(t;\theta), \tag{13}$$

$$C_Y^{(-1)}(y,t\mid y_0,t_0;\theta) = -\frac{(y-y_0)^2}{2}, \tag{14}$$

$$C_Y^{(0)}(y,t\mid y_0,t_0;\theta) = (y-y_0)\times \int_0^1 \mu_Y(y_0+u(y-y_0),t;\theta)\,du, \tag{15}$$

and for $k\ge 1$

$$C_Y^{(k)}(y,t\mid y_0,t_0;\theta) = k\times \int_0^1 G_Y^{(k)}(y_0+u(y-y_0),t\mid y_0,t_0;\theta)\,u^{k-1}du, \tag{16}$$

where

$$G_Y^{(1)}(y,t\mid y_0,t_0;\theta) = -\frac{1}{2}\left[\mu_Y^2(y,t;\theta) + \frac{\partial\mu_Y(y,t;\theta)}{\partial y}\right] - \int_{y_0}^{y}\frac{\partial\mu_Y(\omega,t;\theta)}{\partial t}\,d\omega, \tag{17}$$

and for $k\ge 2$

$$G_Y^{(k)}(y,t\mid y_0,t_0;\theta) = \frac{1}{2}\frac{\partial^2 C_Y^{(k-1)}(y,t\mid y_0,t_0;\theta)}{\partial y^2} - \frac{\partial C_Y^{(k-1)}(y,t\mid y_0,t_0;\theta)}{\partial t} + \frac{1}{2}\sum_{h=1}^{k-2}\binom{k-1}{h}\frac{\partial C_Y^{(h)}(y,t\mid y_0,t_0;\theta)}{\partial y}\times \frac{\partial C_Y^{(k-1-h)}(y,t\mid y_0,t_0;\theta)}{\partial y}. \tag{18}$$

Results and discussion

In this section, we apply Theorem 1 to derive a closed-form expansion for the logarithmic of the transition density $p(r,t \mid r_0,s;\theta)$. By Eq. (2), (9) and (11), we have

$$y = \gamma(r,t;\theta) = \int^r \frac{1}{\sigma(t;\theta)}\,d\omega = \frac{r}{\sigma(t;\theta)}, \tag{19}$$

$$\mu_Y(y,t;\theta) = \frac{\alpha(t;\theta)(\beta(t;\theta)-\sigma(t;\theta)y)}{\sigma(t;\theta)} - \frac{\sigma'(t;\theta)y}{\sigma(t;\theta)}. \tag{20}$$

The coefficient functions in (7) must satisfy the following equations:

$$\hat{D}(r;t;\theta) = D_v(r,t;\theta) = \ln\sigma(t;\theta), \tag{21}$$

and for $k \geq -1$

$$\hat{C}^{(k)}(r,t \mid r_0,t;\theta) = C_Y^{(k)}(\gamma(r,t;\theta),t \mid \gamma(r_0,t_0;\theta),t_0;\theta). \tag{22}$$

For example, the second and third coefficient functions can easily be computed:

$$\hat{C}^{(-1)}(r,t \mid r_0,t;\theta) = -\frac{1}{2}\left(\frac{r}{\sigma(t;\theta)} - \frac{r_0}{\sigma(t_0;\theta)}\right)^2, \tag{23}$$

$$\hat{C}^{(0)}(r,t \mid r_0,t;\theta) = C_Y^{(0)}(\gamma(r,t;\theta),t \mid \gamma(r_0,t_0;\theta),t_0;\theta) \tag{24}$$

where

$$C_Y^{(0)}(y,t \mid y_0,t_0;\theta) = (y-y_0)\times$$

$$\left(\frac{\alpha(t;\theta)(2\beta(t;\theta)-(y+y_0)\sigma(t;\theta))-(y+y_0)\sigma'(t;\theta)}{2\sigma(t;\theta)}\right).$$

The remaining coefficient functions $\hat{C}^{(k)}$, $k = 1,2,\ldots,K,$ satisfying Eq. (22), can be obtained by writing a symbolic computation procedure with MATHEMATICA or MAPLE for solving the integral Eq. (15) and (16). An example of the procedures was written by Rujivan [7].

Next, we define

$$l^{(K,\Delta t)}(r,t \mid r_0,s;\theta) := -\hat{D}(r,t;\theta) - \frac{1}{2}\ln(2\pi\Delta t)$$

$$+\frac{\hat{C}^{(-1)}(r,t \mid r_0,s;\theta)}{\Delta t} + \sum_{k=0}^{K}\frac{\hat{C}^{(k)}(r,t \mid r_0,s;\theta)(\Delta t)^k}{k!},$$

and

$$L_N^{(K,\Delta t)}(\theta) := \sum_{n=1}^{N} l^{(K,\Delta t)}(r_{t_n},t_n \mid r_{t_{n-1}},t_{n-1};\theta), \tag{25}$$

$$\hat{\theta}_N^{(K,\Delta t)} := \arg\max_{\theta\in\Theta} L_N^{(K,\Delta t)}(\theta). \tag{26}$$

It is natural to investigate the distance between $\hat{\theta}_N^{(K,\Delta t)}$ and θ_N^{MLE}. In other words, can we use $\hat{\theta}_N^{(K,\Delta t)}$ as a maximum likelihood estimator to estimate the model parameters? Fortunately, the answer is yes. Theorem 2 in [6] shows that, if Assumptions 1 - 5 imposed in the paper are fulfilled, $\hat{\theta}_N^{(K,\Delta t)} \to \theta_N^{MLE}$ in a probability sense as $K \uparrow \infty$ and $\Delta t \downarrow 0$. From a practical point of view, however, an increase in K results in an increase in running time for solving the optimization problem (26). Hence, for a small Δt, $K = 1$ is appropriate for practical problems [7].

Conclusions

This article provides a closed-form expansion of the transition density of the extended Vasiček process. The obtained closed-form expansion can be used to get AMLEs of the drift and diffusion parameters of the process. In terms of efficiency, Theorem 2 in [6] guarantees that the AMLEs are close to the true MLEs in terms of probability when the number of terms in the expansion is large and the time step size is small. An application of this research is to use the AMLEs for computing fair prices of bonds and options in which their prices depend on the dynamics of short rates as given in Eq. (2). Plugging the AMLEs into Eq. (2) and employing Monte Carlo simulations [8], fair prices can be obtained.

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