

Numerical Study of Unsteady Heat and Fluid Flow through a Curved Rectangular Duct of Small Aspect Ratio

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Abstract

In this paper, a numerical study is presented for the fully developed two-dimensional flow of viscous incompressible fluid through a curved rectangular duct of aspect ratio 0.5 and curvature 0.1. The outer wall of the duct is heated while the inner wall cooled, the top and bottom walls being adiabatic. Numerical calculations are carried out by using the spectral method, and covering a wide range of the Dean number $1000 \leq Dn \leq 10000$ and the Grashof number $100 \leq Gr \leq 2000$. The main concern of the present study is to investigate the nonlinear behavior of the unsteady solutions i.e. whether the unsteady flow is steady-state, periodic, multi-periodic or chaotic, if Dn or Gr is increased. Time evolution calculations as well as their phase spaces show that the unsteady flow is steady-state for $Dn \leq 6400$ and this region increases as Gr becomes large. It is found that the steady-state flow turns into chaotic flow through periodic and multi-periodic flows, if Dn is increased. Typical contours of secondary flow patterns and temperature profiles are also obtained, and it is found that the unsteady flow consists of asymmetric single-, two-, three- and four-vortex solutions. The present study shows that chaotic flow enhances heat transfer more significantly than the steady-state or periodic solutions due to many secondary vortices at the outer concave wall.

Keywords: Curved rectangular duct; secondary flow; time-evolution; periodic solution; chaos

1. Introduction

Investigation of flow and heat transfer through curved ducts and channels has been and continues to be a paramount interest of many researchers because of the

diversity of their practical applications in fluids engineering, such as in fluid transportation, refrigeration turbo-machinery, air conditioning systems, heat exchangers, rocket engines, internal combustion engines

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and blade-to-blade passages in modern gas turbines. In a curved passage, centrifugal forces are developed in the flow due to channel curvature causing a counter rotating vortex motion applied on the axial flow through the curved channel. This creates characteristics spiraling fluid flow in the curved passage known as secondary flow. At a certain critical flow condition and beyond, additional pairs of counter rotating vortices appear on the outer concave wall of curved fluid passages. This flow condition is referred to as Dean's *hydrodynamic instability* and the additional vortices are known as *Dean vortices*, in recognition of the pioneering work in this field by Dean [1]. After that, many theoretical and experimental investigations have been done by keeping this flow in mind; for instance, the articles by Berger et al. [2], Nandakumar and Masliyah [3] and Ito [4] may be referenced.

Early analytical and experimental investigations, such as Baylis [5] and Humphrey et al. [6] concluded that Dean number was solely responsible for secondary flow and Dean instability in curved passages. However later studies with curved rectangular ducts by Cheng et al. [7], Ghia and Sokhey [8] and Sugiyama et al. [9] have shown that the Dean instability is also dependent on the aspect ratio and curvature ratio along with the Dean number. Chandratilleke and Nursubyakto [10] reported a 2-dimensional study to examine the effects of curvature ratio and aspect ratio as well as the wall heat flux. They used toroidal coordinates and utilized a stream function approach with dynamic similarity in axial direction. Yanase et al. [11] investigated flow in a curved rectangular duct of aspect ratio 2 and classified flow range into three different regimes; steady-stable, periodic and chaotic. They used spectral method to see the field response against perturbation and discovered that while for low flow rate the system is confidently stable against perturbation, it will turn into periodic and even chaotic behaviors

for higher flow rates. Norouzi et al. [12] investigated the inertial and creeping flow of a second-order fluid in a curved duct with square cross-section by using finite difference method. The effect of centrifugal force due to the curvature of the duct and the opposing effects of the first and second normal stress difference on the flow field were investigated in that study. Chandratilleke et al. [13] presented a numerical investigation to examine the secondary vortex motion and heat transfer process in fluid flow through curved rectangular ducts of aspect ratios 1 to 6. The study formulated an improved simulation model based on three-dimensional vortex structures for describing secondary flow and its thermal characteristics. Recently, an analytical solution for the incompressible viscous flow through curved ducts with rectangular cross-sections has been presented by Norouzi and Biglari [14] by using perturbation method. In that study, the effect of duct curvature and aspect ratio on flow field was investigated. Very recently, Kun et al. [15] performed an experimental investigation on laminar flows of pseudo-plastic fluids in a square duct of strong curvature using an ultrasonic Doppler velocimetry and microphones, where streamwise velocity in the cross-section of the duct and the fluctuating pressure on the walls were measured for different flow rates. The velocity contours and their development along the duct were presented and compared with benchmark experiments by Taylor, Whitelaw and Yianneskis [16].

Time dependent analysis of fully developed curved duct flows was initiated by Yanase and Nishiyama [17] for a rectangular cross section. In that study, they investigated unsteady solutions for the case where dual solutions exist. The time-dependent behavior of the flow in a curved rectangular duct of large aspect ratio was investigated, in detail, by Yanase et al. [18] numerically. They performed time-evolution calculations of the unsteady solutions with and without

symmetry condition and showed that periodic oscillations appear with symmetry condition while aperiodic time variation without symmetry condition. Wang and Liu [19] performed numerical as well as experimental investigations of periodic oscillations for the fully developed flow in a curved square duct. Flow visualization in the range of Dean numbers from 50 to 500 was conducted in their experiment. They showed, both experimentally and numerically, that a temporal oscillation takes place between symmetric/asymmetric 2-cell and 4-cell flows when there are no stable steady solutions. Yanase et al. [20] performed numerical investigation of isothermal and non-isothermal flows through a curved rectangular duct of aspect ratio 2 and addressed the time-dependent behavior of the unsteady solutions. Recently, Mondal et al. [21, 22] performed numerical prediction of the unsteady solutions by time-evolution calculations for the flow through a curved square duct and discussed the transitional behavior of the unsteady solutions. However, transient behavior of the unsteady solution is not yet resolved for the flow through a curved rectangular duct of small aspect ratio, which motivated the present study to fill up this gap.

One of the most important applications of curved duct flow is to enhance thermal exchange between two sidewalls, because it is possible that the secondary flow may convey heat and then increases heat flux between two sidewalls. Chandratilleke and Nursubyakto [10] presented numerical calculations to describe secondary flow characteristics in the flow through curved ducts of aspect ratios ranging from 1 to 8 that were heated on the outer wall, where they studied for small Dean numbers and compared the numerical results with their experimental data. Yanase et al. [11, 20] studied time-dependent behavior of the unsteady solutions for curved rectangular duct flow and showed that secondary flows enhance heat transfer in the flow. Norouzi et

al. [23] investigated fully developed flow and heat transfer of viscoelastic materials in curved square ducts under constant heat flux. Recently, Fellouah et al. [24] attempted to develop an elementary 3-dimensional (3D) simulation covering the duct aspect ratio 0.5 to 12 with water and air as working fluids. Their model showed reasonable agreement with their own experiments that permitted visualization of vortex formation along channel locations for various Dean numbers. They presented a quantitative criterion for identifying the Dean instability in curved channels using the radial gradient of the axial velocity in the channel. Guo et al. [25] used a laminar incompressible 3D numerical model to explore the interactive effects of geometrical and flow characteristics on heat transfer and pressure drop. They applied entropy generation as a hydro-thermal criterion and reported the influence of the Reynolds number and curvature ratio on the flow profile and the Nusselt number. To the best of the authors' knowledge, however, there has not yet been done detailed investigation on the unsteady flow characteristics for the non-isothermal flow through a curved rectangular duct of aspect ratio 0.5 with large pressure gradient in the axial direction. But from the scientific as well as engineering point of view it is quite interesting to study the unsteady flow behavior in the presence of strong centrifugal and buoyancy forces for small aspect ratio duct, because this type of flow is often encountered in engineering applications such as in gas turbines, metallic industry and exhaustive pipes. The present paper investigates unsteady flow characteristics for the non-isothermal flow through a curved rectangular duct of small aspect ratio (aspect ratio 0.5) by using the spectral method, and covering a wide range of the Dean number and the Grashof number.

2. Model Description and Governing Equations

Consider an incompressible viscous fluid streaming through a curved duct with rectangular cross section whose width and height are $2d$ and $2h$, respectively. The coordinate system with the relevant notations are shown in Fig. 1. It is assumed that the outer wall of the duct is heated while the inner wall cooled, the top and bottom walls being adiabatic. The temperature of the outer wall is $T_0 + \Delta T$ and that of the inner wall is $T_0 - \Delta T$, where $\Delta T > 0$. The x , y and z axes are taken to be in the horizontal, vertical and axial directions, respectively. It is assumed that the flow is uniform in the axial direction and that it is driven by a constant pressure gradient G along the center-line of the duct, i.e. the main flow in the axial direction as shown in Fig. 1.

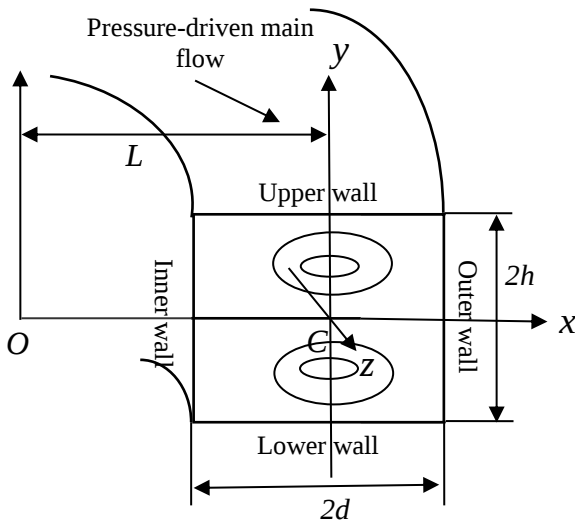


Fig.1. Coordinate system of the curved duct.

The variables are non-dimensionalized by using the representative length d and the representative velocity $U_0 = \nu/d$, where ν is the kinematic viscosity. We introduce the non-dimensional variables defined as:

$$u = \frac{u'}{U_0}, \quad v = \frac{v'}{U_0}, \quad w = \frac{\sqrt{2\delta}}{U_0} w', \quad x = \frac{x'}{d}, \quad \bar{y} = \frac{y'}{d},$$

$$z = \frac{z'}{d}, \quad T = \frac{T'}{\Delta T}, \quad t = \frac{U_0}{d} t', \quad \delta = \frac{d}{L}, \quad P = \frac{P'}{\rho U_0^2},$$

$$G = \frac{\partial P'}{\partial z'} \frac{d}{\rho U_0^2}$$

where u , v , and w are the non-dimensional velocity components in the x , y and z directions, respectively; t is the non-dimensional time, P the non-dimensional pressure, δ the non-dimensional curvature, and temperature is non-dimensionalized by ΔT . Henceforth, all the variables are non-dimensionalized if not specified. The sectional stream function ψ is introduced as:

$$u = \frac{1}{1+\delta x} \frac{\partial \psi}{\partial y}, \quad v = -\frac{1}{1+\delta x} \frac{\partial \psi}{\partial x}. \quad (1)$$

Then the basic equations for the axial velocity w , the stream function ψ and temperature T are derived from the Navier-Stokes equations and the energy equation under the *Boussinesq approximation* as

$$(1+\delta x) \frac{\partial w}{\partial t} + 2 \frac{\partial(w, \psi)}{\partial(x, y)} - D_n + \frac{\delta^2 w}{1+\delta x} = \quad (2)$$

$$(1+\delta x) \Delta_2 w - \frac{2\delta}{(1+\delta x)} \frac{\partial \psi}{\partial y} w + \delta \frac{\partial w}{\partial x}$$

$$\left(\Delta_2 - \frac{\delta}{1+\delta x} \frac{\partial}{\partial x} \right) \frac{\partial \psi}{\partial t} = -\frac{2}{(1+\delta x)} \frac{\partial(\Delta_2 \psi, \psi)}{\partial(x, y)} +$$

$$\frac{2\delta}{(1+\delta x)^2} \left[\frac{\partial \psi}{\partial y} \left(2\Delta_2 \psi - \frac{3\delta}{1+\delta x} \frac{\partial \psi}{\partial x} + \frac{\partial^2 \psi}{\partial x^2} \right) \right.$$

$$\left. - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial x \partial y} \right] + \frac{\delta}{(1+\delta x)^2} \left[3\delta \frac{\partial^2 \psi}{\partial x^2} - \frac{3\delta^2}{1+\delta x} \frac{\partial \psi}{\partial x} \right] \quad (3)$$

$$-\frac{2\delta}{1+\delta x} \frac{\partial}{\partial x} \Delta_2 \psi + 2w \frac{\partial w}{\partial y} + \Delta_2^2 \psi$$

$$-Gr(1+\delta x) \frac{\partial T}{\partial x},$$

$$\frac{\partial T}{\partial t} + \frac{2}{(1+\delta x)} \frac{\partial(T, \psi)}{\partial(x, y)} = \frac{1}{Pr} \left(\Delta_2 T + \frac{\delta}{1+\delta x} \frac{\partial T}{\partial x} \right). \quad (4)$$

where,

$$\Delta_2 \equiv \frac{\partial^2}{\partial x^2} + 4 \frac{\partial^2}{\partial y^2}, \quad \frac{\partial(f,g)}{\partial(x,y)} \equiv \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x}. \quad (5)$$

The Dean number Dn , the Grashof number Gr and the Prandtl number Pr , which appear in Eqs. (2) to (4), are defined as

$$Dn = \frac{Gd^3}{\mu\nu} \sqrt{\frac{2d}{L}}, \quad Gr = \frac{\beta g \Delta T d^3}{\nu^2}, \quad Pr = \frac{\nu}{k} \quad (6)$$

where, μ , ν , β , κ and g are the dynamic viscosity, the kinematic viscosity, the coefficient of thermal expansion, the coefficient of thermal diffusivity and the gravitational acceleration respectively.

The rigid boundary conditions for w and ψ are used as

$$\begin{aligned} w(\pm 1, y) = w(x, \pm 1) = \psi(\pm 1, y) = \psi(x, \pm 1) \\ = \frac{\partial \psi}{\partial x}(\pm 1, y) = \frac{\partial \psi}{\partial y}(x, \pm 1) = 0 \end{aligned} \quad (7)$$

and the temperature T is assumed to be constant on the walls as

$$T(1, y) = 1, \quad T(-1, y) = -1, \quad T(x, \pm 1) = x. \quad (8)$$

The present study is performed for the aspect ratio $\frac{h}{d} = 0.5$ and the curvature $\delta = \frac{d}{L} = 0.1$.

3. Numerical Calculations

3.1 Method of Numerical Calculation

In order to solve the Eqs. (2) to (4) numerically, the spectral method is used (Gottlieb and Orazag [26]). By this method the variables are expanded in the functions consisting of the Chebyshev polynomials, i.e.

$$\left. \begin{aligned} \phi_n(x) &= (1-x^2)C_n(x) \\ \psi_n(x) &= (1-x^2)^2 C_n(x) \end{aligned} \right\} \quad (9)$$

where, $C_n(x) = \cos(n \cos^{-1}(x))$ is the n th order Chebyshev polynomial. $w(x, y, t)$, $\psi(x, y, t)$ and $T(x, y, t)$ are expanded in terms of $\phi_n(x)$ and $\psi_n(x)$ as

$$\left. \begin{aligned} w(x, y, t) &= \sum_{m=0}^M \sum_{n=0}^N w_{mn}(t) \phi_m(x) \phi_n(y), \\ \psi(x, y, t) &= \sum_{m=0}^M \sum_{n=0}^N \psi_{mn}(t) \psi_m(x) \psi_n(y), \\ T(x, y, t) &= \sum_{m=0}^M \sum_{n=0}^N T_{mn}(t) \phi_m(x) \phi_n(y) + x, \end{aligned} \right\} \quad (10)$$

where M and N are the truncation numbers in the x and y directions respectively. The expansion coefficients w_{mn} , ψ_{mn} and T_{mn} are then substituted into the basic Eqs. (2) - (4) and the collocation method is applied. The collocation points are taken to be

$$\begin{aligned} x_i &= \cos \left[\pi \left(1 - \frac{i}{M+2} \right) \right], \\ y_j &= \cos \left[\pi \left(1 - \frac{j}{N+2} \right) \right] \end{aligned} \quad (11)$$

where $i = 1, \dots, M+1$ and $j = 1, \dots, N+1$.

3.2 Numerical Accuracy

The accuracy of the numerical calculations is investigated for the truncation numbers M and N used in this study. To check the dependence of grid size, five types of grid sizes were taken, they are 16×10 , 18×10 , 20×10 , 20×12 and 20×14 . The values of λ and $w(0,0)$ are shown in Table 1 for various values of M and N , where λ is the resistance coefficient and $w(0,0)$ is the axial velocity of the steady solution at $(x, y) = (0, 0)$ for $Dn = 8000$, $Gr = 1000$, $\delta = 0.1$ and aspect ratio 0.5. As seen in Table 1, $M = 20$ and $N = 12$ gives sufficient accuracy of the present numerical solutions.

Table1. The values of λ and $w(0,0)$ for various values of M and N at $Dn = 8000$, $Gr = 1000$, $\delta = 0.1$ for the aspect ratio 0.5.

M	N	λ	$w(0,0)$
16	10	0.154182508	1300.6035547
18	10	0.154195115	1300.6421001
20	10	0.154199641	1300.6362118
20	12	0.154583494	1300.6299397
20	14	0.154369612	1300.6297474

3.3. Time-evolution Calculation

In order to calculate the unsteady solutions, we use the Crank-Nicolson and Adams-Bashforth methods together with the function expansion (10) and the collocation method. Details of this method are discussed in Mondal [27]. By applying the Crank-Nicolson and the Adams-Bashforth methods to the non-dimensional basic equations (2)-(4), and rearranging, we get

$$\left(\frac{1}{\Delta t} - \frac{\Delta_2}{2}\right)w(t + \Delta t) = \left(\frac{1}{\Delta t} + \frac{\Delta_2}{2}\right)w(t) - \delta x \frac{w(t) - w(t - \Delta t)}{\Delta t} \quad (12)$$

$$+ Dn + \delta x \Delta_2 w(t) - \frac{\delta^2 w(t)}{1 + \delta x} + \delta \frac{\partial w(t)}{\partial x} + \frac{3}{2}\bar{P}(t) - \frac{1}{2}\bar{P}(t - \Delta t),$$

$$\left(\frac{1}{\Delta t} - \frac{\Delta_2}{2}\right)\Delta_2 \psi(t + \Delta t) = \left(\frac{1}{\Delta t} + \frac{\Delta_2}{2}\right)\Delta_2 \psi(t) + \frac{\delta}{1 + \delta x} \frac{1}{\Delta t} \left(\frac{\partial \psi(t)}{\partial x} - \frac{\partial \psi(t - \Delta t)}{\partial x} \right) + \frac{\delta}{(1 + \delta x)^2} \quad (13)$$

$$\left\{ -2(1 + \delta x) \frac{\partial}{\partial x} \Delta_2 \psi(t) + 3\delta \frac{\partial^2 \psi(t)}{\partial x^2} - \frac{3\delta^2}{(1 + \delta x)} \frac{\partial \psi(t)}{\partial x} \right\} - Gr(1 + \delta x) \frac{\partial T(t)}{\partial x} + \frac{3}{2}\bar{Q}(t) - \frac{1}{2}\bar{Q}(t - \Delta t)$$

$$\left(\frac{1}{\Delta t} - \frac{\Delta_2}{2\text{Pr}}\right)T(t + \Delta t) = \left(\frac{1}{\Delta t} + \frac{\Delta_2}{2\text{Pr}}\right)T(t) - \delta x \frac{T(t) - T(t - \Delta t)}{\Delta t} + \frac{1}{\text{Pr}} \delta (x \Delta_2 T(t) + \frac{\partial T(t)}{\partial x}) + \frac{3}{2}\bar{R}(t) - \frac{1}{2}\bar{R}(t - \Delta t). \quad (14)$$

In the above formulations, $\bar{P}$, $\bar{Q}$ and $\bar{R}$ are the non-linear terms. Then applying the Adams-Bashforth method for the second term of R. H. S of Eqs. (12), (13) and (14) and simplifying we calculate $w(t + \Delta t)$, $\psi(t + \Delta t)$ and $T(t + \Delta t)$ by numerical computation.

3.4 Resistance Coefficient

The resistance coefficient λ is used as the representative quantity of the flow state. It is also called the *hydraulic resistance coefficient*, and is generally used in fluids engineering, defined as

$$\frac{P_1^* - P_2^*}{\Delta z^*} = \frac{\lambda}{d_h^*} \frac{1}{2} \rho \langle w^* \rangle^2 \quad (15)$$

where quantities with an asterisk denote dimensional ones, $\langle \rangle$ stands for the mean over the cross section of the duct and d_h^* is the *hydraulic diameter*. The mean axial velocity $\langle w^* \rangle$ is calculated by

$$\langle w^* \rangle = \frac{\nu}{4\sqrt{2}\delta d} \int_{-1}^1 dx \int_{-1}^1 w(x, y, t) dy. \quad (16)$$

Since $P_1^* - P_2^* / \Delta z^* = G$, λ is related to the mean non-dimensional axial velocity $\langle w \rangle$ as

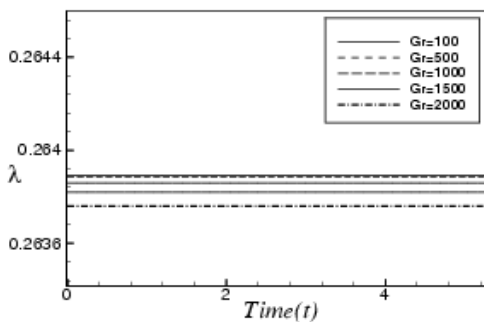
$$\lambda = \frac{8\sqrt{2}\delta Dn}{3\langle w \rangle^2} \quad (17)$$

where $\langle w \rangle = \sqrt{2}\delta d \langle w^* \rangle / \nu$.

4. Results

4.1 Time Evolution of the Unsteady Solutions

In order to investigate the non-linear behavior of the unsteady solutions, time-evolution calculations for the resistance coefficient λ are performed. It is observed that the unsteady flow is a steady-state solution for $Dn \leq 6400$. As for example, time evolution of λ for $Dn = 5000$ is shown in Fig. 2(a), where we see that the flow is steady-state for any value of Gr in the range $100 \leq Gr \leq 2000$. Then we obtain stream lines of secondary flows and isotherms of temperature distributions for $Dn = 5000$ and $Gr = 100, 500, 1000, 1500$ and 2000 as shown in Fig. 2(b). To draw the contours of ψ and T , we use the increments $\Delta\psi = 0.7$ and $\Delta T = 0.2$, respectively. The same increments of ψ and T are used for all the figures in this study, unless specified. The right-hand side of each duct box of ψ and T is in the outside direction of the duct curvature. In the figures of the stream lines, solid lines ($\psi \geq 0$) show that the secondary flow is in the counter clockwise direction while the dotted lines ($\psi < 0$) in the clockwise direction. As seen in Fig. 2(b), the steady-state solution for $Dn = 5000$ is an asymmetric two-vortex solution. Since the flow is steady-state solution for $Dn \leq 6400$, we calculate the unsteady solutions for $Dn > 6400$.



(a)

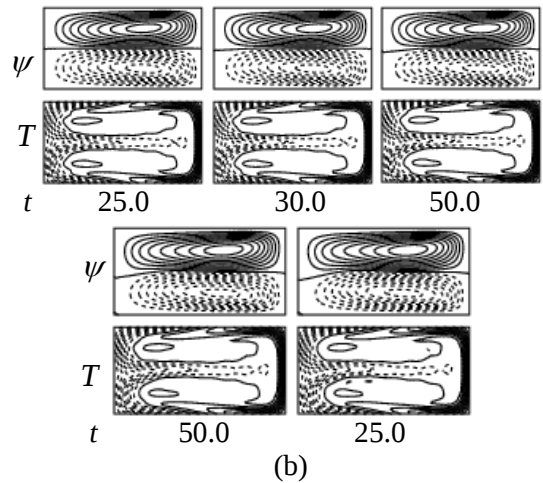
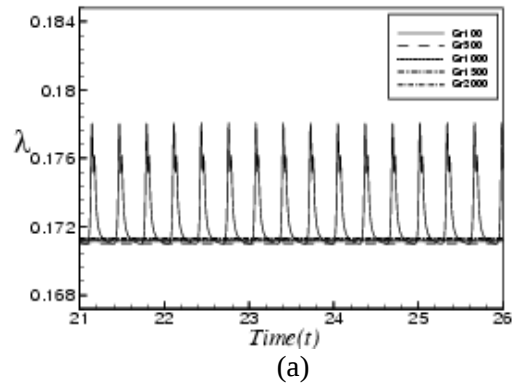
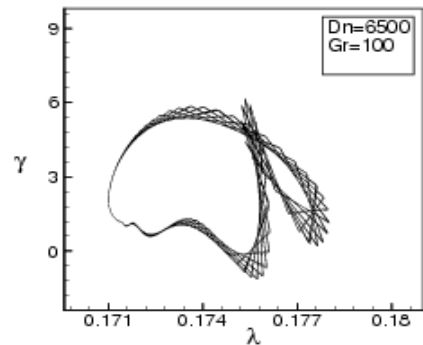


Fig.2. (a) Time evolution of λ for $Dn = 5000$ and $100 \leq Gr \leq 2000$, (b) Stream lines (top) and isotherms (bottom) for $Dn = 5000$ and $Gr = 100, 500, 1000, 1500$ and 2000 , respectively at various time.



(a)



(b)

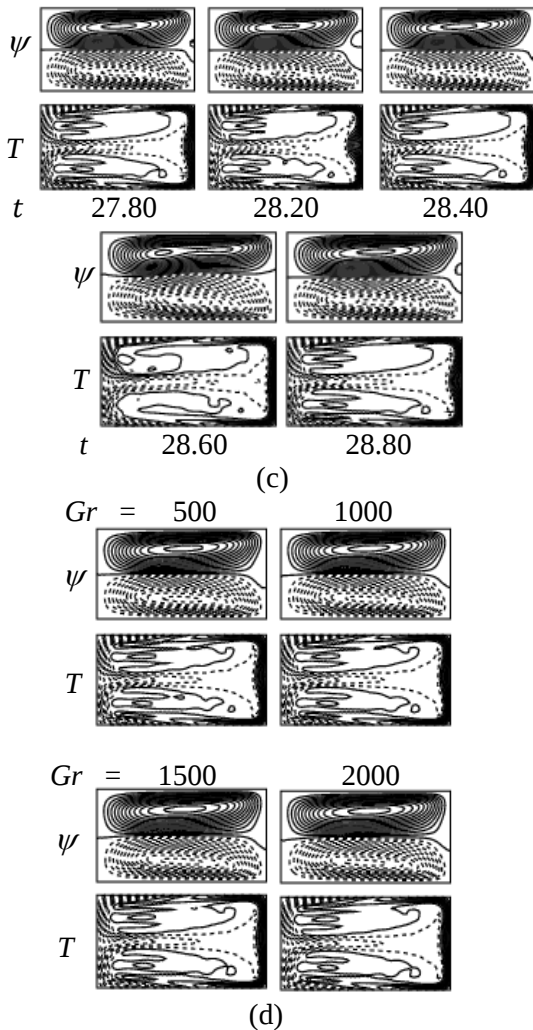


Fig.3. (a) Time evolution of λ for $Dn = 6500$ and $100 \leq Gr \leq 2000$, (b) Phase space for $Gr = 100$ (c) Stream lines (top) and isotherms (bottom) for $Gr = 100$, (d) Stream lines (top) and isotherms (bottom) for $Dn = 6500$ and $Gr = 500, 1000, 1500$ and 2000 .

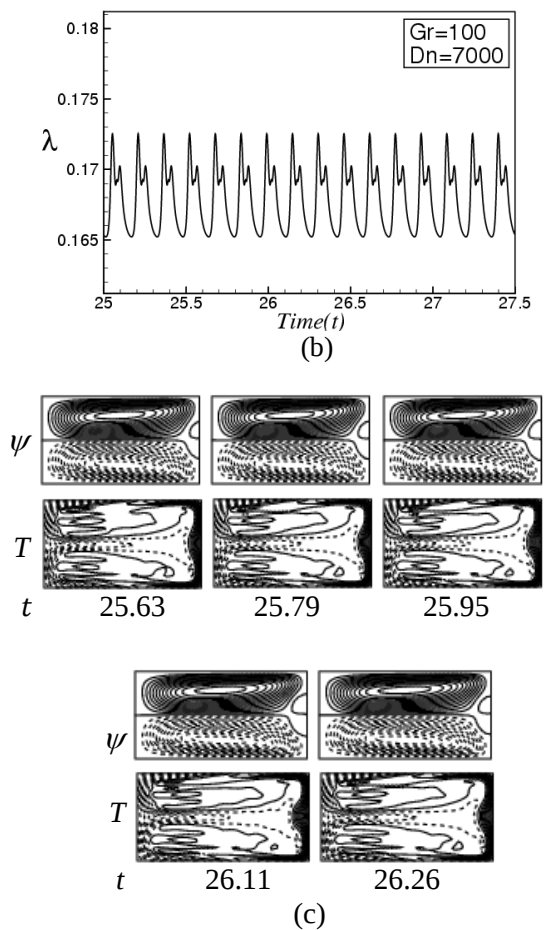
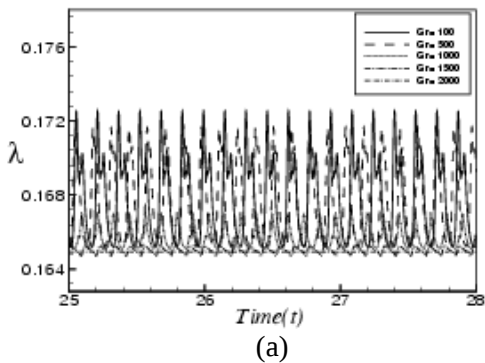
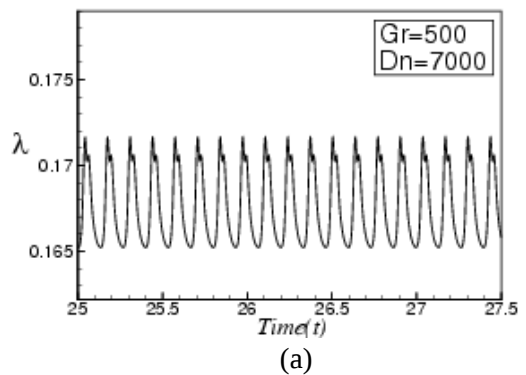


Fig.4. Unsteady solutions for $Dn = 7000$ and $100 \leq Gr \leq 2000$, (a) Time evolution result, (b) Time evolution of λ for $Gr = 100$, (c) Stream lines (top) and isotherms (bottom) for $Dn = 7000$ and $Gr = 100$.



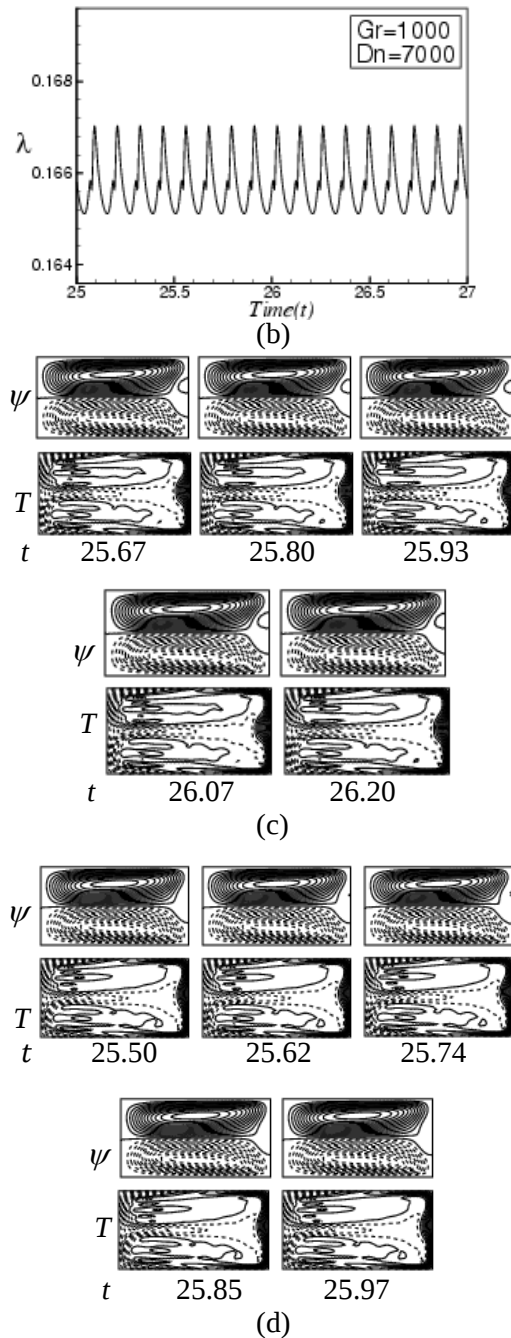


Fig.5. (a) Time evolution of λ $Dn = 7000$ and $Gr = 500$, (b) Time evolution of λ for $Dn = 7000$ and $Gr = 1000$, (c) Stream lines (top) and isotherms (bottom) for $Dn = 7000$ and $Gr = 500$, (d) Stream lines (top) and isotherms (bottom) for $Dn = 7000$ and $Gr = 1000$.

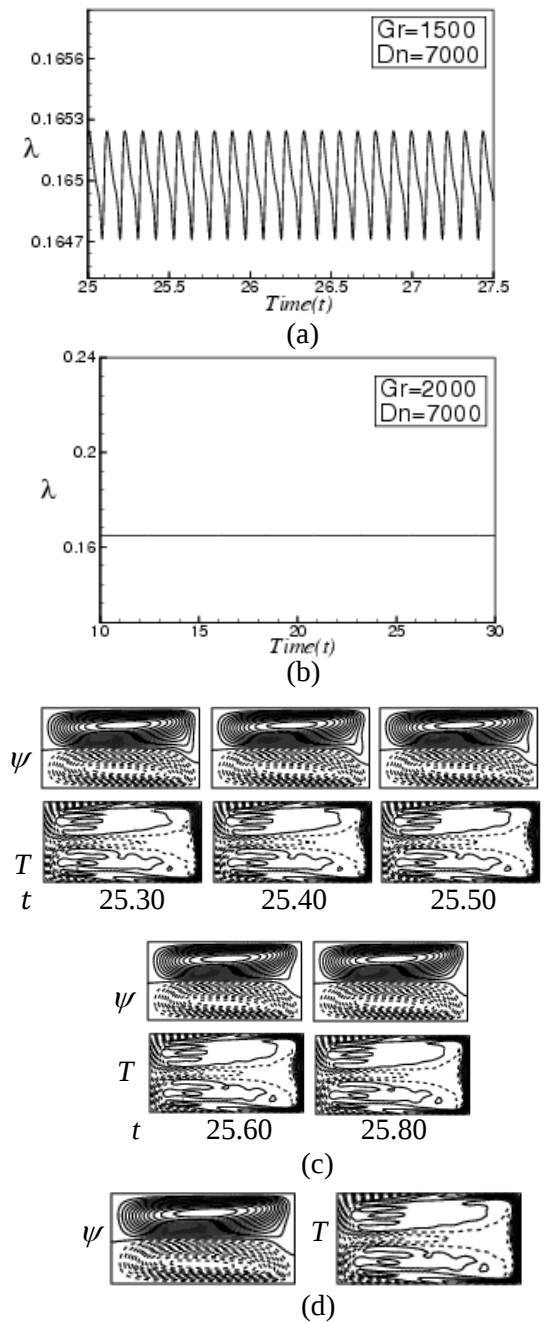


Fig.6. (a) Time evolution of λ for $Dn = 7000$ and $Gr = 1500$, (b) Time evolution of λ for $Dn = 7000$ and $Gr = 2000$, (c) Stream lines (top) and isotherms (bottom) for $Dn = 7000$ and $Gr = 1500$, (d) Stream lines (left) and isotherms (right) for $Dn = 7000$ and $Gr = 2000$.

We studied time evolution of λ for $Dn = 6500$ as shown in Fig. 3(a). As seen in Fig. 3(a), the unsteady flow is multi-periodic for $Gr = 100$ but steady-state for $Gr = 500, 1000, 1500$ and 2000 . In order to see the multi-periodic oscillation more clearly, a phase space of the time change of λ for $Gr = 100$ is shown in Fig. 3(b) in the $\lambda - \gamma$ plane, where $\gamma = \iint \psi dx dy$. Figure 3(b) shows that the flow creates multiple orbits, which suggests that the flow is multi-periodic. Typical contours of secondary flow patterns and temperature profiles are shown in Fig. 3(c), for a single period of oscillation at time $27.80 \leq t \leq 28.80$, and it is found that the flow oscillates between two- and three-vortex solutions. A single contour of each of the secondary flow and isotherms for the steady-state solutions at $Gr = 500, 1000, 1500$ and 2000 are also shown in Fig. 3(d), where we observe that the unsteady flow is an asymmetric two-vortex solution for the steady-state solution at these values of Gr .

Then we investigated time evolutions of λ for $Dn = 7000$ and $100 \leq Gr \leq 2000$ which are shown in Fig. 4(a). Here we find that the unsteady flow is multi-periodic for $Gr = 100, 500$ and 1000 , but periodic at $Gr = 1500$ and steady-state for $Gr = 2000$. The multi-periodic oscillation is explicitly shown in Fig. 4(b) for $Gr = 100$. To observe the pattern variation of secondary vortices and temperature distributions for the multi-periodic oscillation, typical contours of stream lines and isotherms for $Dn = 7000$ and $Gr = 100$ are shown in Fig. 4(c), where we see that the flow is an asymmetric three-vortex solution. In order to see the characteristics of the multi-periodic oscillations for $Gr = 500$ and $Gr = 1000$, we show the unsteady solutions for $Gr = 500$ in Fig. 5(a) and for $Gr = 1000$ in Fig. 5(b). Typical contours of stream lines and isotherms are also shown in Fig. 5(c) and in Fig. 5(d) for $Gr = 500$ and $Gr = 1000$, respectively. As seen in Figs. 5(c) and 5(d), the multi-periodic oscillation for $Gr = 500$ is

a slightly three-vortex solution, while that for $Gr = 1000$ two-vortex solution.

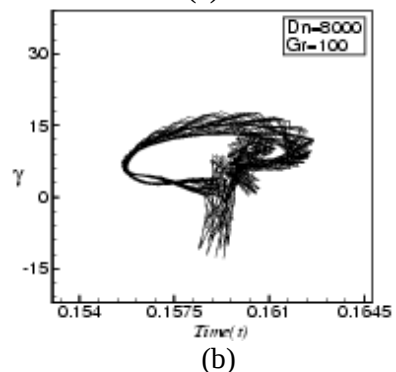
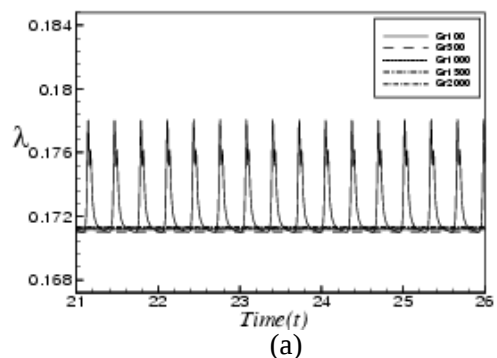
Then we obtained time evolution results for $Dn = 7000$ at $Gr = 1500$ and $Gr = 2000$ as shown in Figs. 6(a) and 6(b) respectively, where we see that the unsteady flow is periodic for $Gr = 1500$ but steady-state for $Gr = 2000$. Thus we observe that the multi-periodic oscillation finally turns into steady-state solution as Gr increases. Typical contours of stream lines and isotherms are shown in Fig. 6(c) for the periodic solution at $Gr = 1500$, for one period of oscillation at time $25.30 \leq t \leq 25.80$, and in Fig. 6(d) for the steady-state solution at $Gr = 2000$. It is found that the unsteady flow is an asymmetric two-vortex solution for both at $Gr = 1500$ and $Gr = 2000$. Then we investigated time evolution for $Dn = 8000$ and $100 \leq Gr \leq 2000$. Figure 7(a) shows that the unsteady flow at $Dn = 8000$ and $Gr = 100$ is a multi-periodic solution. In order to see the multi-periodic oscillation more clearly, a phase space of the time change of λ for $Dn = 8000$ at $Gr = 100$ is shown in Fig. 7(b). As seen in Fig. 7(b), the flow exhibits multiple orbits, which signifies that the unsteady flow presented in Fig. 7(a) for $Gr = 100$ and $Dn = 8000$ is multi-periodic. Typical contours of stream lines and isotherms are also shown in Fig. 7(c) for the corresponding flow parameters and it is seen that the multi-periodic oscillation at $Dn = 8000$ and $Gr = 100$ oscillates between asymmetric three- and four-vortex solution. We show the time evolution of λ for $Dn = 8000$ and $Gr = 500$ in Fig. 8(a) and corresponding phase space in Fig. 8(b). Figure 8(b) confirms that the unsteady flow presented in Fig. 8(a) is purely multi-periodic. Then stream lines of secondary vortices and isotherms of temperature profiles for $Dn = 8000$ and $Gr = 500$ are shown in Fig. 8(c) for one period of oscillation at time $25.49 \leq t \leq 25.94$. As seen in Fig. 8(c), the unsteady flow is a four-vortex solution for $Dn = 8000$ and $Gr = 500$.

It should be mentioned here that the oscillating pattern of multi-periodic solution as well as the Dean vortices generated at the outer wall of the duct for $Dn = 8000$ are different from those at $Dn = 7000$; at the beginning of the secondary vortices, we found developing asymmetric four-vortex solutions, and as time proceeds, we get completely three- and four- vortex solutions. In fact, the periodic oscillation, which is observed in the present study, is a traveling wave solution advancing in the downstream direction which is well justified in the recent investigation by Yanase et al. [28] for a three-dimensional (3D) travelling wave solutions as an appearance of 2D periodic oscillation.

Time evolution of λ for $Dn = 8000$ and $Gr = 1000$ is then performed as shown in Fig. 9(a), which shows that the flow is multi-periodic. Phase space of the multi-periodic oscillation at $Gr = 1000$ is then obtained as depicted in Fig. 9(b). Figure 9(b) shows that the unsteady flow creates a single orbit, which means that the flow is periodic rather than multi-periodic. To observe the periodic change of the flow, typical contours of the stream lines and isotherms are shown in Fig. 9(c) for $Dn = 8000$ and $Gr = 1000$ where we see that the periodic oscillating flow is a three- and four-vortex solution. Time evolutions of λ for $Dn = 8000$ at $Gr = 1500$ and $Gr = 2000$ are shown in Figs. 10(a) and 10(b), respectively. It is seen that the flow oscillation is nearly the same and they are periodic. Then stream lines of secondary vortices and isotherms of temperature distributions are shown in Fig. 10(c) for $Dn = 8000$ and $Gr = 1500$ and in Fig. 10(d) for $Dn = 8000$ and $Gr = 2000$. Figures 10(c) and 10(d) show that the periodic oscillating flows at $Gr = 1500$ and $Gr = 2000$ are weakly three-vortex solutions.

We then calculated time evolutions of λ for $Dn = 9000$ and $100 \leq Gr \leq 2000$ as shown in Fig. 11(a), which shows that the unsteady flow is multi-periodic for all values of Gr in the range. In order to show the

multi-periodic oscillation more clearly, we explicitly show a time evolution result for $Dn = 9000$ and $Gr = 100$ in Fig. 11(b). This multi-periodic oscillation is well justified by drawing a phase space of the time evolution result as shown in Fig. 11(c). Typical contours of stream lines and isotherms are then shown in Fig. 11(d) for $Dn = 9000$ and $Gr = 100$. It is seen that the multi-periodic oscillation at $Dn = 9000$ oscillates between asymmetric three- and four-vortex solutions. Time evolution of λ for $Gr = 500$ and $Gr = 1000$ are shown in Figs. 12(a) and 13(a), respectively. To well understand the flow evolution, we also show corresponding phase spaces in Fig. 12(b) for $Gr = 500$ and in Fig. 13(b) for $Gr = 1000$. It is found that the flow creates multiple orbits, which justifies the multi-periodicity of the unsteady flows for both the cases. Typical contours of stream lines of secondary flow patterns and isotherms of temperature profiles are shown in Fig. 12(c) for $Gr = 500$ and in Fig. 13(c) for $Gr = 1000$, where we see that the secondary flow is an asymmetric four-vortex solution for the multi-periodic oscillation.



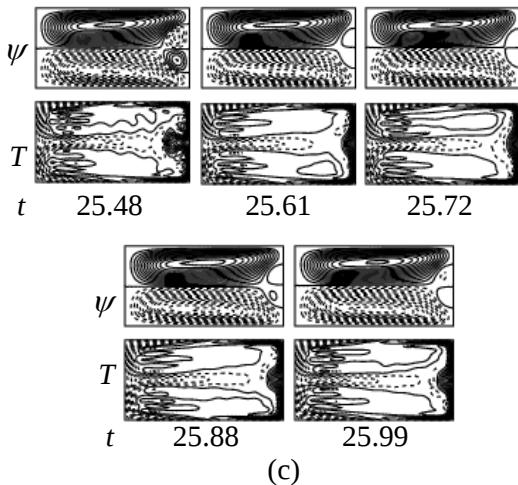


Fig.7. Unsteady solutions for $Dn = 8000$ and $Gr = 100$, (a) Time evolution of λ , (c) Phase space, (d) Stream lines (top) and isotherms (bottom) for $Dn = 8000$ and $Gr = 100$.

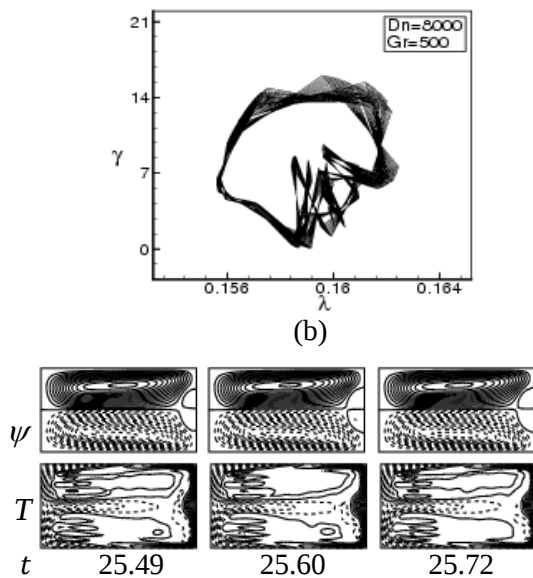
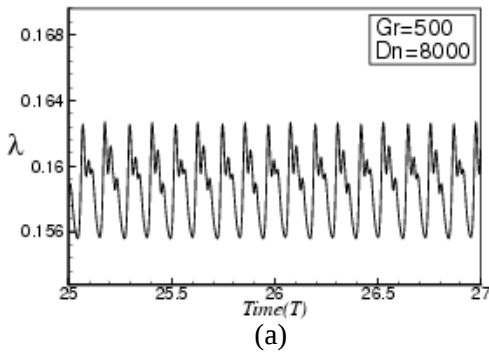


Fig.8. Unsteady solutions for $Dn = 8000$ and $Gr = 500$, (a) Time evolution of λ , (b) Phase space, (c) Stream lines (top) and isotherms (bottom) for $Dn = 8000$ and $Gr = 500$.

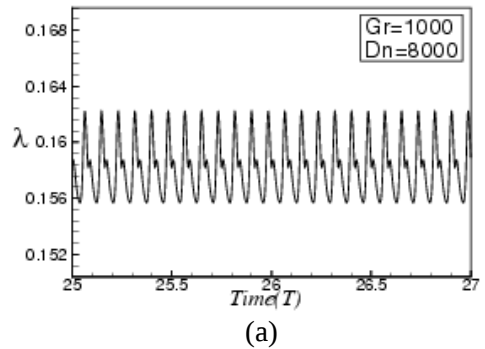


Fig.9. Unsteady solutions for $Dn = 8000$ and $Gr = 1000$, (a) Time evolution of λ , (b) Phase space, (c) Stream lines (top) and isotherms (bottom) for $Dn = 8000$ and $Gr = 1000$.

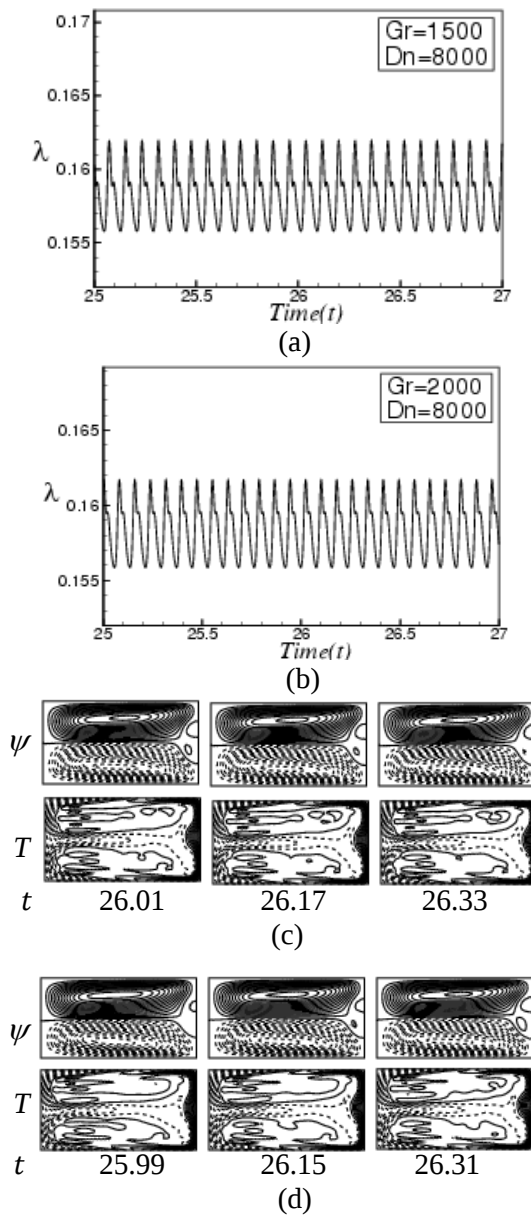


Fig.10. (a) Time evolution of λ for $Dn = 8000$ and $Gr = 1500$, (b) Time evolution for $Dn = 8000$ and $Gr = 2000$, (c) Stream lines (top) and isotherms (bottom) for $Dn = 8000$ and $Gr = 1500$, (d) Stream lines (top) and isotherms (bottom) for $Dn = 8000$ and $Gr = 2000$.

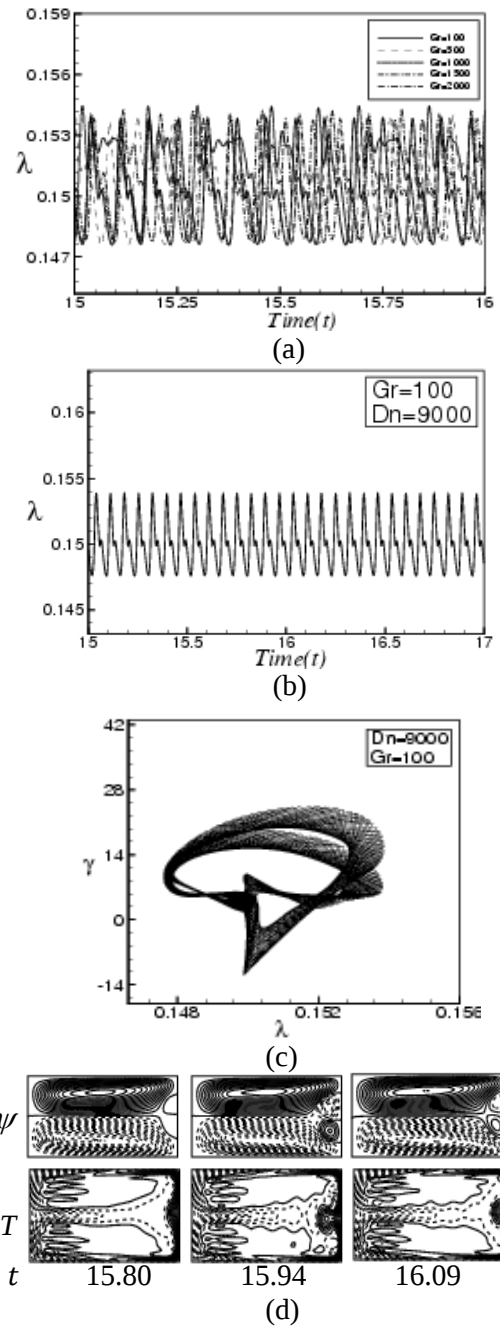
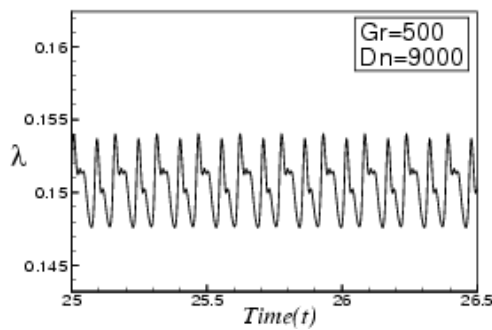
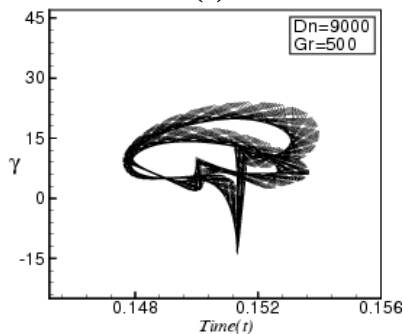


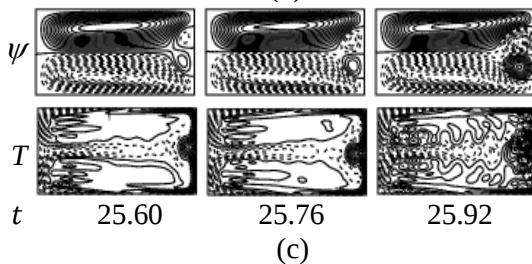
Fig.11. Unsteady solutions for $Dn = 9000$ and $100 \leq Gr \leq 2000$, (a) Time evolution result, (b) Time evolution of λ for $Dn = 9000$ and $Gr = 100$, (c) Phase space for $Dn = 9000$ and $Gr = 100$, (d) Stream lines and isotherms for $Dn = 9000$ and $Gr = 100$.



(a)

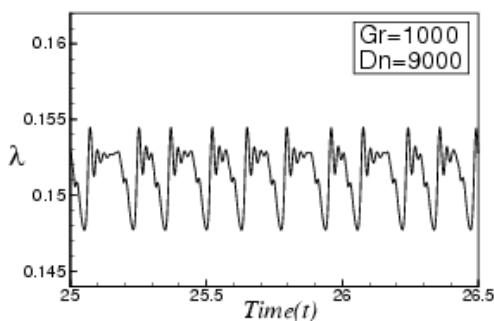


(b)

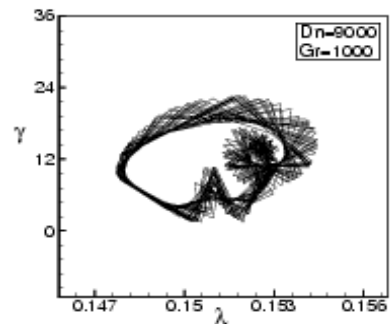


(c)

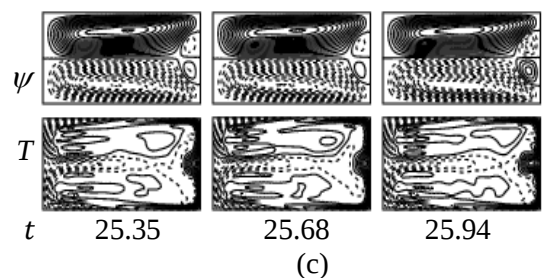
Fig.12. Unsteady solutions for $Dn = 9000$ and $Gr = 500$, (a) Time evolution of λ , (b) Phase space, (c) Stream lines (top) and isotherms (bottom) for $Dn = 9000$ and $Gr = 500$.



(a)

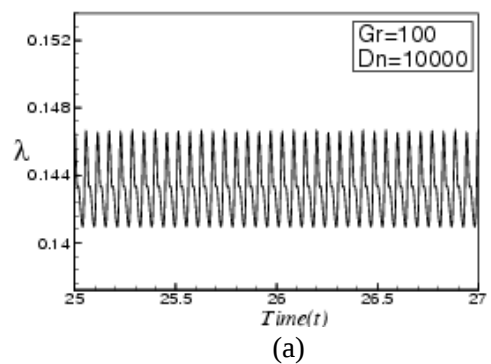


(b)

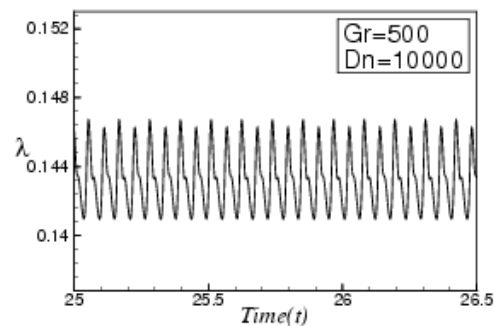


(c)

Fig.13. Unsteady solutions for $Dn = 9000$ and $Gr = 1000$, (a) Time evolution of λ , (b) Phase space, (c) Stream lines and isotherms for $Dn = 9000$ and $Gr = 1000$.



(a)



(b)

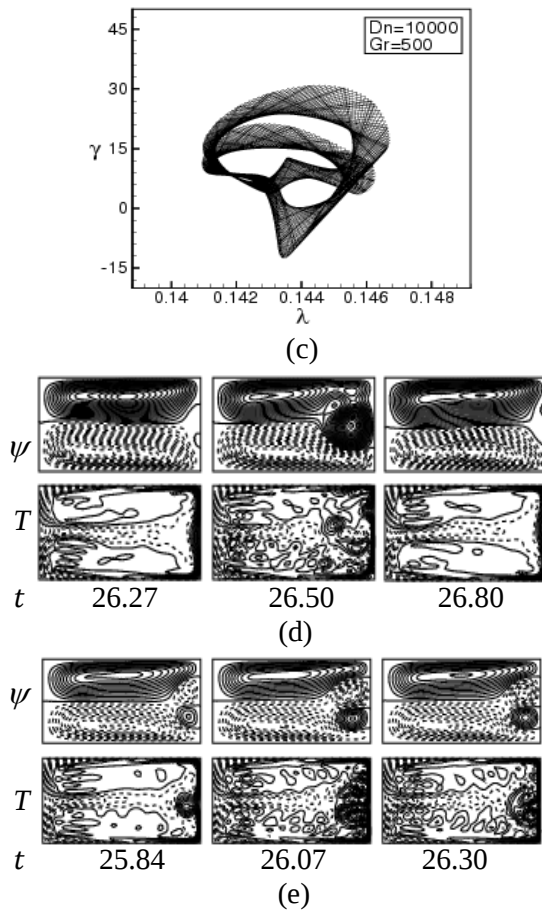


Fig.14. (a) Time evolution of λ for $Dn = 10000$ and $Gr = 100$, (b) Time evolution of λ for $Dn = 10000$ and $Gr = 500$, (c) Phase space for $Dn = 10000$ and $Gr = 500$, (d) Stream lines (top) and isotherms (bottom) for $Dn = 10000$ and $Gr = 100$, (e) Stream lines and isotherms for $Dn = 10000$ and $Gr = 500$.

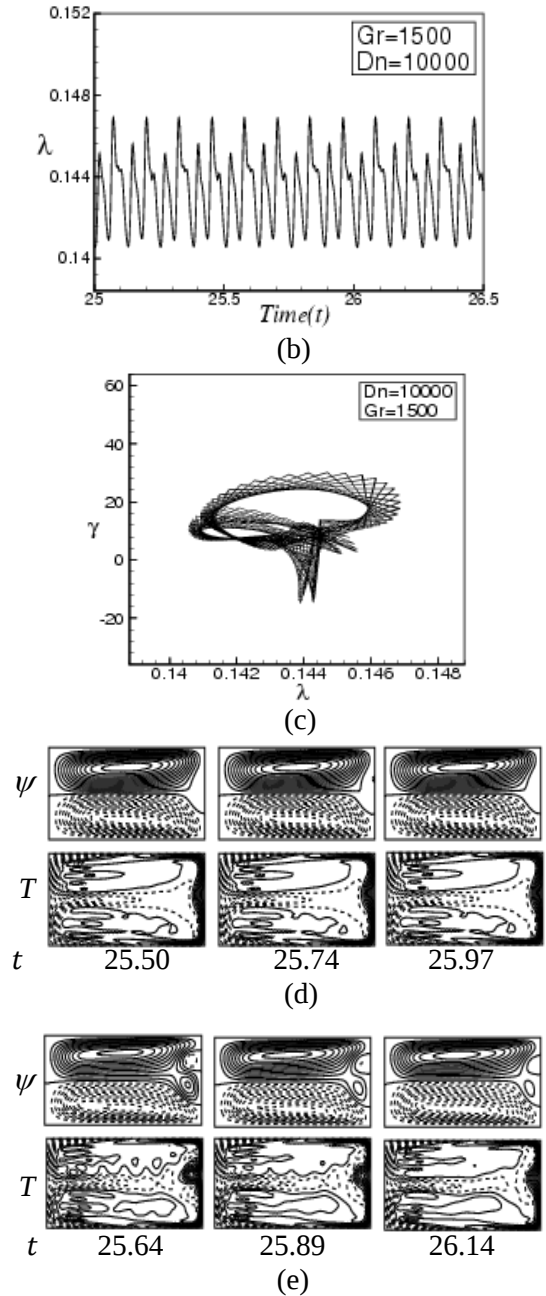
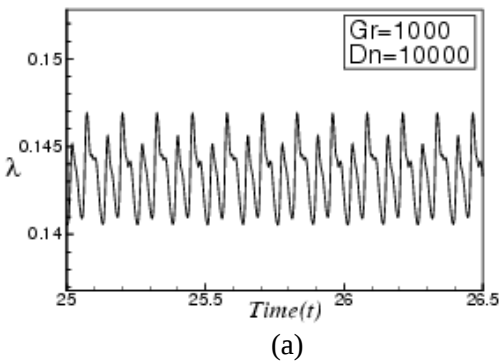


Fig.15. (a) Time evolution of λ for $Dn = 10000$ and $Gr = 1000$, (b) Time evolution of λ for $Dn = 10000$ and $Gr = 1500$, (c) Phase space for $Dn = 10000$ and $Gr = 1500$, (d) Stream lines and isotherms for $Dn = 10000$ and $Gr = 1000$, (e) Stream lines and isotherms for $Dn = 10000$ and $Gr = 1500$.

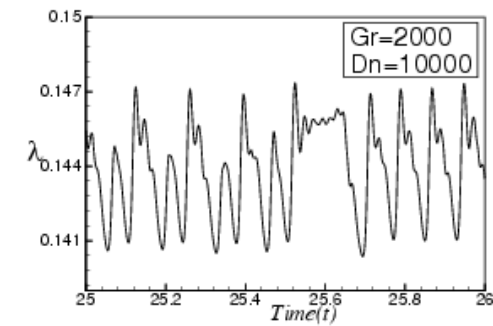
Then we proceeded for the time evolution calculation of λ for $Dn = 10000$ and $100 \leq Gr \leq 2000$. Figures 14(a) and 14(b) show time evolutions of λ for $Gr = 100$ and $Gr = 500$ respectively, where it is found that the flow oscillates periodically for $Gr = 100$ but multi-periodically for $Gr = 500$. The multi-periodic oscillation is well justified by obtaining the phase space as shown in Fig. 14(c). To observe the formation of secondary vortices and temperature distributions for the periodic and multi-periodic oscillations, typical contours of stream lines and isotherms are presented in Figs. 14(d) and 14(e) for $Gr = 100$ and $Gr = 500$ respectively, where it is seen that the periodic flow oscillates between two and four-vortex solutions, while multi-periodic oscillation in the four-vortex solution only. Then we obtained time evolution of λ for $Gr = 1000$ and $Gr = 1500$ at $Dn = 10000$. Figures 15(a) and 15(b) show time evolution results for $Gr = 1000$ and $Gr = 1500$ respectively, where it is seen that the flow oscillates multi-periodically for both $Gr = 1000$ and $Gr = 1500$. The multi-periodic oscillation for $Gr = 1500$ is well justified by drawing the phase space as shown in Fig. 15(c), where we see that the flow exhibits multiple orbits, which confirms that the unsteady flow presented in Fig. 15(a) or in 15(b) is a multi-periodic solution. To observe the change of the secondary vortices and temperature distributions, as time proceeds, typical contours of the stream lines and isotherms are drawn in Figs. 15(d) and 15(e) for $Gr = 1000$ and $Gr = 1500$ respectively, and it is found that the multi-periodic oscillation for $Gr = 1000$ consists of asymmetric two-vortex solution, while that for $Gr = 1500$ asymmetric three- and four-vortex solution. Temperature distribution is observed to be consistent with secondary vortices.

Next, we calculated time evolution of the unsteady solutions for $Gr = 2000$ at $Dn = 10000$. Figure 16(a) shows time evolution of λ for $Gr = 2000$, where it is seen that the

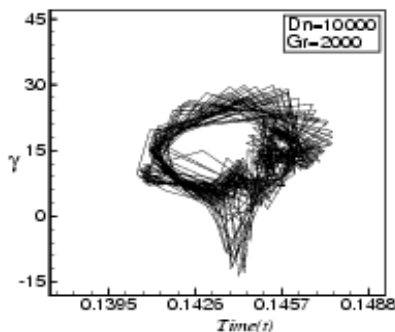
flow oscillates irregularly, that means the flow is chaotic. This type of flow is termed as *transitional chaos* (Mondal et al. [22]). The chaotic oscillation is well proved by drawing the phase space as presented in Fig. 16(b). Figure 16(b) shows that the flow creates an irregular line spectrum, which implies that the flow presented in Fig. 16(a) is chaotic. Then to observe the formation of secondary vortices and temperature distributions for the chaotic oscillation at $Gr = 2000$ and $Dn = 10000$, typical contours of the stream lines and isotherms are presented in Fig. 16(c), where it is seen that the chaotic flow at $Gr = 2000$ and $Dn = 10000$ is a four-vortex solution. In this study, it is found that the temperature distribution is well consistent with the secondary vortices, and convective heat transfer is significantly enhanced as the secondary vortices become stronger. The present study also shows that there is a strong interaction between the heating-induced buoyancy force and the centrifugal instability in the curved passage that stimulates fluid mixing and consequently enhance heat transfer in the fluid.

4.2 Phase Diagram of the Unsteady Solutions

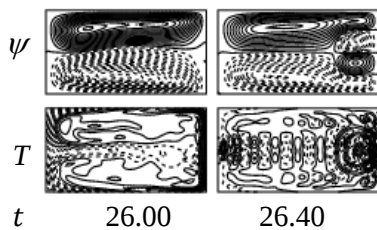
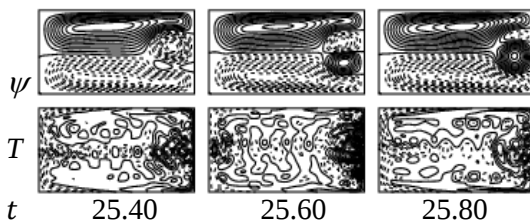
Finally, the complete unsteady solutions, obtained by the time evolution computations in the present study, is shown by a phase diagram in Fig. 17 in the Dean number vs. Grashof number plane ($Dn-Gr$ plane) for $1000 \leq Dn \leq 10000$ and $100 \leq Gr \leq 2000$ for the flow through a curved rectangular flow of aspect ratio 0.5. In this figure, the circles indicate steady-state solutions, crosses periodic solutions and triangles chaotic solutions. Figure 17 shows that the unsteady flow becomes steady-state for any value of Gr when $Dn \leq 6400$; for $Dn > 6400$, however, the flow becomes periodic or multi-periodic first and then chaotic as Dn is increased.



(a)



(b)



(c)

Fig.16. (a) Time evolution of λ for $Dn = 10000$ and $Gr = 2000$, (b) Phase space for $Dn = 10000$ and $Gr = 2000$, (c) Stream lines (top) and isotherms (bottom) for $Dn = 10000$ and $Gr = 2000$.

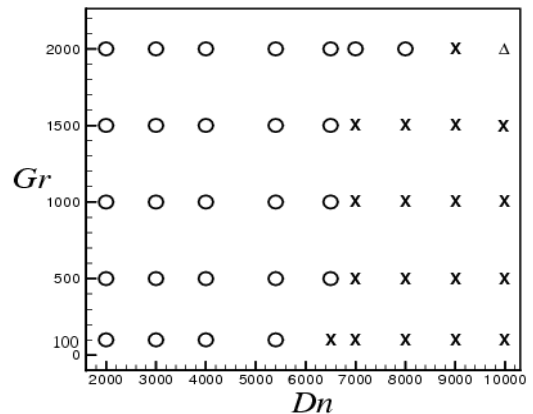


Fig.17. Unsteady solutions at a glance in the $Dn - Gr$ plane for $2000 \leq Dn \leq 10000$ and $100 \leq Gr \leq 2000$ for the curved rectangular duct of aspect ratio 0.5 and curvature 0.1.

5. Discussion

In this section, a brief discussion on the past and present study of time dependent solutions for curved duct flow as well as the plausibility of applying two-dimensional (2D) calculations for studying curved duct flows will be given. It has been shown by many experimental and numerical investigations that curved duct flows easily attain asymptotic fully developed 2D flow at most 270° from the inlet (Wang and Yang [29]). Recently, Mondal et al. [21, 22] investigated detailed numerical study on the unsteady solutions through a curved square duct over a wide range of curvature and showed that steady flow turns into chaos via periodic or multi-periodic flows if Dn is increased no matter what the curvature is. Yanase et al. [11] investigated non-isothermal flows with convective heat transfer through a curved rectangular duct of aspect ratio 2, where they classified the flow regions into three different regimes; steady-state, periodic and chaotic. They showed that for low flow rate the system is confidently stable while it turns into periodic and even chaotic for higher flow rates. Very recently, Mondal et al. [30] performed spectral numerical study to investigate

unsteady flow characteristics for flow through curved rectangular ducts of aspect ratios ranging 1 to 3, and showed that the steady-state flow turns into chaotic flow through various flow instabilities if the aspect ratio is increased. However, there is no known work studying the unsteady solutions with convective heat transfer through a curved rectangular duct of aspect ratio less than 1 (one). In this paper, we investigated detailed numerical study on unsteady solutions through a curved rectangular duct of aspect ratio 0.5, where it is found that the flow is steady-state up to a large value of Dn and this region increases as the Grashof number becomes large. The present study also shows that the existence of the chaotic solution is not found for $Dn \leq 10000$ at $Gr < 2000$; however, at large values of Gr and Dn , for example $Dn = 10000$ and $Gr = 2000$, a chaotic solution is attained.

On the assumption of two-dimensionality in curved duct flows, Wang and Yang [29] showed that for an oscillating flow, there exists a close similarity between the flow observation at 270° and 2D calculation. There is some evidence showing that the occurrence of chaotic or turbulent flow may be predicted by 2D analysis. Yamamoto et al. [31] investigated helical pipe flows with respect to 2D perturbations and compared the results with their experimental data (Yamamoto et al. [32]). There was a good agreement between the numerical results and the experimental data, which shows that even the transition to turbulence can be predicted by 2D analysis to some extent. The transition to chaos of the periodic oscillation, obtained by the 2D calculation in the present study, may correspond to destabilization of travelling waves in the curved duct flows. Our present result, therefore, may contribute to the study of flows through a curved duct of small aspect ratio, and thus can give a firm framework for the three-dimensional study of curved duct flows.

6. Conclusions

In this paper, a comprehensive numerical study is presented for the unsteady solutions of the flow through a curved rectangular duct of aspect ratio 0.5 and curvature 0.1. Numerical calculations are carried out by using the spectral method, and covering a wide range of the Dean number, $1000 \leq Dn \leq 10000$, and the Grashof number, $100 \leq Gr \leq 2000$.

Time evolution calculations as well as their phase spaces show that the flow is steady-state up to a large value of Dn and this region increases as Gr becomes large. It is found that the unsteady flow is a steady-state solution for $Dn \leq 6400$ for all values of Gr in the range $100 \leq Gr \leq 2000$ and also for $Dn = 6500$ when $500 \leq Gr \leq 2000$, and for $Dn = 7000$ when $Gr = 2000$. However, the unsteady flow becomes periodic at $Dn = 7000$ when $Gr = 1500$ and at $Dn = 8000$ when $1500 \leq Gr \leq 2000$, but multi-periodic for $Dn = 6500$ at $Gr = 100$; for $Dn = 7000$ and $Dn = 8000$ when $100 \leq Gr \leq 1000$; for $Dn = 9000$ when $100 \leq Gr \leq 2000$ and at $Dn = 10000$ for $500 \leq Gr \leq 1500$. In the present study, it is found that the existence of the chaotic solution was not found for $Dn \leq 10000$ if $Gr < 2000$; however, at large values of Gr and Dn , for example $Dn = 10000$ and $Gr = 2000$, the chaotic solution was obtained. We obtained stream lines of secondary flows and isotherms of temperature profiles for all types of solutions, and it is found that there exist an asymmetric two-vortex solution for the steady-state solution, while asymmetric two-, three-, and four-vortex solutions for the periodic and multi-periodic solutions. For chaotic solution, on the other hand, we obtain asymmetric four-vortex solutions only. The present study also shows that chaotic flow enhances heat transfer more effectively than the steady-state or periodic solutions as the secondary flow becomes stronger.

7. Acknowledgement

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