Research Article

A Note on Normal and Comaximal Lattices

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Abstract

In this paper 1-distributive lattices which is a generalization of distributive lattices were discussed. We give a characterization of 1-distributive lattices. The Separation Theorem for the element 1 is proven. Normal lattices and comaximal lattices are also disscussed. It is shown that the class of comaximal lattices is a subclass of normal lattices but not the converse. The two classes are same if the lattices are 1-distributive.

Keywords: 1-distributive lattice; prime ideal normal lattice; comaximal lattice
AMS Subject Classification: 06A12, 06A99, 06B10

1. Introduction

Lattice theories play an important role for the study of universal algebra. We refer the reader to the monographs [2, 3] for lattice theories. There are many research works on modular and distributive lattices. A lattice **L** is called *distributive* if for any $a,b,c \in L$,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

or equivalently,

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

Now a natural question is: when it is possible to generalize the results of distributive lattices for any lattices? In this paper we generalize some results of distributive lattices with (the largest element) 1. A lattice **L** with 1 is called 1-distributive if for any $a,b,c \in L$,

$$a \vee b = 1 = a \vee c \Rightarrow a \vee (b \wedge c) = 1.$$

The pentagonal lattice $\mathcal{P}_5$ (see the diagram in Figure 1) is 1-distributive but not distributive. Thus, not every 1-distributive lattice is a distributive lattice. The diamond lattice $\mathcal{M}_3$ (see the diagram in Figure 1) is not 1-distributive.

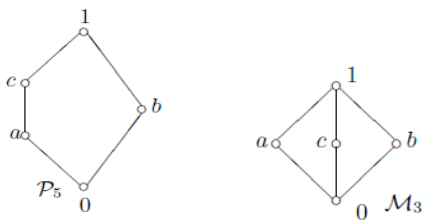

Fig.1. The pentagonal lattice $\mathbf{P}_5$ and the diamond lattice $\mathbf{M}_3$.

In Section 2, we study 1-distributive lattices. Like as distributive lattices we give a characterization of 1-distributive lattices. We also prove a separation theorem for the element 1.

In [1] Cornish has introduced normal lattices in presence of distributivity. In Section 3, we generalize a part of his result. We study the class of normal lattices in general. We also study the class of comaximal lattices. We

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show that the class of comaximal lattices is a proper subclass of the class of normal lattices. We also show that these two classes are equivalent in 1-distributive lattices.

2. 1-distributive Lattices.

There are very few works on 1-distributive lattices. For 1-distributive lattices, we refer the reader to [4].

By the definition, clearly we have the following results.

Lemma 2.1.

(a) Every distributive lattice with 1 is 1-distributive.
(b) Every sublattice with the 1 of a 1-distributive lattice is 1-distributive.

Thus, the class of distributive lattices with 1 is a proper subclass of 1-distributive lattices. Another important subclass of lattices is modular lattices. A lattice **L** is called *modular* if for any $a,b,c\in L$ with $c\leq a$ implies

$$a\wedge(b\vee c)=(a\wedge b)\vee c.$$

Now we have the following characterization of 1-distributive lattices.

Theorem 2.2.

Let **L** be a modular lattice with 1. Then the following are equivalent:

(a) **L** is 1-distributive;
(b) **L** contains no sublattice with the 1 isomorphic to either $\mathcal{M}_3$ or $\mathcal{N}$ given in the Figure 2.

Proof.

(a)$\Rightarrow$(b). Let **L** be 1-distributive. By the definition, the lattices $\mathcal{M}_3$ and $\mathcal{N}$ are not 1-distributive. By Lemma 2.1.(b) every sublattice with 1 of a 1-distributive lattice is 1-distributive. Therefore, **L** has no sublattice isomorphic to $\mathcal{M}_3$ or $\mathcal{N}$.

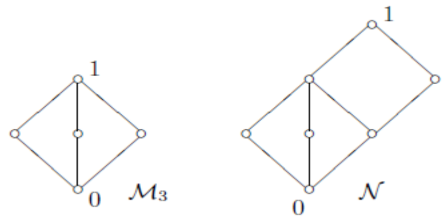

Fig.2. Two modular but not 1-distributive lattices.

(b)$\Rightarrow$(a). Let **L** be a modular lattice with 1. Assume **L** is not 1-distributive. We construct a sublattice with 1 of **L** which is isomorphic to one of the lattices $\mathcal{M}_3$ or $\mathcal{N}$ given by the diagrams in Figure 2. Since **L** is not 1-distributive, there are non-comparable elements $d,e,f\in L$ such that $d\vee e=1=d\vee f$ and $d\vee(e\wedge f)<1.$

Put

$$\begin{aligned}
p&:=(d\vee e)\wedge(e\vee f)\wedge(d\vee f)\\
q&:=(d\wedge e)\vee(e\wedge f)\vee(d\wedge f)\\
u&:=(d\vee q)\wedge p\\
v&:=(e\vee q)\wedge p\\
w&:=(f\vee q)\wedge p\\
r&:=d\vee(e\wedge f)=d\vee q.
\end{aligned}$$

Then clearly, u,v,w are non-comparable, $d\vee p=1,\ d\vee q<1,\ e\vee q=e\vee(d\wedge f)$ and $q,u,v,w\leq p$. Hence $q<u,v,w$ and $q<p$. Now

$$\begin{aligned}
u\vee v&=((d\vee q)\wedge p)\vee((e\vee q)\wedge p)\\
&=((d\vee q)\vee((e\vee q)\wedge p))\wedge p
\end{aligned}$$

By modularity where $(e\vee q)\wedge p\leq p$

$$=((d\vee q)\vee((e\wedge p)\vee q))\wedge p$$

By modularity where $q<p$

$$\begin{aligned}
&=((d\vee q)\vee(e\wedge p)\vee q)\wedge p\\
&=((d\vee q)\vee(e\wedge p))\wedge p\\
&=((d\vee(e\wedge f)\vee(e\wedge(d\vee f))\wedge p\\
&=(d\vee(e\wedge f)\vee e)\wedge p\quad[\because d\vee f=1]\\
&=(d\vee e)\wedge p\\
&=p\quad[\because d\vee e=1]
\end{aligned}$$

Similarly, $v \vee w = p$ and $w \vee u = p$.

Again

$$u \wedge v = ((d \vee q) \wedge p) \wedge ((e \vee q) \wedge p)$$
$$= (d \vee q) \wedge (e \vee q) \wedge p$$
$$= ((d \wedge (e \vee q)) \vee q) \wedge p$$

By modularity where $q \leq e \vee q$

$$= ((d \wedge (e \vee (d \wedge f))) \vee q) \wedge p$$
$$= ((d \wedge e) \vee (d \wedge f) \vee q) \wedge p$$

By modularity where $d \wedge f \leq d$

$$= q \wedge p$$
$$= q.$$

Similarly, $v \wedge w = q$ and $w \wedge u = q$.

Case 1. When $p = 1$. Then

$$u = d \vee q = d \vee (e \wedge f) = r.$$

Hence the set $\{q, u, v, w, 1\}$ with the operations of **L** forms a sublattice, say, **A** with 1 of **L** (see the diagram in Figure 3) which is isomorphic to M_3.

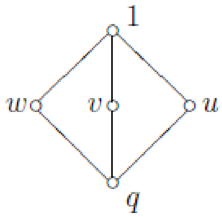

Figure 3.

Case 2. When $p \neq 1$. We have

$$u = (d \vee q) \wedge p = (d \vee (e \wedge f)) \wedge p$$
$$= r \wedge p.$$

Thus, $u < r$. Now

$$r \vee p = d \vee (e \wedge f) \vee p = 1.$$
$$r \vee v = (d \vee (e \wedge f)) \vee ((e \vee q) \wedge p)$$
$$= d \vee (e \wedge f) \vee (p \wedge q) \vee e$$
$$= d \vee e \vee (e \wedge f) \vee (p \wedge q)$$

By modularity, where $e \leq p = e \vee f$

$$= 1, \text{ since } d \vee e = 1.$$

Similarly, $r \vee w = 1$. Also

$$r \wedge v = (d \vee q) \wedge (e \vee q) \wedge p$$
$$= (d \vee (e \wedge f)) \wedge (e \vee (f \wedge d)) \wedge p$$
$$= (((e \vee (f \wedge d)) \wedge d) \vee (e \wedge f)) \wedge p$$

By modularity $e \wedge f \leq e \vee (f \wedge d)$

$$= ((d \wedge e) \vee (f \wedge d) \vee (e \wedge f)) \wedge p$$

By modularity where $f \wedge d \leq d$

$$= q \wedge p = q.$$

Similarly, $r \wedge w = q$.

Hence the set $\{q, u, v, w, p, r, 1\}$ with the operations of **L** forms a sublattice, say, **B** with 1 of **L** (see the diagram in Figure 4) which is isomorphic to $\mathcal{N}$.

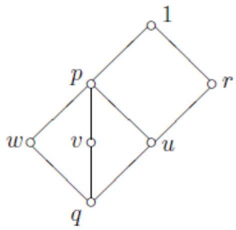

Fig.4.

♦

Let **L** be a lattice. A non-empty subset *I* of *L* is called an *ideal* of **L** if

(i) $a \in L$ and $b \in I$ with $a \leq b$ implies $a \in I$.

(ii) $a, b \in I$ implies $a \vee b \in I$.

An ideal *I* of **L** is called a *proper ideal* if $I \neq L$. A *minimal ideal* *I* of **L** is a proper ideal which does not contain any other proper ideal, that is, if there is a proper ideal *J* of **L** such that $J \subseteq I$, then $J = I$. A proper ideal *P* of **L** is called a *prime ideal* if for any $a, b \in L$ such that $a \wedge b \in P$ implies either $a \in P$ or $b \in P$.

Now we have the following Separation Theorem for the element 1.

Theorem 2.3. Let **L** be a 1-distributive lattice and *I* be an ideal of *L* such that $1 \notin I$. Then there is a prime ideal *P* of L such that $I \subseteq P$ and $1 \notin P$.

Proof. Let **L** be a 1-distributive lattice and let I be an ideal of **L** such that $[1) \cap I = \phi$. Suppose

$$\mathcal{X} = \{J \mid J \text{ is an ideal of L containing } I \text{ and } [1) \cap J = \phi\}$$

Then $\mathcal{X}$ satisfies all the conditions of the Zorn's Lemma. Hence there is a maximal element, say, M of $\mathcal{X}$. Now we show that M is a prime ideal of **L**. If M is not prime, then there are $a, b \in L \setminus M$ such that $a \wedge b \in M$. By the maximality of M, we have $1 \in M \vee (a]$ and $1 \in M \vee (b]$. This implies $m \vee a = 1 = m \vee b$ for some $m \in M$. Thus $m \vee (a \wedge b) = 1 \in M$ as **L** is 1-distributive, which is a contradiction. Hence M is prime.

3. Normal and Comaximal Lattices

A lattice **L** is called a *normal lattice* if every prime ideal of L contains a unique minimal prime ideal. The pentagonal lattice $\mathcal{P}_5$ (see the diagram in Figure 1) is a normal lattice, because it has only two prime ideals $(b]$ and $(c]$ which are also minimal prime ideals.

Two ideals P and Q of a lattice **L** are said to be *comaximal* if $P \vee Q = L$. A lattice **L** with 0 is said to be a *comaximal lattice* if any two minimal prime ideals of **L** are comaximal. The pentagonal lattice $\mathcal{P}_5$ (see the diagram in Figure 1) is comaximal, because it has only two minimal prime ideals $(b]$ and $(c]$ where $(b] \vee (c] = L$.

Theorem 3.1. Every comaximal lattice is normal.

Proof. Let **L** be a comaximal lattice. If **L** is not normal, then there is a prime ideal P of **L** such that P contains two distinct minimal prime ideals Q and R (say) of **L**. In this case $Q \vee R \subseteq P \neq L$, which contradicts the fact that **L** is comaximal. Thus **L** is normal. ♦

The converse of the above theorem not necessarily true. That is, not every normal lattice is a comaximal lattice. For counterexample, if we consider the lattice $\mathbf{L}_1$ given by the following diagram (see Figure 5), then the ideals $(a]$ and $(b]$ are only the prime ideals. This shows that every prime ideal contains a unique minimal prime ideal. Thus, $\mathbf{L}_1$ is normal. But $\mathbf{L}_1$ is not comaximal as $(a] \vee (b] \neq L$. Observe that the lattice $\mathbf{L}_1$ is not 1-distributive as it has a sublattice with 1 isomorphic to the diamond lattice M_3.

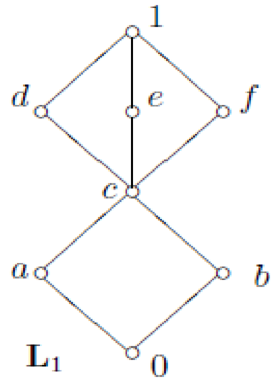

Fig.5. A normal but not comaximal lattice.

Now we have the following characterization of comaximal lattices.

Theorem 3.2. Every 1-distributive normal lattice is a comaximal lattice.

Proof. Let **L** be a 1-distributive normal lattice. Suppose **L** is not comaximal. Then there are two minimal distinct prime ideals P and Q of **L** such that $P \vee Q \neq L$. Thus $1 \notin P \vee Q$. Then by Theorem 2.3, there is a prime ideal M containing $P \vee Q$. This shows that **L** is not a normal lattice, a contradiction. Hence **L** is a comaximal lattice.

4. References

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