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Embedded Pseudo-Runge-Kutta Methods for First and Second Order Initial Value Problems

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ABSTRACT

In this paper, we have proposed an embedded pseudo-Runge-Kutta method (EPRKM) of order three. We have developed two implicit embedded pseudo Runge-Kutta methods (EPRKM) for the solution of first order initial value problems of which the first one is fully implicit and the other is diagonally implicit. Also, a two-step explicit embedded pseudo-Runge-Kutta method for solving second order initial value problems is also derived. This development is an effort to minimize the computational cost of classical Runge-Kutta (RK) method. A classical s-stage RK method requires s function evaluations (slopes) per step. The present methods require four evaluations in the first step and from the second step onwards it requires only two evaluations per step. The cost of computation in the form of number of function evaluations (slopes) are compared via Table 1, from which it is evident that our proposed developments are cost-efficient. The proposed methods are tested on two initial value problems (IVPs) and the errors obtained by these methods are compared with the errors by RK third order (RK3), third order implicit Runge-Kutta (IRK3) method and Runge-Kutta-Nyström 3-stage method (RKN3) which are shown in Tables 2 and 3. The convergence and stability properties are established and the governing theorems are also derived. Intervals and regions of absolute stability of the proposed methods are also derived, and regions are depicted in Fig. 1. The proposed methods have wide scope of applicability for the numerical solution of IVPs which are frequently occurred in physics, chemistry, molecular dynamics, fluid dynamics and several branches of engineering.

Keywords: Initial value problem; Interval of absolute stability; Pseudo Runge-Kutta method; Runge-Kutta method; Stability properties

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1. Introduction

The differential equation is an important mathematical tool which is used to model several phenomena in physics, heat flow, fluid flow, vibration, chemical reaction, nuclear reaction and several areas of economics and finance. In this paper, we shall be focused on the numerical solution of an initial value problem (IVP) in a first order differential equation, given by

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0, x \in [x_0, b], \tag{1.1}$$

and an IVP in a second order differential equation

$$\begin{aligned} \frac{d^2y}{dx^2} &= f(x, y, y'), \\ y(x_0) &= y_0, \\ y'(x_0) &= y'_0, x \in [x_0, b], \end{aligned} \tag{1.2}$$

where $y = y(x)$ is an unknown solution function and $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is assumed that it satisfies the Lipschitz condition so that there exists a unique solution of IVP (1.1). Though several analytical tools have been developed, but they are inadequate to solve an initial value problem, particularly in a non-linear ordinary differential equation. So, an extensive amount of research has been done to find the numerical solution of IVP. There are several numerical methods such as the Euler method, Taylor series method, Runge-Kutta formulae, and Predictor-Corrector formulae [1–3]. The Runge-Kutta formulae is one of the widely used methods to find the numerical solution of IVPs that arise in the various fields of applied mathematics. An excellent book written by Butcher [4] covers the whole development of Runge-Kutta methods. Some significant contributions can also be found in the literature (see [2] and [5]). Several types of Runge-Kutta methods have been developed on the basis of stability properties and truncation error bounds [6, 7]. Modifications over existing classical RK methods have been envisaged to form new high order accurate RK methods in the last two decades. Simos [8] developed an RK method to find the solution of oscillatory IVP (having a periodic solution) called the exponentially fitted RK method. Another form of modified RK method is known as the symplectic RK method was given by Simos and Vigo-Aguiar [9]. Van de Vyver [10] constructed an exponentially fitted symplectic implicit RK method based on the Gauss method. Van de Vyver [11] first constructed a symplectic RK-Nyström method with increased phase-leg order. An excellent review on RK methods by Z. Kalogiratou et al. [6] includes the whole short history of several modified RK methods. These include standard Runge-Kutta-Nyström (RKN), exponentially fitted RK-Nyström, Partition RK method (PRK), exponentially trigonometrically fitted PRK, symplectic PRK, and RK with minimum phase-leg, etc.

The classic RK method of order s involves s function evaluations per time step. So, an attempt to reduce function evaluation per step results in the development of the pseudo RK method. The pseudo RK method was first introduced by Byrne [12].

The aim of the present paper is to propose a new type of pseudo RK method of order three called the embedded pseudo RK method. The key benefit of such construction is to reduce the computational cost and increase the size of the stability region. These methods are not self-starting and they require four slopes (function evaluations) in the first time-step. But from the second step onwards, only two evaluations per step are required. Several research works have been done in this direction and many authors have attempted to increase the efficiency of the RK method [6]. Goeken and Johnson [13] proposed a class of RK methods based on the numerical evaluations of f and its derivative f' to find the solution of a first order IVP. Wu [14] proposed a class of explicit RK formulae with reduced number of function evaluations. Byrne et al. [15] proposed a pseudo RK method involving two-points. Several research papers in literature for pseudo RK methods can be seen in [16–18]. Recently, the numerical solver based on the operational matrix approach is used to solve the IVPs in ordinary differential equations of integral and fractional order (see [19, 20] and the references cited therein). The Runge-Kutta-Nyström method is used to find the numerical solution of second order IVPs (1.2) directly in which the first order derivative is missing i.e. the autonomous form of IVP (1.2). In this paper, we have derived an embedded pseudo Runge-Kutta method given in Subsection 2.3 for solving am IVP (1.2). Though, the proposed methods in Subsections 2.1, 2.2 and 2.3 require initially two previous values to compute the next iteration, but they require just the previous value after the first step; in other words, all the methods are two-step methods. The great advantage of these methods over the classical RK methods is the reduction of computation cost. This can be understood via the Table 1. Also, the point wise errors obtained here are smaller than errors obtained by the classical RK methods which are shown in numerical experiments. Another important aspect of the proposed developments is that they maintain absolute stability in large intervals of step size h as compared to the classical RK method, i.e. for a given initial value problem, we have more options to select larger values of step-size h than in the RK method.

2. Derivation of the Method

In this section, we derive the embedded pseudo Runge-Kutta method (EPRKM) for the numerical solution of IVPs of the first order and second order ordinary differential equations (ODE). A general one step method to solve the IVP

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0, \qquad (2.1)$$

can be written as

$$\begin{aligned} y_{n+1} &= y_n + h\phi(x_n, y_n; h), \qquad (2.2) \\ n &= 0, 1, 2, \ldots N-1, \end{aligned}$$

where $\phi(x_n, y_n; h)$ is a continuous function of starting iterates x_n, y_n and step size h. For an autonomous initial value problem, we may take Eq. (2.2) as

$$\begin{aligned} y_{n+1} &= y_n + h\phi(y_n, f; h), \\ n &= 0, 1, 2, \ldots N-1. \qquad (2.3) \end{aligned}$$

In this section, we restrict ourselves to autonomous IVPs.

For a general s-stage two-step pseudo Runge-Kutta method, we choose

$$\overline{\phi}(\overline{y}_n, f; h) = \sum_{i=1}^{s} b_i k_i + \sum_{i=1}^{s} b'_i k_{-i}, \qquad (2.4)$$

$$\overline{\phi}(\overline{y}_n, f; h) = \phi_1(y_n, f; h) + \phi_2(y_n, f; h). \qquad (2.5)$$

Here, we consider a general two-stage two-step pseudo implicit Runge-Kutta method of the form (2.3) and (2.4), where the slopes (function evaluations) can be taken as

$$k_1 = f(y_n), \qquad (2.6)$$

$$k_{-1} = f(y_{n-1}), \qquad (2.7)$$

$$k_i = f(y_n + h\sum_{j=1}^{i} a_{ij}k_j),\ i = 2, 3, \ldots, s, \qquad (2.8)$$

$$k_{-i} = f(y_{n-1} + h\sum_{j=1}^{i} a'_{ij}k_{-j}). \quad (2.9)$$

The augmented Butcher table representation of the s-stage two-step pseudo implicit Runge-Kutta method is given as:

$$\begin{array}{c|cccc|c|cccc}
c_1 & & & & & c_{-1} & & & & \\
c_2 & a_{21} & a_{22} & \cdots & & c_{-2} & a'_{21} & a'_{22} & \cdots & \\
\vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & & \\
c_s & a_{s1} & a_{s2} & \cdots & a_{ss} & c_{-s} & a'_{s1} & a'_{s2} & \cdots & a'_{ss} \\
\hline
 & b_1 & b_2 & \cdots & b_s & & b'_1 & b'_2 & \cdots & b'_s
\end{array}$$

Particularly in this paper, we concentrate on the derivation of pseudo implicit two-stage two-step Runge-Kutta method and for which we consider

$$\phi(y_n, f; h) = \sum_{i=1}^{2} b_i k_i + \sum_{i=1}^{2} b'_i k_{-i}, \quad (2.10)$$

where

$$k_1 = f(y_n), \quad (2.11)$$
$$k_{-1} = f(y_{n-1}), \quad (2.12)$$
$$k_2 = f(y_n + h(a_{21}k_1 + a_{22}k_2)), \quad (2.13)$$
$$k_{-2} = f(y_{n-1} + h(a'_{21}k_{-1} + a'_{22}k_{-2})). \quad (2.14)$$

Applying Taylor's series approximation in (2.13) and (2.14), we can derive the governing equations of the coefficients. In the following Subsections 2.1 and 2.2 we will derive EPIRK of order three.

2.1 Embedded pseudo implicit two-step Runge-Kutta method (EPIRKM) of order three

Consider the autonomous initial value problem

$$\frac{dy}{dx} = f(y), y(x_0) = y_0. \quad (2.15)$$

Let us consider the formula

$$\overline{y}_{n+1} = \overline{y}_n + h\overline{\phi}(\overline{y}_n, f; h), \quad (2.16)$$

where,

$$\overline{\phi} = [\phi_1, \phi_2]^T,$$
$$\overline{y}_n = [y_n, y_{n-1}]^T,$$
$$\overline{y}_{n+1} = [y_{n+1}, 0]^T,$$

and

$$\overline{\phi}(\overline{y}_n, f; h) = [a_1(k_1 + k_{-1}) + b(k_2 - k_{-2})], \quad (2.17)$$

$$k_1 = f(y_n) = f_n, \quad (2.18)$$
$$k_{-1} = f(y_{n-1}) = f_{n-1}, \quad (2.19)$$
$$k_2 = f(y_n + a_{21}h(k_1 + k_2)), \quad (2.20)$$
$$k_{-2} = f(y_{n-1} + a'_{21}h(k_{-1} + k_{-2})). \quad (2.21)$$

Let us assume k_2 in the series form:

$$k_2 = A_1 + hA_2 + h^2A_3 + \cdots .$$

Substituting the value of k_2 in Eq. (2.20) and equating the coefficient of like power of h followed by using Taylor's series expansion, we obtain the coefficients which are given below:

$$A_1 = f_n, \quad (2.22)$$
$$A_2 = a_{21}f_n f_{y_n} + a_{21}A_1 f_{y_n}, \quad (2.23)$$

$$A_3 = a_{21}A_2 f_{y_n} + \frac{1}{2}a_{21}^2 f_n^2 f_{y_n y_n} + \frac{1}{2}a_{21}^2 A_1^2 f_{y_n y_n} + a_{21}^2 f_n A_1 f_{y_n y_n}. \quad (2.24)$$

Similarly, if we assume k_{-2} in the form

$$k_{-2} = B_1 + hB_2 + h^2B_3 + \cdots .$$

Substituting the value of k_{-2} in Eq. (2.21) and equating the coefficients of like powers of h we get

$$B_1 = f_{n-1}, \quad (2.25)$$
$$B_2 = a'_{21}f_{n-1}f_{y_{n-1}} + a'_{21}B_1 f_{y_{n-1}}, \quad (2.26)$$
$$B_3 = a'_{21}B_2 f_{y_{n-1}} + \frac{1}{2}(a'_{21})^2 B_1^2 f_{y_{n-1}y_{n-1}}. \quad (2.27)$$

Let $a'_{21} = a_{21}$. So, by Taylor's series expansion of $y(x_n + h)$, we have

$$\begin{aligned} y_{n+1} = y(x_n + h) &= y(x_n) + hy'(x_n) \\ &+ \frac{1}{2}h^2 y''(x_n) + \frac{1}{6}h^3 y'''(x_n) \\ &+ O(h^4), \end{aligned} \tag{2.28}$$

using (2.16),

$$y_{n+1} - y_n = y(x_n + h) - y(x_n), \tag{2.29}$$

$$\begin{aligned} y_{n+1} - y_n &= 2a_1 hy'(x_n) + (-a_1 + b \\ &+ 2b(a_{21} - a'_{21}))h^2 y''(x_n) \\ &+ (\frac{a_1}{2} - \frac{b}{2} + a_{22}b + b(a_{11}^2 \\ &- a_{22}^2))h^3 y'''(x_n) + O(h^4). \end{aligned} \tag{2.30}$$

Comparing (2.28) with (2.30), we get the following order conditions

$$\begin{aligned} 2a_1 &= 1, \\ -a_1 + b + 2b(a_{21} - a'_{21}) &= \frac{1}{2}, \\ \frac{1}{2}a_1 - \frac{1}{2}b + 2ba'_{21} + 2b(a_{21}^2 - a'^2_{21}) &= \frac{1}{6}. \end{aligned} \tag{2.31}$$

From equation (2.31), we get

$$a_1 = \frac{1}{2},\ a_{21} = \frac{6-b}{24b},\ a'_{21} = \frac{-6+11b}{24b}, \tag{2.32}$$

where b is a free parameter. So, the proposed EPIRK method of order $O(h^4)$ is obtained by Eqs. (2.16)-(2.21) together with the constants given in (2.32).

2.2 Embedded pseudo diagonally implicit two-step Runge-Kutta method (EPDIRKM) of order three

Consider the autonomous initial value problem

$$\frac{dy}{dx} = f(y), y(x_0) = y_0. \tag{2.33}$$

In case of diagonally implicit method, we consider the following evaluation functions:

$$k_1 = f(y_n) \tag{2.34}$$

$$k_{-1} = f(y_{n-1}), \tag{2.35}$$

$$k_2 = f(y_n + a_{11}hk_2), \tag{2.36}$$

$$k_{-2} = f(y_{n-1} + a_{22}hk_{-2}). \tag{2.37}$$

The approximate solution of (2.33) is given by

$$y_{n+1} = y_n + h[a_1(k_1 + k_{-1}) + b(k_2 - k_{-2})]. \tag{2.38}$$

Putting k_2 in the form

$$k_2 = A_1 + hA_2 + h^2 A_3 + \cdots,$$

and applying Taylor's series expansion in (2.36) we get first three coefficients in k_2 as

$$A_1 = f(y_n), \tag{2.39}$$

$$A_2 = a_{11}A_1 f_{y_n}, \tag{2.40}$$

$$A_3 = a_{11}A_2 f_{y_n} + \frac{1}{2}a_{11}A_1^2 f_{y_n y_n}. \tag{2.41}$$

Similarly, the expansion of k_{-2} of the form

$$k_{-2} = B_1 + hB_2 + h^2 B_3 + \cdots,$$

where

$$B_1 = f_{n-1}, \tag{2.42}$$

$$B_2 = a_{22}B_1 f_{y_{n-1}}, \tag{2.43}$$

$$B_3 = a_{22}B_2 f_{y_{n-1}} + \frac{1}{2}a_{22}^2 B_1^2 f_{y_{n-1}y_{n-1}}. \tag{2.44}$$

Taylor's series expansion of $y(x_n + h)$ is

$$\begin{aligned} y(x_{n+1}) = &y(x_n) + hy'(x_n) + \frac{1}{2}h^2 y''(x_n) \\ &+ \frac{1}{6}h^3 y'''(x_n) + O(h^4), \end{aligned} \tag{2.45}$$

using (2.38), we get

$$y_{n+1} - y_n = y(x_n + h) - y(x_n) \quad (2.46)$$

$$\begin{aligned} y_{n+1} - y_n &= 2a_1 h y'(x_n) + (-a_1 + ba_{11} \\ &+ b - ba_{22})h^2 y''(x_n) + (\frac{a_1}{2} - \frac{b}{2} \\ &+ a_{22}b + b(a_{11}^2 - a_{22}^2))h^3 y'''(x_n) \\ &+ O(h^4). \end{aligned} \quad (2.47)$$

Comparing (2.45) and (2.47), the order conditions are

$$2a_1 = 1, \quad (2.48)$$

$$ba_{11} - ba_{22} + b = 1, \quad (2.49)$$

$$\frac{a_1}{2} - \frac{b}{2} + a_{22}b + b(a_{11}^2 - a_{22}^2) = \frac{1}{6}. \quad (2.50)$$

Solving Eqs. (2.48)-(2.50) we get

$$a_1 = \frac{1}{2}, \; a_{11} = \frac{-12 + 13b - 6b^2}{12b(b-2)}, \quad a_{22} = \frac{12 - 23b + 6b^2}{12b(b-2)}, \quad (2.51)$$

where b is a free parameter. Eqs (2.34)-(2.38) together with the values of coefficients given in (2.51) govern the desired EPDIRK method of order three. Here, truncation error is of order $O(h^4)$.

2.3 Embedded Pseudo two-stage two-step explicit Runge-Kutta method for second order IVP

In this section, we derive an explicit embedded pseudo Runge-Kutta for a second order initial value problem:

$$\frac{d^2y}{dx^2} = f(x, y, y'), y(x_0) = y_0, y'(x_0) = y'_0. \quad (2.52)$$

Like the methods proposed in Subsections 2.1 and 2.2, this method is not self-starting; it requires four slopes in the first step and from the second step onwards it requires only two slopes (evaluation functions) per step which means that it incurs less computation cost than the classical RK method of order three. Consider the two-step two-stage pseudo RK formula

$$y_{n+1} = y_n + hy'_n + [a(k_1 + k_{-1}) + b(k_2 - k_{-2}), \quad (2.53)$$

$$y'_{n+1} = y'_n + \frac{1}{h}[a^*(k_1 + k_{-1}) + b^*(k_2 - k_{-2}))], \quad (2.54)$$

where,

$$\begin{aligned} k_1 &= \frac{h^2}{2} f(x_n, y_n, y'_n), \\ k_{-1} &= \frac{h^2}{2} f(x_{n-1}, y_{n-1}, y'_{n-1}), \\ k_2 &= \frac{h^2}{2} f(x_n + c_2 h, y_n + c_2 h y'_n + a_{21}k_1, \\ &\quad y'_n + \frac{b_{21}}{h} k_1), \\ k_{-2} &= \frac{h^2}{2} f(x_{n-1} + c_3 h, y'_{n-1} + c_3 h y_{n-1} \\ &\quad + a_{31}k_{-1}, y'_{n-1} + \frac{b_{31}}{h} k_{-1}). \end{aligned}$$

Choosing the conditions, $a_{21} = c_2$, $b_{21} = 2c_2$, $a_{31} = c_3$, $b_{31} = 2c_3$ and expanding k_2 and k_{-2} by Taylor's series, we get the following order conditions

$$2a = 1, \quad (2.55)$$

$$-a + bc_2 + b - bc_3 = \frac{1}{3}, \quad (2.56)$$

$$a + b(c_2^2 - 1) + c_3 b(2 - c_3) = \frac{1}{6}, \quad (2.57)$$

$$2a^* = 2. \quad (2.58)$$

The system of Eqs (2.55)-(2.58) has two degrees of freedom. Letting b and b^* as free parameters and solving the system, one can get:

$$c_2 = \frac{25 - 12b}{60b} = a_{21}, \quad (2.59)$$

$$c_3 = \frac{-25+48b}{60b} = a_{31}, \tag{2.60}$$

$$a = \frac{1}{2}, a^* = 1. \tag{2.61}$$

Since not all coefficients of h^4 in y'_{n+1} and h^3 in y_{n+1} are equal, the local truncation error in solution $y(x)$ is of order $O(h^4)$ and of order $O(h^3)$ in $y'(x)$. Eqs (2.53)-(2.54), together with the values of coefficients given in (2.59)-(2.61), established the desired pseudo Runge-Kutta method for IVP (2.52).

If the free parameters b and b^* are taken as $b = 1$ and $b^* = \frac{1}{2}$, then the governing EPRK method is:

$$\begin{aligned}
k_1 &= \frac{h^2}{2!} f(x_n, y_n, y'_n),\\
k_{-1} &= \frac{h^2}{2!} f(x_{n-1}, y_{n-1}, y'_{n-1}),\\
k_2 &= \frac{h^2}{2!} f(x_n + \frac{13}{60}h, y_n + \frac{13}{60}hy'_n + \frac{13}{60}k_1,\\
&\quad y'_n + \frac{13}{30h}k_1),\\
k_{-2} &= \frac{h^2}{2!} f(x_{n-1} + \frac{23}{60}h, y_{n-1} + \frac{23}{60}hy'_{n-1}\\
&\quad + \frac{23}{60}k_1, y'_{n-1} + \frac{23}{30h}k_{-1}),\\
y_{n+1} &= y_n + hy'_n + \frac{1}{2}k_1 + \frac{1}{2}k_{-1} + k_2 - k_{-2},\\
y'_{n+1} &= y'_n + \frac{1}{h}\left(k_1 + k_{-1} + \frac{1}{2}k_2 - \frac{1}{2}k_{-2}\right).
\end{aligned}$$

3. Stability Properties

Definition 3.1. The single step method

$$\overline{y}_{n+1} = \overline{y}_n + h\overline{\phi}(x_n, y_n; h) \tag{3.1}$$

is said to be consistent if $\forall \epsilon > 0$ there exist a step-size $h(\epsilon) > 0$ such that $|T_n| < \epsilon$, for $0 < h < h(\epsilon)$, for every pair of points $(x_n, y(x_n))$,$(x_{n+1}, y(x_{n+1}))$ on the solution curve in domain $[x_0, b]$, where

$$T_n = \frac{y(x_{n+1}) - y(x_n)}{h} - \phi(x_n, y(x_n); h).$$

Lemma 3.2. *The function f is said to satisfy the Lipschitz condition in the domain $D = \{x = x_0 + nh : x_0 \le nh \le b\}$ if $\|f(x)-f(y)\| \le L\|x-y\|, \forall x, y \in D$, where $L \ge 0$ is the Lipschitz constant.*

Theorem 3.3. *If $f(x, y)$ is Lipschitz continuous and $\overline{y}_n$, $\overline{z}_n$ are any two points on the solution curve, then the function $\overline{\phi}(y_n, f; h)$ is Lipschitz continuous in D with respect to the first component. i.e.*

$$\|\overline{\phi}(\overline{y}_n, f; h) - \overline{\phi}(\overline{z}_n, f; h)\| \le \widehat{L}_\phi \|\overline{y}_n - \overline{z}_n\|.$$

Proof. First, we prove that

$$\|k_i(y_n) - k_i(z_n)\| \le \alpha_i \|y_n - z_n\|, \forall i \tag{3.2}$$

$$\|k_{-i}(y_n) - k_{-i}(z_n)\| \le \beta_i \|y_n - z_n\| \forall i. \tag{3.3}$$

Consider,

$$\begin{aligned}
\|k_1(y_n) - k_1(z_n)\| &= \|f(y_n) - f(z_n)\|\\
&\le L\|y_n - z_n\|\\
&= \alpha_1 \|y_n - z_n\|,
\end{aligned}$$

i.e. (3.2) is true for $i = 1$. For, $i = 2$, consider,

$$\begin{aligned}
&\|k_2(y_n) - k_2(z_n)\|\\
=&\|f(y_n + h\sum_{j=1}^{2} a_{ij}k_j(y_n))\\
&- f(z_n + h\sum_{j=1}^{2} a_{ij}k_j(z_n)\|\\
\le& L\|y_n + h(a_{21}k_1(y_n) + a_{22}k_2(y_n)) - z_n\\
&- h(a_{21}k_1(z_n) + a_{22}k_2(z_n))\|\\
\le& L(\|y_n - z_n\| + h|a_{21}|\|k_1(y_n) - k_1(z_n)\|\\
&+ h|a_{22}|\|k_2(y_n) - k_2(z_n)\|)
\end{aligned}$$

and so

$$\begin{aligned}
&\|k_2(y_n) - k_2(z_n)\|\\
&\le L\left(\frac{1 + h|a_{21}|}{1 - hL|a_{22}|}\right)\|y_n - z_n\|
\end{aligned}$$

$$= \alpha_2 \|y_n - z_n\|, \tag{3.4}$$

where

$$\alpha_2 = L\left(\frac{1 + h|a_{21}|}{1 - hL|a_{22}|}\right),$$

i.e. the inequality (3.2) is true for $i = 2$ also. So, by induction it can be shown that:

$$\begin{aligned}&\|k_i(y_n) - k_i(z_n)\| \\ &\quad\le \|f(y_n) + h\sum_{j=1}^{i} a_{ij}k_j(y_n) \\ &\quad - f(z_n) - h\sum_{j=1}^{i} a_{ij}k_j(z_n)\|\end{aligned}$$

$$\begin{aligned}&\|k_i(y_n) - k_i(z_n)\| \\ &\quad= L\|y_n - z_n\| \\ &\quad+ Lh\|\sum_{j=1}^{i-1} a_{ij}\left(k_j(y_n) - k_j(z_n)\right)\| \\ &\quad+ Lh\|a_{ii}\left(k_i(y_n) - k_i(z_n)\right)\|\end{aligned}$$

$$\begin{aligned}&\|k_i(y_n) - k_i(z_n)\| \\ &\quad\le L\left(\frac{1 + h|a_{ij}|}{1 - hL|a_{ii}|}\right)\|y_n - z_n\|\end{aligned}$$

$$\|k_i(y_n) - k_i(z_n)\| \le \alpha_i\|y_n - z_n\|,$$

where

$$\alpha_i = L\left(\frac{1 + h|a_{ij}|}{1 - hL|a_{ii}|}\right),$$

i.e. the inequality (3.2) holds $\forall\, i$. Similarly we consider

$$\begin{aligned}&\|k_{-1}(y_{n-1}) - k_{-1}(z_{n-1})\| \\ &\quad= \|f(y_{n-1}) - f(z_{n-1})\| \\ &\quad\le L\|y_{n-1} - z_{n-1}\|,\end{aligned}$$

$$\|k_{-1}(y_{n-1}) - k_{-1}(z_{n-1})\| = \beta_1\|y_{n-1} - z_{n-1}\|. \tag{3.5}$$

Repeating the similar procedure as in k_i we can prove that:

$$\|k_{-i}(y_{n-1}) - k_{-i}(z_{n-1})\| \le \beta_i\|y_{n-1} - z_{n-1}\|. \tag{3.6}$$

Consider,

$$\begin{aligned}\overline{\phi}(\overline{y}_n, f; h) &= \sum_{i=1}^{s} b_i k_i(y_n) + \sum_{i=1}^{s} b_i k_{-i}(y_{n-1}) \\ &= \phi_1(y_n, f; h) + \phi_2(y_{n-1}, f; h).\end{aligned} \tag{3.7}$$

First we take,

$$\begin{aligned}&\|\phi_1(y_n, f; h) - \phi_1(z_n, f; h)\| \\ &\quad= \|\sum_{i=1}^{s} b_i k_i(y_n) - \sum_{i=1}^{s} b_i k_i(z_n)\| \\ &\quad\le \sum_{i=1}^{s} |b_i|\|k_i(y_n) - k_i(z_n)\|\end{aligned}$$

$$\begin{aligned}&\|\phi_1(y_n, f; h) - \phi_1(z_n, f; h)\| \\ &\quad\le \sum_{i=1}^{s} \alpha_i|b_i|\|y_n - z_n\|.\end{aligned} \tag{3.8}$$

Similarly,

$$\begin{aligned}&\|\phi_2(y_{n-1}, f; h) - \phi_2(z_{n-1}, f; h)\| \\ &\quad= \|\sum_{i=1}^{s} b_i k_{-i}(y_{n-1}) - b_i k_{-i}(z_{n-1})\| \\ &\quad\le \sum_{i=1}^{s} |b_i|\|k_{-i}(y_{n-1}) - k_{-i}(z_{n-1})\| \\ &\quad\le \sum_{i=}^{s} \beta_i|b_i|\|(y_{n-1}) - (z_{n-1})\|.\end{aligned} \tag{3.9}$$

Next, we will show that

$$\|\overline{\phi}(\overline{y}_n, f; h) - \overline{\phi}(\overline{z}_n, f; h)\| \le \widehat{L}_\phi\|\overline{y}_n - \overline{z}_n\|,$$

where,

$$\overline{\phi} = (\phi_1, \phi_2), \overline{y}_n = (y_n, y_{n-1}), \overline{z}_n = (z_n, z_{n-1}).$$

Consider,

$$\begin{aligned}
&\|\overline{\phi}(\overline{y}_n, f; h) - \overline{\phi}(\overline{z}_n, f; h)\| \\
&= \|\phi_1(y_n, f; h) - \phi_1(z_n, f; h) \\
&+ \phi_2(y_{n-1}, f; h) - \phi_2(z_{n-1}, f; h)\| \\
&\leq \|\phi_1(y_n, f; h) - \phi_1(z_n, f; h)\| \\
&+ \|\phi_2(y_{n-1}, f; h) - \phi_2(z_{n-1}, f; h)\| \\
&\leq \sum_{i=1}^{s} \alpha_i |b_i| \|y_n - z_n\| \\
&+ \sum_{i=1}^{s} \beta_i |b_i| \|y_{n-1} - z_{n-1}\| \\
&\leq \sum_{i=1}^{s} \alpha_i |b_i| \|\overline{y}_n - \overline{z}_n\| \\
&+ \sum_{i=1}^{s} \beta_i |b_i| \|\overline{y}_n - \overline{z}_n\| \\
&\leq \left(\sum_{i=1}^{s} \alpha_i |b_i| + \sum_{i=1}^{s} \beta_i |b_i| \right) \|y_n - z_n\|,
\end{aligned}$$

where

$$\|\overline{y}_n - \overline{z}_n\| = \max\left(\|y_n - z_n\|, \|y_{n-1} - z_{n-1}\|\right).$$

Thus,

$$\|\overline{\phi}(\overline{y}_n, f; h) - \overline{\phi}(\overline{z}_n, f; h)\| \leq \widehat{L}_\phi \|\overline{y}_n - \overline{z}_n\|, \tag{3.10}$$

where,

$$\widehat{L}_\phi = \sum_{i=1}^{s} \left(\alpha_i |b_i| + \beta_i |b_i|\right).$$

from inequality (3.10), we conclude that

$$\phi(y_n, f; h)$$

is Lipschitz continuous in the first component y_n. □

Remark: If for the limit $h \to 0$ and $n \to \infty$, sequence $\{x_n\}$ is convergent say

$$\lim_{n\to\infty} x_n = x.$$

Then,

$$\lim_{n\to\infty} T_n = y'(x) - \phi(x, y; 0). \tag{3.11}$$

So, the numerical method (3.1) is consistent if and only if $\phi(x, y; 0) = f(x, y)$, or $\phi(y, f; 0) = f(y)$ (for autonomous IVP).

Definition 3.4. An iteration method $\overline{y}_{n+1} = \overline{y}_n + h\overline{\phi}(x_n, f; h)$ is stable if for each initial value problem satisfying the Lipschitz condition there exist positive constants h_0 and M such that the difference between two numerical solutions y_n and z_n such that $\|y_n - z_n\| \leq M\|y_0 - z_0\|$, $\forall 0 \leq h \leq h_0$.

Theorem 3.5. *If $\phi(y, f; h)$ satisfies the Lipschitz condition in y then the method*

$$\overline{y}_{n+1} = \overline{y}_n + h\overline{\phi}(y, f; h)$$

is stable. i.e.

$$\|y_{n+1} - z_{n+1}\| \leq M\|y_0 - z_0\|,$$

where, $M = (1 + |h|L)^{n+1}$ and

$$\begin{aligned}
y_{n+1} &= y_n + h\phi(y_n, f; h), \\
z_{n+1} &= z_n + h\phi(z_n, f; h).
\end{aligned}$$

Proof.

$$\begin{aligned}
\|y_{n+1} - z_{n+1}\| &= \|y_n + h\phi(y_n, f; h) - z_n \\
&- h\phi(z_n, f; h)\| \\
&\leq \|y_n - z_n\| + h\|\phi(y_n, f; h) \\
&- \phi(z_n, f; h)\| \\
&\leq \|y_n - z_n\| + h\widehat{L}\|y_n - z_n\| \\
&\text{(Using Theorem 3.3)}
\end{aligned}$$

$$\begin{aligned}
\|y_{n+1} - z_{n+1}\| &\leq (1 + h\widehat{L})\|y_n - z_n\|, \forall n \\
&\leq (1 + h\widehat{L})(1 + h\widehat{L})\|y_{n-1} \\
&- z_{n-1}\| \\
&\leq (1 + h\widehat{L})^{n+1}\|y_0 - z_0\|.
\end{aligned}$$

□

Remark: If we choose an initial guess such that $\|y_0 - z_0\| < \epsilon$ then $\|y_{n-1} - z_{n-1}\| < \epsilon$.

Theorem 3.6. *If $\phi(y, f; h)$ satisfies the Lipschitz condition in y and h and the iteration method (3.1) is consistent and if $e_n = y(x_n) - y_n$, $n \geq 0$ denotes the error at the n^{th} iteration where x_0 is the initial point, then the error bound $\|e_n\|$ exists and is given by*

$$\|e_n\| \leq (1 + hL_\phi)^n(\|e_0\| + h\widetilde{L}),$$

i.e. error is bounded.

Proof. Let $e_n = y(x_n) - y_n, e_{n+1} = y(x_{n+1}) - y_{n+1}$ from (3.1)

$$y_{n+1} = y_n + h\phi(y_n, f; h), \tag{3.12}$$

by mean value theorem,

$$y(x_{n+1}) = y(x_n) + hy'(x_n + \theta h), 0 \leq \theta \leq 1. \tag{3.13}$$

By subtracting equation (3.12) from (3.13), we get the error as

$$e_{n+1} = e_n + h[\phi(y_n, f; h) - y'(x_n + \theta h)]. \tag{3.14}$$

$$\begin{aligned}\|e_{n+1}\| \leq \|e_n\| + h\|&\phi(y_n, f; h) \\ &- \phi(y(x_n), f; h)\| \\ &+ h\|\phi(y(x_n), f; h) - \phi(y(x_n), f; 0)\| \\ &+ h\|\phi(y(x_n), f; 0) - y'(x_n + \theta h)\|,\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}\|\phi(y_n, f; h) - \phi(y(x_n), f; h)\| \\ &\leq L_\phi\|y_n - y(x_n)\| \\ &= L_\phi\|e_n\|.\end{aligned}$$

Similarly,

$$\|\phi(y(x_n), f; h) - \phi(y(x_n), f; 0)\| \leq L_1 h,$$

(as $\phi(y(x_n), f; h)$ is Lipschitz continuous with respect to h) and

$$\begin{aligned}\|\phi(y(x_n), f; 0) - y'(x_n + \theta h)\| \\ &\leq L_2\theta h\|y'(x_n + \theta_1 h)\|, \\ &\leq L_3 h, (\text{where } 0 < \theta_1 < 1)\end{aligned}$$

where $L_3 = \theta\|y'(x_n + \theta_1 h)\|$. So,

$$\begin{aligned}\|e_{n+1}\| \leq &\|e_n\| + hL_\phi\|e_n\| + hL_1 + hL_3 \\ \|e_{n+1}\| \leq &\|e_n\|(1 + L_\phi h) + h(L_1 + L_3) \\ \|e_{n+1}\| \leq &(1 + L_\phi h)^{n+1}\|e_0\| + h(L_1 + L_3) \\ &[1 + (1 + L_\phi h) + (1 + L_\phi h)^2 + \cdots \\ &+ (1 + L_\phi h)^n] \\ \|e_{n+1}\| \leq &(1 + L_\phi h)e^{Lb}\|e_0\| \\ &+ h(L_1 + L_3)\left[\frac{e^{Lb} - 1}{hL_\phi}\right] \\ \|e_{n+1}\| \leq &(1 + L_\phi h)^n e^{Lb}\|e_0\| \\ &+ h(L_1 + L_3)\left[\frac{(1 + L_\phi h)^n - 1}{hL_\phi}\right],\end{aligned}$$

or

$$\|e_n\| \leq (1 + L_\phi h)^n \left[\|e_0\| + h(L_1 + L_3)\right],$$

or

$$\|e_n\| \leq (1 + L_\phi h)^n[\|e_0\| + h\widetilde{L}].$$ □

3.1 Absolute Stability

In this section, we compute the interval of absolute stability for the proposed methods given in Subsections 2.1 and 2.2. For the interval of absolute stability of EPRKM proposed in Subsections 2.1 and 2.2, we apply these methods on the Dahlquist test problem:

$$\frac{dy}{dx} = \lambda y, \lambda = \alpha + i\beta, \alpha < 0. \tag{3.16}$$

The method derived in Subsection 2.1 given in Eqs (2.16)-(2.21) along with coefficients

(2.32) is a two-step method i.e. the governing difference equation is of the second order. The characteristic polynomials $\phi(z, w)$, to test the absolute stability are given by: For, EPIRKM in Subsection 2.1:

$$\phi(z, w) = w^2 - \left(1 + \frac{z}{2} + 0.8\frac{(z + 0.270833z^2)}{1 - 0.270833z}\right) w - \left(\frac{z}{2} - 0.8\frac{(z + 0.145833z^2)}{1 - 0.145833z}\right). \quad (3.17)$$

On the other hand, for EPDIRKM in Section (2.2):

$$\phi(z, w) = w^2 - \left(1 + \frac{z}{2} + \frac{0.94z}{1 - 0.42499z}\right) w - \left(\frac{z}{2} - \frac{0.94z}{1 - 0.3611z}\right), \quad (3.18)$$

where, $z = \lambda h$.

The region of absolute stability is defined as the set of all those values of z for which the roots of the stability polynomial lie inside the unit circle, i.e. $\{z \in C : |w| < 1\}$ and it is displayed in Figure 1. If $z = \lambda h$ is real, then the intervals of absolute stability are (-3.75, 0) and (-1.82, 0) for the EPRK methods given in Subsection 2.1 and 2.2, respectively. So, these methods are not A-stable as they are not absolutely stable in the whole negative complex plane. However, they have large regions of absolute stability. To find the interval of absolute stability we follow the Routh-Hurwitz criterion. The truncation error of embedded Runge-Kutta methods proposed in Subsections 2.1, 2.2 and 2.3 are of order $O(h^4)$. So, their local truncation errors satisfy $|T.E.| \leq O(h^4)$, which infers that all the three proposed methods are of order three. The cost of computation for the proposed methods is shown in Table 1. In Table 1, we have made a comparison of a number of function evaluations between our proposed methods and existing classical Runge-Kutta method of order 3 (explicit and implicit). It is quite evident that our proposed method is $O(h^4)$ like the classical RK3 method and Runge-Kutta-Nyström method of order 3 (RK3) but requires fewer function evaluations which ensure that the proposed methods are computationally efficient. Although, there are two fewer slope computations in IRK3 than our proposed methods, this causes a large in error. As seen in Table 2, it is evident that IRK3 has a much larger error than EPDIRK proposed in Subsection 2.2.

Table 1. Comparison of order of the method and total No. of function evaluations (slopes) incurred in N time-steps.

Case	Order	Total no. of function evaluations
RK3	$O(h^4)$	3N
IRK3	$O(h^4)$	2N
RKN3	$O(h^4)$	3N
EPRKM (2.1)	$O(h^4)$	2N+2
EPRKM (2.2)	$O(h^4)$	2N+2
EPRKM (2.3)	$O(h^4)$	2N+2

4. Numerical Experiments

First, we consider the autonomous initial value problem in first order ODE given by

$$\frac{dy}{dx} = -y^2, y(0) = 1, x \in [0, 1], \quad (4.1)$$

where $y(x)$ is an unknown solution. We apply the embedded implicit pseudo-Runge-Kutta method proposed in Subsection 2.2. Using the Eqs. (2.34)-(2.38) together with (2.51) and taking $h = 0.1$ for the numerical solution for various value of x, errors are calculated. Further, the point wise errors are compared with the errors obtained by the classical RK third order method (RK3) which is depicted in Table 2. From Table 2, it is quite obvious that errors are smaller than in the RK3 method which shows the high accuracy of the proposed method. Next, we consider an IVP in second order ODE, given by

$$\frac{d^2y}{dx^2} = (1 + x^2)y, y(0) = 1, y'(0) = 0. \quad (4.2)$$

We apply the method proposed in Subsection 2.3 to the IVP (4.2). The solutions at various values of x and local point wise errors are calculated. Further, the point wise errors are compared with the errors obtained by 3-stage Runge-Kutta-Nyström (RKN3) method in Table 3. From Table 3, it is evident that errors obtained here are smaller than the errors by RKN3, which along with the fact that the proposed method involves less cost, ensures the superiority of the method given in Subsection 2.3.

Table 2. Absolute error (point wise) in solution at different values of for the IVP (4.1), with total of function evaluations.

Method →	RK3	IRK3	EPDIRK
x ↓	N=27	N=18	N=20
0.2	4.18(-5)	4.59(-3)	4.14(-6)
0.3	4.74(-5)	7.41(-3)	3.58(-6)
0.4	4.88(-5)	9.13(-3)	1.83(-6)
0.5	4.80(-5)	1.01(-2)	1.92(-7)
0.6	4.61(-5)	1.06(-2)	2.08(-6)
0.7	4.37(-5)	1.09(-2)	3.71(-6)
0.8	4.11(-5)	1.09(-2)	5.03(-6)
0.9	3.85(-5)	1.08(-2)	6.07(-6)
1.0	3.60(-5)	1.07(-2)	6.86(-6)

Note: Notation $a(b)$ means $a \times 10^b$.

5. Conclusion

In this paper, we have successfully proposed two embedded implicit pseudo-Runge-Kutta methods of order three for the numerical solution of a first order IVP. Another explicit type embedded pseudo Runge-Kutta method for a second-order IVP is also derived. These methods expend much less cost of computation in the form of evaluation functions (slopes) which is quite evident from Table 1. These methods are also tested on numerical experiments. Two IVPs, one is of order one and the other one is of order two, are solved using the methods given in Subsection 2.2 and Subsection 2.3. The results so obtained are compared error

Table 3. Absolute error (point wise) in solution $y(x)$ at different values of x ($h = 0.1$) for the IVP (4.2), with N total of function evaluations.

Method →	RKN3	Our Method (Sec. 2.3)
x ↓	N=27	N=20
0.2	4.18(-6)	3.51(-6)
0.3	4.24(-6)	6.64(-6)
0.4	4.36(-6)	3.52(-6)
0.5	4.53(-6)	4.92(-8)
0.6	4.77(-6)	5.70(-6)
0.7	5.09(-5)	1.25(-6)
0.8	5.50(-6)	2.14(-5)
0.9	6.02(-6)	1.21(-5)
1.0	6.69(-6)	0.00000

Note: Notation $a(b)$ means $a \times 10^b$.

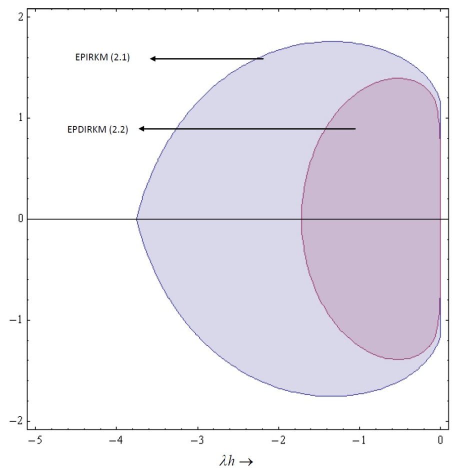

Fig. 1. Stability region for EPIRKM (2.1) and EPDIRK (2.2).

wise with the results obtained by the classical RK3, IRK3 and RKN3 methods. These comparisons are made in Table 2 and Table 3, which show that the errors are smaller. Thus the proposed methods are more efficient, accurate and superior to the existing Runge-Kutta methods. Stability properties and governing theorems establish convergence and stability. The proposed methods are also applied to the test problems to find the interval of absolute stability which are significantly large in size.

To improve the order of accuracy, we may use the Richardson active or passive extrapolation. The combination of EPRKM with the Richardson extrapolation scheme can be the scope of further research in this direction.

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