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Original research article

Fekete-Szegö Problem for Certain Classes of Analytic Functions Associated with Petal Type Domain and Modified Sigmoid Function

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ABSTRACT

In this work, the authors studied the Fekete-Szegö problems for certain classes of analytic functions associated with petal type domain and modified sigmoid function. The initial coefficient bounds have been obtained and discussed the relevant connection to Fekete-Szegö inequalities. The results give birth to some corollaries.

Keywords: Analytic function; Univalent function; Subordination; Fekete-Szegö inequalities; Modified sigmoid function; Petal type domain.

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1. Introduction

Let A be the class of function $f(z)$ analytic in the open unit disc $\mathbb{D} = \{z: |z| < 1\}$ and of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \qquad (1.1)$$

normalized by $f(0) = f'(0) - 1 = 0$. Recall that, $S \subset A$ is the univalent function which has the starlike and convex functions as its subclasses which satisfies $Re\left(\frac{zf'(z)}{f(z)}\right) > 0$ and $Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0$. The usual classes of functions aforementioned have been used to define various subclasses of analytic functions by many scholars and their interesting results can not be ignored.

Two functions f and g are said to be subordinate to each other, written as $f \prec g$, if there exists a schwartz function ω such that

$$f(z) = g(\omega(z)) \qquad (1.2)$$

where $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \mathbb{D}$. Let P denote the class of analytic functions such that $p(0) = 1$ and $p \prec \frac{1+z}{1-z}$, $z \in \mathbb{D}$ (see [1]).

Goodman [2] initiated the concept of conic domain to generalize a convex function which generated the first parabolic region as an image domain of analytic functions. He introduced and studied the class of uniformly convex functions which satisfy

$$UCV = Re\left(1 + (z - \Psi)\frac{f''(z)}{f'(z)}\right) > 0, \qquad z, \Psi \in \mathbb{D}.$$

Ma and Minda [3] gave a characterization of the class UCV which satisfy

$$UCV = Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right|, \qquad z \in \mathbb{D}.$$

The characterization gave birth to the first parabolic characterization of the class UCV which satisfy

$$\Omega = \{\omega > |\omega - 1|\} \qquad (1.3)$$

and (1.3) was later generalized by Kanas and Wisniowska [4, 5] to

$$\Omega_k = \{\omega > k|\omega - 1|, k \geq 0\}.$$

The Ω_k represents the right half plane for $k = 0$, hyperbola for $k \in (0,1)$, parabolic for $k = 1$ and elliptic regions for $k > 1$.

Many researchers have worked tirelessly on generalized conic domains and their results are too voluminous to discuss (see for example [6], [7] and so on).

Moreover, the petal type region $\Omega[A, B], -1 \leq B < A \leq 1$ was also generalized from (1.3) to

$$\Omega(A,B) = \{u + iv: [(B^2-1)(u^2+v^2) - 2(AB-1)u + (A^2-1)]^2 > (-2(b+1)(u^2+v^2) + 2(A+B+C)u - 2(A+1))^2 + 4(A-B)^2v^2\}$$

by Noor and Malik [8].

Recently, Murugusundaramoorthy et al. [9] studied the Fekete-Szegö problems for space of logistic sigmoid functions based on quasi-subordination for the classes $\mathcal{S}_q^*(\alpha, \Phi)$, $\mathcal{M}_q(\alpha, \Phi)$ and $\mathcal{L}_q(\alpha, \Phi)$ in which interpretations satisfy

$$\frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)} - 1 \prec_q \Phi(z) - 1, \qquad (1.4)$$

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1 \prec_q \Phi(z) - 1, \qquad (1.5)$$

and

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} - 1 \prec_q \Phi(z) - 1, \qquad (1.6)$$

where $\alpha \geq 0$, $\Phi(z)$ is the logistic sigmoid function and the results obtained are added to literature.

A function $p(z)$ is said to be in the class $UP[A,B]$; if and only if

$$p < \frac{(A+1)\tilde{p}(z)-(A-1)}{(B+1)\tilde{p}(z)-(B-1)}, -1 \leq B < A \leq 1, \tag{1.7}$$

where $\tilde{p} = 1 + \frac{2}{\pi^2}\left(log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2$, $z \in \mathbb{D}$.

Varying $A = 1$ and $B = -1$ "in (1.7)," one may obtain the classes of functions studied by Goodman [1] and Kanas [10].

Also, the classes $ST[A,B]$ and $UCV[A,B]$ are uniformly Janowski starlike and convex functions satisfies

$$R\left(\frac{(B-1)\frac{zf'(z)}{f(z)}-(A-1)}{(B+1)\frac{zf'(z)}{f(z)}-(A+1)}\right) > \left|\frac{(B-1)\frac{zf'(z)}{f(z)}-(A-1)}{(B+1)\frac{zf'(z)}{f(z)}-(A+1)} - 1\right| \tag{1.8}$$

and

$$R\left(\frac{(B-1)\frac{(zf'(z))'}{f'(z)}-(A-1)}{(B+1)\frac{(zf'(z))'}{f'(z)}-(A+1)}\right) > \left|\frac{(B-1)\frac{(zf'(z))'}{f'(z)}-(A-1)}{(B+1)\frac{(zf'(z))'}{f'(z)}-(A+1)} - 1\right| \tag{1.9}$$

See Noor and Malik [8]. Setting $A = 1$ and $B = -1$ (1.8) and (1.9) obtains the classes of functions studied by Goodman [2] and Ronning [11].

The relevant connection to Fekete-Szegö inequalities is a way of maximizing the non-linear functional $|a_3 - \lambda a_2^2|$ for various subclasses of univalent function theory. See [12], [13], [14] [15], [16], [17], [18] and so on.

We recall the definition and properties of a sigmoid function as a special function that deals with an information process inspired by the way a nervous system such as the brain processes information. It contains large numbers of highly interconnected processing elements (neurones) working together to solve a specific problem. It has application in real analysis, topology, differential equations, algebra and so on. Special functions can be trained by example and categorized into three classes of functions namely, sigmoid, ramp and threshold functions. The familiar function among all these is the sigmoid; because of its gradient descendent learning algorithm, it can be evaluated in different ways but majorly by truncated series expansion.

A sigmoid function of the form

$$h(z) = \frac{1}{1+e^{-z}} \tag{1.10}$$

is differentiable and has the following characteristics:

(i) it outputs real numbers between 0 and 1
(ii) it maps a very large output domain to a small range of inputs
(iii) it never loses information because it is a one-to-one function and
(iv) it increases monotonically.

The characteristics aforementioned are very useful in geometric function theory.

Fadipe et al. [19] modified $h(z)$ in (1.10) to

$$\Phi(z) = \frac{2}{1+e^{-z}} \tag{1.11}$$

which has a series expansion of the form

$$\Phi(z) = 2h(z) = 1 + \sum_{m=1}^{\infty}\frac{(-1)^m}{2^m}\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}z^n\right)^m = 1 + \frac{z}{2} - \frac{z^3}{24} + \frac{z^5}{240} + \ldots \tag{1.12}$$

Some properties were proved; see details in [9], [20], [21], [22] and so on.

Motivated by earlier works by Goodman [2], Fadipe et al. [19], Olatunji [20] and Malik et al. [23], in this work, the authors aim is to obtain the Fekete-Szegö inequalities for certain classes of analytic functions associated with petal type domain and modified sigmoid function. The results are new and give birth to some corollaries.

For the purpose of the main results, the following definitions are

Definition 1.1: A function $f \in A$ is said to be in the class $U\mathcal{S}^*[\Phi,\alpha,A,B], -1 \le B < A \le 1$, if and only if

$$R\left(\frac{(B-1)\left[\frac{zf'(z)}{f(z)}+\alpha\frac{z^2f''(z)}{f(z)}\right]-(A-1)}{(B+1)\left[\frac{zf'(z)}{f(z)}+\alpha\frac{z^2f''(z)}{f(z)}\right]-(A+1)}\right) > \left|\frac{(B-1)\left[\frac{zf'(z)}{f(z)}+\alpha\frac{z^2f''(z)}{f(z)}\right]-(A-1)}{(B+1)\left[\frac{zf'(z)}{f(z)}+\alpha\frac{z^2f''(z)}{f(z)}\right]-(A+1)}-1\right| \tag{1.13}$$

and $\alpha \ge 0$.

Definition 1.2: A function $f \in A$ is said to be in the class $U\mathcal{M}[\Phi,\alpha,A,B], -1 \le B < A \le 1$, if and only if

$$R\left(\frac{(B-1)\left[(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right]-(A-1)}{(B+1)\left[(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right]-(A+1)}\right) > \left|\frac{(B-1)\left[(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right]-(A-1)}{(B+1)\left[(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right]-(A+1)}-1\right| \tag{1.14}$$

and $\alpha \ge 0$.

Definition 1.3: A function $f \in A$ is said to be in the class $U\mathcal{M}[\Phi,\alpha,A,B], -1 \le B < A \le 1$, if and only if

$$R\left(\frac{(B-1)\left[\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha}\right]-(A-1)}{(B+1)\left[\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha}\right]-(A+1)}\right) > \left|\frac{(B-1)\left[\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha}\right]-(A-1)}{(B+1)\left[\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha}\right]-(A+1)}-1\right| \tag{1.15}$$

and $\alpha \ge 0$.

2. Main Results

Theorem 2.1: *Let* $p \in UP[\Phi,A,B]$, $-1 \le B < A \le 1$ *and of the form* $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ *Then* ,denoting the coefficients of z, z^2, z^3 by p_1, p_2, p_3 respectively, the absolute values of the coefficients are bounded as such

$$|p_1| \le \frac{A-B}{\pi^2},$$

$$|p_2| \le \frac{(A-B)(B+1)}{\pi^4} + \frac{(A-B)+2(B+1)}{12\pi^2},$$

$$|p_3| \le \frac{(A-B)(B+1)^2}{\pi^6} + \frac{(B+1)^2}{6\pi^4} + \frac{13[(A-B)+2(B+1)]}{180\pi^2}. \tag{2.1}$$

Proof. For $\Phi \in P$ and of the form $\Phi(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m}\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n\right)^m$, consider

$$\Phi(z) = \frac{1+\omega(z)}{1-\omega(z)}, \tag{2.2}$$

where $\omega(z)$ is such that $\omega(0) = 0$ and $|\omega(z)| < 1$. It follows easily that

$$\omega(z) = \frac{\Phi(z)-1}{\Phi(z)+1} = \frac{z}{4} - \frac{z^2}{16} - \frac{z^3}{192} - \frac{5z^4}{768} - \ldots \tag{2.3}$$

Now, if

$$\tilde{p}(z) = 1 + R_1 z + R_2 z^2 + R_3 z^3 + \ldots,$$

then from (2.3), one may obtain

$$\tilde{p}(\omega(z)) = 1 + R_1\omega(z) + R_2(\omega(z))^2 + R_3(\omega(z))^3 + \ldots$$

$$= 1 + R_1\left(\frac{z}{4} - \frac{z^2}{16} - \frac{z^3}{192} - \frac{5z^4}{768} - \cdots\right) + R_2\left(\frac{z}{4} - \frac{z^2}{16} - \frac{z^3}{192} - \frac{5z^4}{768} - \ldots\right)^2 + R_3\left(\frac{z}{4} - \frac{z^2}{16} - \frac{z^3}{192} - \frac{5z^4}{768} - \ldots\right)^3 + \ldots$$

,
where $R_1 = \frac{8}{\pi^2}, R_2 = \frac{16}{3\pi^2}$ and $R_3 = \frac{184}{45\pi^2}$ see [10]. Using these, the series reduces to

$$\tilde{p}(\omega(z)) = 1 + \frac{2z}{\pi^2} - \frac{z^2}{6\pi^2} - \frac{13z^3}{90\pi^2} + \ldots \quad (2.4)$$

Since $p \in UP[\Phi, A, B]$, so from relations (1.2), (1.7) and (2.4), one may have

$$p(z) = \frac{(A+1)\tilde{p}(\omega(z)) - (A-1)}{(B+1)\tilde{p}(\omega(z)) - (B-1)}$$

$$= 1 + \frac{(A-B)z}{\pi^2} - \left(\frac{(A-B)(B+1)}{\pi^4} + \frac{(A-B)+2(B+1)}{12\pi^2}\right)z^2 - \left(\frac{(A-B)(B+1)^2}{\pi^6} + \frac{(B+1)^2}{6\pi^4} + \frac{13[(A-B)+2(B+1)]}{180\pi^2}\right)z^3 + \ldots \quad (2.5)$$

If $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, then equating coefficients of z and z^2, to obtain

$$p_1 = \frac{A-B}{\pi^2} \quad (2.6)$$

$$p_2 = -\frac{(A-B)(B+1)}{\pi^4} - \frac{(A-B)+2(B+1)}{12\pi^2} \quad (2.7)$$

$$p_3 = -\frac{(A-B)(B+1)^2}{\pi^6} - \frac{(B+1)^2}{6\pi^4} - \frac{13[(A-B)+2(B+1)]}{180\pi^2} \quad (2.8)$$

, which completes the proof of Theorem 2.1.

Theorem 2.2: *Let* $f \in US^*[\Phi, \alpha, A, B]$, $-1 \leq B < A \leq 1, \alpha \geq 0$ *and of the form* (1.1). *Then*, denoting the coefficients of z, z^2, z^3 by p_1, p_2, p_3 respectively, the absolute values of the coefficients are bounded as such

$$\left|a_3 - \mu a_2^2\right| \leq \frac{(A-B)}{2\pi^2(1+3\alpha)}\left|\frac{(A-B)}{(1+2\alpha)\pi^2} - \frac{(B+1)}{\pi^2} - \frac{(A-B)+2(B+1)}{12(A-B)} - \frac{2(A-B)(1+3\alpha)\mu}{(1+2\alpha)^2\pi^2}\right|, \quad (2.9)$$

for a real number μ.

Proof. If $f \in US^*[\Phi, \alpha, A, B], -1 \leq B < A \leq 1$ and $\alpha \geq 0$ then it follows from (1.2), (1.7) and (1.13)

$$\frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)} = \frac{(A+1)\tilde{p}(\omega(z)) - (A-1)}{(B+1)\tilde{p}(\omega(z)) - (B-1)}, \quad (2.10)$$

where $\omega(z)$ is such that $\omega(0) = 0$ and $|\omega(z)| < 1$. The right hand side of (2.10) gets its series form from (2.5) and reduces to

$$\frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)} = 1 + \frac{(A-B)z}{\pi^2} - \left(\frac{(A-B)(B+1)}{\pi^4} + \frac{(A-B)+2(B+1)}{12\pi^2}\right)z^2 - \left(\frac{(A-B)(B+1)^2}{\pi^6} + \frac{(B+1)^2}{6\pi^4} + \frac{13[(A-B)+2(B+1)]}{180\pi^2}\right)z^3 + \ldots \quad (2.11)$$

If $f = z + \sum_{k=2}^{\infty} a_k(\alpha)z^k$, then the left hand side of (2.10) gives

$$\frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)} = 1 + (1+2\alpha)a_2 z + [2(1+3\alpha)a_3 - (1+2\alpha)a_2^2]z^2 + \ldots \quad (2.12)$$

From (2.11), (2.12) and comparison of coefficients of z and z^2 we get

$$a_2 = \frac{(A-B)}{(1+2\alpha)\pi^2} \quad (2.13)$$

and

$$2(1+3\alpha)a_3 - (1+2\alpha)a_2^2 = -\frac{(A-B)(B+1)}{\pi^4} - \frac{(A-B)+2(B+1)}{(12\pi^2}. \quad (2.14)$$

This implies, by using (2.13)

$$a_3 = \frac{(A-B)}{2\pi^2(1+3\alpha)}\left[\frac{(A-B)}{\pi^2(1+2\alpha)} - \frac{(B+1)}{\pi^2} - \frac{(A-B)+2(B+1)}{12(A-B)}\right] \quad (2.15)$$

which completes the proof of Theorem 2.2.

Theorem 2.3: *Let* $f \in \mathcal{UM}[\Phi, \alpha, A, B]$, $-1 \leq B < A \leq 1, \alpha \geq 0$ *and of the form* (1.1). *Then*, denoting the coefficients of z, z^2, z^3 by p_1, p_2, p_3 respectively, the absolute values of the coefficients are bounded as such

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)}{2\pi^2(1+2\alpha)}\left|\frac{(A-B)(1+3\alpha)}{(1+\alpha)^2\pi^2} - \frac{(B+1)}{\pi^2} - \frac{(A-B)+2(B+1)}{12(A-B)} - \frac{2(A-B)(1+2\alpha)\mu}{(1+\alpha)^2\pi^2}\right|, \quad (2.16)$$

for a real number μ.

Proof. If

$f \in \mathcal{UM}[\Phi, \alpha, A, B], -1 \leq B < A \leq 1$ and $\alpha \geq 0$ then it follows from (1.2), (1.7) and (1.14)

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = \frac{(A+1)\bar{p}(\omega(z)) - (A-1)}{(B+1)\bar{p}(\omega(z)) - (B-1)}, \quad (2.17)$$

where $\omega(z)$ is such that $\omega(0) = 0$ and $|\omega(z)| < 1$. The right hand side of (2.17) gets its series form from (2.5) and reduces to

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + \frac{(A-B)z}{\pi^2} - \left(\frac{(A-B)(B+1)}{\pi^4} + \frac{(A-B)+2(B+1)}{12\pi^2}\right)z^2 - \left(\frac{(A-B)(B+1)^2}{\pi^6} + \frac{(B+1)^2}{6\pi^4} + \frac{13[(A-B)+2(B+1)]}{180\pi^2}\right)z^3 + \ldots \quad (2.18)$$

If $f = z + \sum_{k=2}^{\infty} a_k(\alpha)z^k$, then the left hand side of (2.18) gives

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + (1+\alpha)a_2 z + [2(1+2\alpha)a_3 - (1+3\alpha)a_2^2]z^2 + \ldots \quad (2.19)$$

From (2.18) and (2.19), comparison of coefficients of z and z^2 gives

$$a_2 = \frac{(A-B)}{(1+\alpha)\pi^2} \quad (2.20)$$

and

$$2(1+2\alpha)a_3 - (1+3\alpha)a_2^2 = -\frac{(A-B)(B+1)}{\pi^4} - \frac{(A-B)+2(B+1)}{12\pi^2}. \quad (2.21)$$

This implies, by using (2.20)

$$a_3 = \frac{1}{2\pi^2(1+2\alpha)}\left[\frac{(A-B)^2(1+3\alpha)}{\pi^2(1+\alpha)^2} - \frac{(A-B)(B+1)}{\pi^2} - \frac{(A-B)+2(B+1)}{12}\right], \quad (2.22)$$

which completes the proof of Theorem 2.3.

Theorem 2.4: *Let* $f \in \mathcal{UL}[\Phi, \alpha, A, B]$, $-1 \leq B < A \leq 1, \alpha \geq 0$ *and of the form* (1.1). *Then*, denoting the coefficients of z, z^2, z^3 by p_1, p_2, p_3 respectively, the absolute values of the coefficients are bounded as such

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)}{4\pi^2(3-2\alpha)}\left|\frac{(A-B)[3(4-3\alpha)-(2-\alpha)^2]}{(2-\alpha)^2\pi^2} - \frac{2(B+1)}{\pi^2} - \frac{(A-B)+2(B+1)}{6(A-B)} - \frac{4(A-B)(3-2\alpha)\mu}{(2-\alpha)^2\pi^2}\right|, \quad (2.23)$$

for a real number μ.

Proof. If $f \in \mathcal{UL}[\Phi, \alpha, A, B], -1 \leq B < A \leq 1$ and $\alpha \geq 0$ then it follows from (1.2), (1.7) and (1.15)

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha} = \frac{(A+1)\bar{p}(\omega(z))-(A-1)}{(B+1)\bar{p}(\omega(z))-(B-1)} \tag{2.24}$$

where $\omega(z)$ is such that $\omega(0)=0$ and $|\omega(z)|<1$. The right hand side of (2.24) gets its series form from (2.4) and reduces to

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf'\prime(z)}{f'(z)}\right)^{1-\alpha} = 1+\frac{(A-B)z}{\pi^2}-\left(\frac{(A-B)(B+1)}{\pi^4}+\frac{(A-B)+2(B+1)}{12\pi^2}\right)z^2-\left(\frac{(A-B)(B+1)^2}{\pi^6}+\frac{(B+1)^2}{6\pi^4}+\frac{13[(A-B)+2(B+1)]}{180\pi^2}\right)z^3+\ldots \tag{2.25}$$

If $f=z+\sum_{k=2}^{\infty}a_k(\alpha)z^k$, then the left hand side of (2.24) gives

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha}=1+(2-\alpha)a_2z+\frac{1}{2}[((2-\alpha)^2-3(4-3\alpha))a_2^2+4(3-2\alpha)a_3]z^2+\cdots \tag{2.26}$$

From (2.25) and (2.26), comparison of coefficients of z and z^2 gives

$$a_2=\frac{(A-B)}{(2-\alpha)\pi^2} \tag{2.27}$$

and

$$\frac{1}{2}[((2-\alpha)^2-3(4-3\alpha))a_2^2+4(3-2\alpha)a_3]=-\frac{(A-B)(B+1)}{\pi^4}-\frac{(A-B)+2(B+1)}{(12\pi^2}. \tag{2.28}$$

This implies, by using (2.27)

$$a_3=\frac{(A-B)}{4\pi^2(3-2\alpha)}\left[\frac{(A-B)[3(4-3\alpha)-(2-\alpha)^2]}{(2-\alpha)^2\pi^2}-\frac{2(B+1)}{\pi^2}-\frac{(A-B)+2(B+1)}{6(A-B)}\right], \tag{2.29}$$

which completes the proof of Theorem 2.4.

Taking $A=1$ and $B=-1$ in Theorem 2.1 to obtain

Corollary 2.1: *Let* $p\in UP[\Phi,1,-1]$, $-1\le B<A\le 1$*and of the form* $p(z)=1+\sum_{n=1}^{\infty}p_nz^n$. *Then*, denoting the coefficients of z,z^2,z^3 by p_1,p_2,p_3respectively, the absolute values of the coefficients are bounded as such

$$|p_1|\le\frac{2}{\pi^2},$$

$$|p_2|\le\frac{1}{6\pi^2},$$

and

$$|p_3|\le\frac{13}{90\pi^2}.$$

Setting $A=1$ and $B=-1$ in Theorem 2.2 obtains

Corollary 2.2: *Let* $f\in US^*[\Phi,\alpha,1,-1]$, $-1\le B<A\le 1,\alpha\ge 0$*and of the form* (1.1). *Then*, denoting the coefficients of z,z^2,z^3 by p_1,p_2,p_3respectively, the absolute values of the coefficients are bounded as such

$$|a_3-\mu a_2^2|\le\frac{1}{\pi^2(1+3\alpha)}\left|\frac{2}{(1+2\alpha)\pi^2}-\frac{1}{12}-\frac{4(1+3\alpha)\mu}{(1+2\alpha)^2\pi^2}\right|,$$

for a real number μ.

Putting $\alpha=0$ in Corollary 2.2, one may have

Corollary 2.3: *Let* $f\in US^*[\Phi,0,1,-1]$, $-1\le B<A\le 1,\alpha\ge 0$ *and of the form* (1.1). *Then*, denoting the coefficients of z,z^2,z^3 by p_1,p_2,p_3respectively, the absolute values of the coefficients are bounded as such

$$|a_3-\mu a_2^2|\le\frac{1}{\pi^2}\left|\frac{2}{\pi^2}-\frac{1}{12}-\frac{4\mu}{\pi^2}\right|,$$

for a real number μ.

Taking $\alpha=1$ in Corollary 2.2 gets

Corollary 2.4: *Let* $f \in \mathcal{US}^*[\Phi, 1,1,-1]$, $-1 \leq B < A \leq 1, \alpha \geq 0$ *and of the form* (1.1). *Then*, denoting the coefficients of z, z^2, z^3 by p_1, p_2, p_3 respectively, the absolute values of the coefficients are bounded as such

$$|a_3 - \mu a_2^2| \leq \frac{1}{4\pi^2}\left|\frac{2}{3\pi^2} - \frac{1}{12} - \frac{16\mu}{9\pi^2}\right|,$$

for a real number μ.

Inserting $A = 1$ and $B = -1$ in Theorem 2.3 we obtain

Corollary 2.5: *Let* $f \in \mathcal{UM}[\Phi, \alpha, 1,-1]$, $-1 \leq B < A \leq 1, \alpha \geq 0$*and of the form* (1.1). *Then*, denoting the coefficients of z, z^2, z^3 by p_1, p_2, p_3 respectively, the absolute values of the coefficients are bounded as such

$$|a_3 - \mu a_2^2| \leq \frac{1}{\pi^2(1+2\alpha)}\left|\frac{2(1+3\alpha)}{(1-\alpha)^2\pi^2} - \frac{1}{12} - \frac{4(1+2\alpha)\mu}{(1+\alpha)^2\pi^2}\right|,$$

for a real number μ.

Putting $\alpha = 0$ in Corollary 2.5 we get

Corollary 2.6: *Let* $f \in \mathcal{UM}[\Phi, 0,1,-1]$, $-1 \leq B < A \leq 1, \alpha \geq 0$ *and of the form* (1.1). *Then*, denoting the coefficients of z, z^2, z^3 by p_1, p_2, p_3 respectively, the absolute values of the coefficients are bounded as such

$$|a_3 - \mu a_2^2| \leq \frac{1}{\pi^2}\left|\frac{2}{\pi^2} - \frac{1}{12} - \frac{4\mu}{\pi^2}\right|,$$

for a real number μ.

Setting $\alpha = 1$ in Corollary 2.5, one may have

Corollary 2.7: *Let* $f \in \mathcal{UM}[\Phi, 1,1,-1]$, $-1 \leq B < A \leq 1, \alpha \geq 0$ *and of the form* (1.1). *Then*, denoting the coefficients of z, z^2, z^3 by p_1, p_2, p_3 respectively, the absolute values of the coefficients are bounded as such

$$|a_3 - \mu a_2^2| \leq \frac{1}{3\pi^2}\left|\frac{2}{\pi^2} - \frac{1}{12} - \frac{3\mu}{\pi^2}\right|,$$

for a real number μ.

Putting $A = 1$ and $B = -1$ in Theorem 2.4 we get

Corollary 2.8: *Let* $f \in \mathcal{UL}[\Phi, \alpha, 1,-1]$, $-1 \leq B < A \leq 1, \alpha \geq 0$ *and of the form* (1.1). *Then*, denoting the coefficients of z, z^2, z^3 by p_1, p_2, p_3 respectively, the absolute values of the coefficients are bounded as such

$$|a_3 - \mu a_2^2| \leq \frac{1}{2\pi^2(3-2\alpha)}\left|\frac{2[3(4-3\alpha)-(2-\alpha)^2]}{(2-\alpha)^2\pi^2} - \frac{1}{6} - \frac{8(3-2\alpha)\mu}{(2-\alpha)^2\pi^2}\right|,$$

for a real number μ.

Taking $\alpha = 0$ in Corollary 2.8 to obtain

Corollary 2.9: *Let* $f \in \mathcal{UL}[\Phi, 0,1,-1]$, $-1 \leq B < A \leq 1, \alpha \geq 0$ *and of the form* (1.1). *Then*, denoting the coefficients of z, z^2, z^3 by p_1, p_2, p_3 respectively, the absolute values of the coefficients are bounded as such

$$|a_3 - \mu a_2^2| \leq \frac{1}{6\pi^2}\left|\frac{4}{\pi^2} - \frac{1}{6} - \frac{6\mu}{\pi^2}\right|,$$

for a real number μ.

Inserting $\alpha = 1$ in Corollary 2.8, one may have

Corollary 2.10: *Let* $f \in \mathcal{UL}[\Phi, 1,1,-1]$, $-1 \leq B < A \leq 1, \alpha \geq 0$ *and of the form* (1). *Then*, denoting the coefficients of z, z^2, z^3 by p_1, p_2, p_3 respectively, the absolute values of the coefficients are bounded as such

$$|a_3 - \mu a_2^2| \leq \frac{1}{2\pi^2}\left|\frac{4}{\pi^2} - \frac{1}{6} - \frac{8\mu}{\pi^2}\right|,$$

for a real number μ.

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