Weighted Likelihood Estimator of Scale Parameter for the Two-parameter Weibull Distribution with a Contamination

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Abstract

In this article, we propose weighted likelihood estimator (WLE) that assigns zero weights to observations with small likelihood. We also examine the robustness properties of the maximum likelihood estimator (MLE) and WLE for the scale parameter of the two-parameter Weibull distribution when the data set has a contamination. We investigated the performance of the WLE as compared to the MLE and found that the WLE outperforms the MLE with respect to the relative bias and quadratic risk values.

Keywords: Weibull Distribution; Weighted Likelihood Estimator; Maximum Likelihood Estimator; Contamination; Outlier.

1. Introduction

The Weibull distribution is widely used in reliability and life data analysis due to its usefulness in many fields including engineering [1], biomedical sciences [2], and ecology [3]. This distribution is useful in describing wear-out or fatigue failures [4]. Other practical applications of the two-parameter Weibull distribution include wind energy assessment, estimation of rainfall amount, and analysis of lifetime of materials. Depending on the values of the parameters, the Weibull distribution can be used to model many of life behaviors. For the values of the shape parameter and the scale parameter affect the distribution characteristics such as the shape of the probability density function, the reliability and the failure rate. The shape parameter is significant for applocatuons of the Weibull distribution because it identifies the shape of the probability density function plot of the Weibull distribution. For the Weibull distribution with $\beta<1$, the probability decreases exponentially from infinity. In terms of the failure rate, data that fit this distribution have many initial failures which decrease over time as the incomplete items are removed from the sample. At $\beta=1$, the Weibull distribution reduces to the exponential distribution. For $\beta>1$, the probability density function is unimodal and is skewed to the right. For the Weibull distribution with $1<\beta<2$, this distribution rises to the top quickly, then falls over time. The failure rate increases overall, with the most rapid increase occurring initially. This shape is demonstrative of early were-out failures. When $\beta=2$, the Weibull distribution models is a linearly increasing failure rate, where the risk of were-out failure increasea steadily over the product of lifetimes. When

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$3<\beta<4$, the probability density function has a roughly bell shape simialr to that of the normal distribution. This form of the Weibull distribution models quick wearing out failures during the final period of product life,which is when most failures happen. Moreover, a change in the scale parameter is the same as a change of the abscissa scale, a change of the unit or data. If the scale parameter increases while the shape is fitted, the value of scale has an effect of spreading out the distribution to the right, and its height decreases while the value of scale has no effect on the shape of the distribution. If the scale parameter decreases while the shape is fitted, the values of scale has an effect of pushing in the distribution to the left and its height increases while the value of scale has no effect on the shape of the distribution. Additionally, the scale parameter has the same units as the variable *x*, such as hours, miles and cycles. An assumption of a shape parameter is appropriate for many real lifetime analysis problems. For example, in engineering applicarions, we often assume that the shape parameter is one and get an exponential distribution [5]. Therefore, we would like to study the two-parameter Weibull distribution assuming that the shape parameter is known. As we mentioned above, Weibull distribution plays a central role in lifetime models. If a dataset is contaminated with outliers, the maximum likelihood estimator (MLE) can be very unreliable [6]. The problem of estimating the parameters of the Weibull distribution when a proportion of the observations are outliers is quite important in reliability applications. The weighted likelihood estimator (WLE) was proposed for a robust estimation of the exponential distribution parameters by [7]. The WLE was introduced by [8] and it has been applied to a problem of robust estimation of parameters. The weighted likelihood method was introduced as a generalization of the local likelihood method, and it can be global, as demonstrated in [9]. The weighted likelihood method from [7] yields α–trimmed mean type estimators of the parameter of interest. We continue this investigation by applying this technique to the Weibull distribution in order to obtain a robust estimation of the scale parameter. We assume that the shape parameter is known and that a dataset shows a contamination. This method assigns zero weights to observations with small likelihood. We also examine the robustness properties of the MLE and WLE for the parameters of the Weibull distribution with contamination. In addition, the values of the shape parameter have a marked effect on the failure rate of the Weibull distribution, and inferences can be drawn about a population of failure characteristics just by considering whether the values of the shape parameter is less than, equal to, or greater than one. When $\beta < 1$, a failure rate that decreases with time is also known as infantile or early–life failures. The Weibull distributions with $\beta = 1$ have a constant failure rate, a property indicative of useful life or random failures. For $\beta > 1$, the Weibull distributions have failure rates that increase with time that is also known as wearing out failures. The simulation studies are extended to compare the MLE and WLE based on the relative bias and the quadratic risk values. In this study, we $\beta > 1$. The rest of this paper is organized as follows. In section 2 we define the WLE for the Weibull distribution. We investigate the robustness of this estimator in Section 3. In section 4, we compare the WLE and MLE in terms of relative bias and quadratic risk. The last section concludes the paper.

2. Proposed Weighted Likelihood Estimator

The distribution function for the two-parameter Weibull distribution is

$$F(x;\delta,\beta) = 1 - e^{-\left(x/\delta\right)^{\beta}} \qquad ; x \geq 0, \delta > 0, \beta > 0,$$

and the probability density function is

$$f(x;\delta,\beta) = \frac{\beta}{\delta^{\beta}} x^{\beta-1} e^{-(x/\delta)^{\beta}} \quad ; x \ge 0, \delta > 0, \beta > 0,$$

where δ is the scale parameter and β is the shape parameter of the distribution.

Let $x^{(n)} = \{x_1, x_2, \ldots, x_n\}$ be sample values from a distribution with a density function $f(x;\delta,\beta)$. The weighted likelihood estimators (WLE) of $\{\delta,\beta\}$ are obtained by maximizing the weighted likelihood function

$$L(\delta,\beta \mid x^{(n)}) = \sum_{i=1}^{n} w_i(x^{(n)}) \ln\left(f(x_i;\delta,\beta)\right),$$

where $w_i(x^{(n)}),\ 1 \le i \le n$ are the weights which depend on the sample. If all the weights are equal to one, then the resulting estimator is the maximum likelihood estimator (MLE).

Our goal is to estimate the scale parameter δ of the Weibull distribution. We assume that the shape parameter β is known. Following the idea presented in [7], we set the weight w_i that corresponds to the i^{th} observation to 1 if its estimated likelihood is sufficiently large and set it to 0 elsewhere. To be more precise, we let

$$w_i = \begin{cases} 1 & \text{if} \quad f(x_i;\hat{\delta},\beta) > C \\ 0 & \text{otherwise,} \end{cases}$$

where $\hat{\delta}$ is the MLE of the parameter δ. (Recall that β is assumed to be known.) This means that we delete all improbable observations from the sample and reject only extreme order statistics. Now, we need to choose *C*. Following the ideas of [7], we suggest that *C* not be considered as a constant. Rather, assume that *C* is chosen from the condition of a small probability of rejection of an observation when we sample from the non-contamination Weibull distribution with a cumulative distribution function, $F(x;\delta,\beta)$. Hence, we define *C* by the given pre-assigned small probability α as

$$P\left[f\left(\max_{1\le i\le n} X_i;\hat{\delta},\beta \right) < C \right] = \alpha.$$

So we get, $C \approx \dfrac{\alpha\beta}{n\hat{\delta}}$. Let the weighted likelihood estimator $\tilde{\delta}$ of the parameter δ be defined as the solution to the equation

$$\frac{\sum_{k=1}^{m} \partial f(x_{i_k};\delta,\beta)}{\partial\delta} = 0,$$

where $x_{i_1}, x_{i_2}, \ldots, x_{i_m}$ are the remaining observations in the sample after applying our procedure ($w_{i_k} = 1$). The WLE of δ is

$$\tilde{\delta} = \left(\frac{1}{m} \sum_{k=1}^{m} x_{i_k}^{\beta} \right)^{\frac{1}{\beta}}.$$

3. Robustness Properties of the WLE

Assume that the sample $(x_1, x_2, \ldots, x_n)$ is taken from a population that follows a distribution with the distribution function $G_\varepsilon(x)$ to be defined now. We define the ε–contamination model as

$$G_\varepsilon(x) = (1-\varepsilon)F(x;\delta,\beta) + \varepsilon F_1(x;\delta_1,\beta_1)$$

where $F(x;\delta,\beta)$ is the Weibull distribution with parameters (δ,β), a contamination $F_1(x;\delta_1,\beta_1)$ is the Weibull distribution with parameters (δ_1,β_1), where

$\delta_1 = \delta(1+\Delta), \beta_1 = \beta(1+\Delta), \Delta, \Delta > 0$;

ε denotes the contamination proportion, and $0 \le \varepsilon \le 1$. Under the ε–contamination model we assume that β is known. Let $\hat{\delta} = \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{\beta} \right)^{\frac{1}{\beta}}$ be the MLE of the scale parameter δ. We assume that

$$\delta_\varepsilon = \delta\left((1-\varepsilon)\Gamma\left(1+\frac{1}{\beta}\right) + \varepsilon(1+\Delta)\Gamma\left(1+\frac{1}{\beta(1+\Delta)}\right) \right).$$

By the strong law of large numbers, $\lim_{n\to\infty} \hat{\delta} = \delta_\varepsilon$. The estimate $\tilde{\delta}$ converges in probability to some value $\overline{\delta_\varepsilon}$, which can be calculated as a limit of the expected values truncated at the point $A = -\delta_\varepsilon \left(\ln \frac{\alpha}{n} \right)^{\frac{1}{\beta}}$ of the distribution $\overline{G_\varepsilon(x)} = \frac{G(x)}{G(A)}$ where $0 \le x \le A$. So we get

$$\overline{\delta_\varepsilon} = \frac{\delta}{G(A)}\left((1-\varepsilon)IG_1 + \varepsilon(1+\Delta)IG_2\right)$$

where

$$IG_1 = \Gamma\left(\frac{1}{\beta}+1\right)P\left(\frac{1}{\beta}+1, A_u\right),$$

$$IG_2 = \Gamma\left(\frac{1}{\beta_1}+1\right)P\left(\frac{1}{\beta_1}+1, A_v\right),$$

$$G(A) = 1 - e^{-\left(\frac{A}{\delta}\right)^{\beta}} - \varepsilon\left(e^{-\left(\frac{A}{\delta_1}\right)^{\beta_1}} - e^{-\left(\frac{A}{\delta}\right)^{\beta}} \right),$$

$$A_u = \frac{\delta_\varepsilon^{\beta} \ln \frac{n}{\alpha}}{\delta^{\beta}}, \quad A_v = \frac{\delta_\varepsilon^{\beta_1} \ln \frac{n}{\alpha}}{\delta_1^{\beta_1}},$$

$$P\left(\frac{1}{\beta}+1, A_u\right) = \frac{\gamma\left(\frac{1}{\beta}+1, A_u\right)}{\Gamma\left(\frac{1}{\beta}+1\right)}, \text{ and}$$

$$P\left(\frac{1}{\beta_1}+1, A_v\right) = \frac{\gamma\left(\frac{1}{\beta_1}+1, A_v\right)}{\Gamma\left(\frac{1}{\beta_1}+1\right)}.$$

Therefore, an exact calculation of the gain in the bias and the reduction of the risk of the proposed estimator in comparison with the MLE is impossible. Indeed, these integrals cannot be evaluated in the closed form. Furthermore, the solution is cumbersome. Thus, we cannot easily compare the relative bias of $\tilde{\delta}$ with the relative bias of $\hat{\delta}$. A relevant conclusion may not be possible concerning the gain in bias using these precise formulas even if we expand $\overline{\delta_\varepsilon}$ in powers of ε. We shall confine ourselves to asymptotic analysis. Hence, the probability of rejecting an observation in the contamination model is asymptotically equal to

$$P[X_1 > A] = 1 - P[X_1 \le A] = 1 - G(A) \approx \frac{\alpha}{n}.$$

The asymptotic distribution of $\tilde{\delta}$ is equal to the distribution of the $\alpha-$generalized trimmed sample mean where we defined α–generalized trimmed sample (10)
mean as

$$Y = \left(\frac{1}{n(1-\alpha)} \sum_{k=1}^{n(1-\alpha)} Y_k^{\beta} \right)^{\frac{1}{\beta}}$$

of the random sample of size $n(1-\alpha)$ from the distribution concentrated on the interval $(0, A)$. The probability density of this distribution is positive only in the interval $(0, A)$ and has the form

$$f_A(x;\delta,\beta) = \left(\overline{G_\varepsilon(x)}\right)' = \frac{G(x)'}{G(A)} \quad ;0 < x \le A.$$

In classical robustness studies under the assumption that the sample is taken from the distribution $G_\varepsilon(x)$ with fixed ε, one would find the limits in the probability of the estimates according to the WLE and MLE for all sample close to 1, and compare their biases. We will also compare the quadratic risks of these estimates. The quadratic risk of an estimator is the expected squared distance between the estimator and the parameter being the quadratic risk is an omnibus measure of the performance of an estimator. The quadratic risk of an estimator takes into consideration the bias and the precision of the estimator.

Result 1

The Weibull distribution contamination has a relative bias of the maximum likelihood estimator $(\hat{\delta})$ that can be described as

$$(1-\varepsilon)\Gamma\left(1+\frac{1}{\beta}\right) + \varepsilon(1+\Delta)\Gamma\left(1+\frac{1}{\beta_1}\right) - 1.$$

Also, the Weibull distribution contamination has a relative bias of the weighted likelihood estimator $(\tilde{\delta})$ that can be described as

$$\frac{1}{G_\alpha}\big((1-\varepsilon)IG_1+\varepsilon(1+\Delta)IG_2\big)-1$$

where

$$IG_1=\Gamma\left(\frac{1}{\beta}+1\right)P\left(\frac{1}{\beta}+1,A_u\right),$$

$$IG_2=\Gamma\left(\frac{1}{\beta_1}+1\right)P\left(\frac{1}{\beta_1}+1,A_v\right),\ G_\alpha\approx 1-\frac{\alpha}{n},$$

$$A_u=\frac{\delta_\varepsilon^\beta\ln\frac{n}{\alpha}}{\delta^\beta},\ A_v=\frac{\delta_\varepsilon^{\beta_1}\ln\frac{n}{\alpha}}{\delta_1^{\beta_1}},$$

$$P\left(\frac{1}{\beta}+1,A_u\right)=\frac{\gamma\left(\frac{1}{\beta}+1,A_u\right)}{\Gamma\left(\frac{1}{\beta}+1\right)},\ \text{and}$$

$$P\left(\frac{1}{\beta_1}+1,A_v\right)=\frac{\gamma\left(\frac{1}{\beta_1}+1,A_v\right)}{\Gamma\left(\frac{1}{\beta_1}+1\right)}.$$

Result 2

The Weibull distribution contamination has a quadratic risk of the maximum likelihood estimator $(\hat{\delta})$ that can be described as

$$\frac{\delta^2}{n}\left(E_2-E_1^2+(1-E_1)^2\right).$$

Also, the contamination is the Weibull distribution has a quadratic risk of the weighted likelihood estimator $(\tilde{\delta})$ that can be described as

$$\frac{\delta^2((1-\varepsilon)IG_3+\varepsilon(1+\Delta)^2IG_4)}{n(1-\alpha)G_\alpha}+\frac{\delta^2}{n(1-\alpha)}-\frac{2\delta^2((1-\varepsilon)IG_1+\varepsilon(1+\Delta)IG_2)}{n(1-\alpha)G_\alpha}$$

where

$$E_1=(1-\varepsilon)\Gamma\left(1+\frac{1}{\beta}\right)+\varepsilon(1+\Delta)\Gamma\left(1+\frac{1}{\beta(1+\Delta)}\right),$$

$$IG_1=\Gamma\left(\frac{1}{\beta}+1\right)P\left(\frac{1}{\beta}+1,A_u\right),$$

$$E_2=(1-\varepsilon)\Gamma\left(1+\frac{2}{\beta}\right)+\varepsilon(1+\Delta)^2\Gamma\left(1+\frac{2}{\beta(1+\Delta)}\right),$$

$$IG_2=\Gamma\left(\frac{1}{\beta_1}+1\right)P\left(\frac{1}{\beta_1}+1,A_v\right)$$

$$G_\alpha\approx 1-\frac{\alpha}{n},\ A_u=\frac{\delta_\varepsilon^\beta\ln\frac{n}{\alpha}}{\delta^\beta},\ A_v=\frac{\delta_\varepsilon^{\beta_1}\ln\frac{n}{\alpha}}{\delta_1^{\beta_1}},$$

$$IG_3=\Gamma\left(\frac{2}{\beta}+1\right)P\left(\frac{2}{\beta}+1,A_u\right),$$

$$IG_4=\Gamma\left(\frac{2}{\beta_1}+1\right)P\left(\frac{2}{\beta_1}+1,A_v\right),$$

$$P\left(\frac{1}{\beta}+1,A_u\right)=\frac{\gamma\left(\frac{1}{\beta}+1,A_u\right)}{\Gamma\left(\frac{1}{\beta}+1\right)},$$

$$P\left(\frac{1}{\beta_1}+1,A_v\right)=\frac{\gamma\left(\frac{1}{\beta_1}+1,A_v\right)}{\Gamma\left(\frac{1}{\beta_1}+1\right)},$$

$$P\left(\frac{2}{\beta}+1,A_u\right)=\frac{\gamma\left(\frac{2}{\beta}+1,A_u\right)}{\Gamma\left(\frac{2}{\beta}+1\right)},\ \text{and}$$

$$P\left(\frac{2}{\beta_1}+1,A_v\right)=\frac{\gamma\left(\frac{2}{\beta_1}+1,A_v\right)}{\Gamma\left(\frac{2}{\beta_1}+1\right)}.$$

4. Results and Discussion

We generated data sets of size n= 25, 50, 100 from the ε–contamination model. The central distribution is the Weibull distribution with the scale parameter $\delta=1$ and the shape parameter β = 2.5, 5, 10. The contamination has the Weibull distribution with the scale parameter $\delta_1 = \delta(1+\Delta)$ and the shape parameter $\beta_1 = \beta(1+\Delta)$ where Δ = 1, 3, 5, the contamination proportion ε = 0.01 0.05, 0.10 and the values of α =0.01, 0.03, 0.05, 0.07, 0.10. All of the following simulation results are based on 10,000 replicates by using programs written in the R statistical programming language [10]. A simulation study was carried out to analyze the performance of the MLE and WLE for the Weibull distribution contamination based on the simulated and asymptotic relative bias and the quadratic risk of the estimation for the scale parameter δ. The simulated and the asymptotic relative bias and the quadratic risk of the MLE do not depend on α for any of the sample sizes. The WLE seems to yield the best performance in the term of the relative bias and the quadratic risk for the estimation of the δ for all cases. Additionally, when Δ gets large, the relative bias and the quadratic risk of the WLE is smaller than those of the MLE when α increases.. In most of the cases, the simulated and the asymptotic relative biases and the quadratic risks of the MLE and the WLE decrease as the sample sizes increase. The simulated relative bias of the MLE is greater than that of the WLE when Δ increases. The simulated and the asymptotic relative bias of the MLE are greater than those of the WLE for all values of α. On the other hand, the simulated and the asymptotic relative bias of the MLE and WLE are smallest at $\Delta = 1$, and Δ does not affect the changes in the shape parameter. The simulated and the asymptotic relative bias and the quadratic risk of the MLE increase as β increases. While the asymptotic relative bias of the WLE decreases as β increases. The simulated quadratic risk of the MLE is close to that of the WLE when $\Delta = 1$ and α is small. At $\Delta = 3$ and 5, the simulated quadratic risk of the MLEis greater than that of the WLE for all values of α. Additionally, the asymptotic quadratic risk of the MLE is also greater than that of the WLE for small α. Hence, the WLE outperforms the MLE in terms of the relative bias and the quadratic risk for the estimation of δ. The results are summarized numerically in Tables 1 – 3.

The two-parameter Weibull distribution function has been used widely in many fields such as wind energy assessment, estimation of rainfall amount, and analysis of lifetime of materials. We also apply the MLE and the WLE to the data set on the system data for the scenario lifetime time (years) data. This data is taken from [11]. The lifetimes are 30.20, 36.55, 25.11, 39.35, 27.57, 25.91, 31.50, 29.24, 18.39, 16.65, 21.85, 24.88, 31.61, 18.74, 19.63, 28.98, 11.10, 21.66, 26.04, 25.07, 23.48, 28.21, 25.21, 25.12, 27.76, 23.47, 23.51, 24.39, 21.93, 37.63, 20.32, 28.17, 24.66, 30.13, 21.42, 17.21, 19.98, 33.09, 16.04, 17.96, 19.57, 22.91, 25.69, 23.47, 16.91, 27.20, and 27.23. We use the MLE method to fit this data set. We fit the data set for the Weibull distribution using the function fitdist in the *R* package. The MLE of the scale and the shape parameter are 27.007 and 4.579 with the standard error of 0.911 and 0.493, respectively. Then we create 5% of the contamination in the data set with the Weibull distribution using the shape parameter 27.007(1+3) and the shape parameter 4.579(1+3). We change the last two of the observations 27.20, 27.23 to 100.2, 107.6, respectively. Then, we apply the MLE and WLE to the new data set, assuming that the shape parameter is 4.579. The MLE of the scale parameter is 52.782 with the standard error of 4.930, but the WLE of the scale parameter is 26.741 with the standard error of 0.774. The results from the WLE are close to those from the MLE obtained from the data set without any outlier. Therefore, the WLE outperforms the MLE when the contamination is present in this data set.

Table1. The simulated(sim) and the asymptotic(asy) relative bias and quadratic risk of the MLE and WLE for $W(1,2.5)+\varepsilon W(1(1+\Delta),2.5(1+\Delta))$.

n	Δ	α	relative bias						quadratic risk					
			ε =0.01		ε =0.05		ε =0.10		ε =0.01		ε =0.05		ε =0.10	
			sim	asy	sim	asy	sim	asy	sim	asy	sim	asy	sim	asy
25	1	MLE	0.014	-0.103	0.014	-0.065	0.015	-0.018	0.008	0.007	0.007	0.008	0.007	0.009
		0.01	0.013	-0.110	0.012	-0.089	0.013	-0.057	0.008	0.007	0.007	0.007	0.008	0.007
		0.03	0.010	-0.116	0.009	-0.103	0.010	-0.081	0.008	0.007	0.007	0.007	0.008	0.007
		0.05	0.008	-0.120	0.007	-0.111	0.008	-0.094	0.008	0.008	0.007	0.008	0.008	0.008
		0.07	0.006	-0.123	0.005	-0.117	0.006	-0.104	0.008	0.008	0.007	0.008	0.008	0.008
		0.09	0.004	-0.126	0.003	-0.122	0.005	-0.112	0.008	0.009	0.007	0.009	0.008	0.009
	3	MLE	0.085	-0.084	0.086	0.033	0.083	0.179	0.041	0.009	0.041	0.022	0.040	0.038
		0.01	0.003	-0.123	0.005	-0.157	0.005	-0.200	0.009	0.007	0.008	0.007	0.008	0.007
		0.03	0.000	-0.126	0.001	-0.159	0.001	-0.201	0.008	0.007	0.007	0.007	0.007	0.007
		0.05	-0.002	-0.129	-0.001	-0.160	0.000	-0.202	0.007	0.008	0.007	0.008	0.007	0.008
		0.07	-0.003	-0.131	-0.002	-0.162	-0.002	-0.203	0.007	0.009	0.007	0.009	0.007	0.009
		0.09	-0.004	-0.133	-0.003	-0.163	-0.003	-0.204	0.007	0.010	0.007	0.010	0.007	0.010
	5	MLE	0.184	-0.064	0.185	0.133	0.187	0.378	0.168	0.015	0.168	0.052	0.169	0.098
		0.01	0.003	-0.123	0.001	-0.157	0.003	-0.202	0.010	0.007	0.011	0.007	0.009	0.007
		0.03	-0.001	-0.126	-0.002	-0.158	0.000	-0.202	0.007	0.007	0.008	0.007	0.007	0.008
		0.05	-0.002	-0.128	-0.003	-0.159	-0.001	-0.202	0.007	0.008	0.008	0.008	0.007	0.008
		0.07	-0.003	-0.130	-0.004	-0.160	-0.002	-0.202	0.007	0.009	0.007	0.009	0.007	0.009
		0.09	-0.004	-0.132	-0.005	-0.161	-0.004	-0.202	0.007	0.010	0.007	0.010	0.007	0.010
50	1	MLE	0.018	-0.103	0.018	-0.065	0.017	-0.018	0.004	0.003	0.004	0.004	0.004	0.005
		0.01	0.016	-0.108	0.016	-0.083	0.015	-0.046	0.004	0.003	0.004	0.003	0.004	0.003
		0.03	0.014	-0.112	0.013	-0.094	0.013	-0.065	0.004	0.004	0.004	0.004	0.004	0.004
		0.05	0.012	-0.115	0.011	-0.100	0.011	-0.076	0.004	0.004	0.004	0.004	0.004	0.004
		0.07	0.011	-0.117	0.010	-0.105	0.010	-0.085	0.004	0.004	0.004	0.004	0.004	0.004
		0.09	0.010	-0.119	0.009	-0.109	0.009	-0.091	0.004	0.004	0.004	0.004	0.004	0.004
	3	MLE	0.097	-0.084	0.096	0.033	0.094	0.179	0.029	0.005	0.028	0.011	0.028	0.019
		0.01	0.005	-0.122	0.004	-0.157	0.003	-0.199	0.004	0.003	0.004	0.003	0.003	0.003
		0.03	0.003	-0.124	0.002	-0.158	0.002	-0.201	0.003	0.004	0.003	0.004	0.003	0.004
		0.05	0.002	-0.125	0.001	-0.159	0.001	-0.201	0.003	0.004	0.003	0.004	0.003	0.004
		0.07	0.001	-0.127	0.001	-0.159	0.000	-0.202	0.003	0.005	0.003	0.005	0.003	0.005
		0.09	0.001	-0.128	0.000	-0.160	0.000	-0.202	0.003	0.005	0.003	0.005	0.003	0.005
	5	MLE	0.215	-0.064	0.219	0.133	0.213	0.378	0.131	0.008	0.133	0.026	0.129	0.049
		0.01	0.001	-0.122	0.002	-0.157	0.003	-0.201	0.003	0.003	0.003	0.003	0.004	0.003
		0.03	0.001	-0.124	0.001	-0.158	0.002	-0.202	0.003	0.004	0.003	0.004	0.003	0.004
		0.05	0.000	-0.125	0.000	-0.158	0.001	-0.202	0.003	0.004	0.003	0.004	0.003	0.004
		0.07	0.000	-0.126	0.000	-0.158	0.001	-0.202	0.003	0.005	0.003	0.005	0.003	0.005
		0.09	-0.001	-0.127	-0.001	-0.159	0.000	-0.202	0.003	0.005	0.003	0.005	0.003	0.005
100	1	MLE	0.018	-0.103	0.018	-0.065	0.019	-0.018	0.002	0.002	0.002	0.002	0.002	0.002
		0.01	0.017	-0.107	0.016	-0.078	0.018	-0.038	0.002	0.002	0.002	0.002	0.002	0.002
		0.03	0.015	-0.109	0.014	-0.087	0.016	-0.053	0.002	0.002	0.002	0.002	0.002	0.002
		0.05	0.014	-0.111	0.013	-0.092	0.015	-0.062	0.002	0.002	0.002	0.002	0.002	0.002
		0.07	0.013	-0.113	0.012	-0.096	0.014	-0.068	0.002	0.002	0.002	0.002	0.002	0.002
		0.09	0.012	-0.114	0.011	-0.099	0.013	-0.074	0.002	0.002	0.002	0.002	0.002	0.002
	3	MLE	0.099	-0.084	0.100	0.033	0.101	0.179	0.020	0.002	0.020	0.006	0.021	0.009
		0.01	0.004	-0.122	0.005	-0.156	0.004	-0.198	0.002	0.002	0.002	0.002	0.002	0.002
		0.03	0.003	-0.123	0.004	-0.157	0.003	-0.200	0.002	0.002	0.002	0.002	0.002	0.002
		0.05	0.002	-0.124	0.004	-0.158	0.003	-0.200	0.002	0.002	0.002	0.002	0.002	0.002
		0.07	0.002	-0.124	0.003	-0.158	0.002	-0.201	0.002	0.002	0.002	0.002	0.002	0.002
		0.09	0.002	-0.125	0.003	-0.158	0.002	-0.201	0.002	0.003	0.002	0.003	0.002	0.003
	5	MLE	0.238	-0.064	0.240	0.133	0.240	0.378	0.105	0.004	0.107	0.013	0.107	0.025
		0.01	0.003	-0.122	0.002	-0.157	0.002	-0.201	0.002	0.002	0.002	0.002	0.002	0.002
		0.03	0.003	-0.123	0.002	-0.157	0.002	-0.201	0.002	0.002	0.002	0.002	0.002	0.002
		0.05	0.003	-0.123	0.002	-0.158	0.002	-0.202	0.002	0.002	0.002	0.002	0.002	0.002
		0.07	0.002	-0.124	0.002	-0.158	0.001	-0.202	0.002	0.002	0.002	0.002	0.002	0.002
		0.09	0.002	-0.125	0.002	-0.158	0.001	-0.202	0.002	0.003	0.002	0.003	0.002	0.003

Table2. The simulated(sim) and the asymptotic(asy) relative bias and quadratic risk of the MLE and WLE for $W(1,5)+\varepsilon W(1(1+\Delta),5(1+\Delta))$.

n	Δ	α	relative bias						quadratic risk					
			ε =0.01		ε =0.05		ε =0.10		ε =0.01		ε =0.05		ε =0.10	
			sim	asy	sim	asy	sim	asy	sim	asy	sim	asy	sim	asy
25	1	MLE	0.035	-0.072	0.035	-0.033	0.036	0.017	0.008	0.002	0.009	0.004	0.009	0.005
		0.01	0.002	-0.091	0.002	-0.125	0.002	-0.166	0.002	0.002	0.002	0.002	0.002	0.002
		0.03	0.001	-0.093	0.000	-0.127	0.001	-0.169	0.002	0.003	0.002	0.003	0.002	0.003
		0.05	0.000	-0.095	-0.001	-0.129	-0.001	-0.171	0.002	0.004	0.002	0.004	0.002	0.004
		0.07	-0.001	-0.097	-0.002	-0.130	-0.001	-0.173	0.002	0.005	0.002	0.005	0.002	0.005
		0.09	-0.002	-0.098	-0.002	-0.132	-0.002	-0.174	0.002	0.006	0.002	0.006	0.002	0.006
	3	MLE	0.247	-0.052	0.247	0.067	0.240	0.216	0.279	0.005	0.278	0.019	0.269	0.036
		0.01	0.002	-0.092	0.003	-0.128	0.002	-0.174	0.005	0.002	0.006	0.002	0.004	0.002
		0.03	0.000	-0.094	0.001	-0.129	0.000	-0.174	0.004	0.003	0.004	0.003	0.003	0.003
		0.05	-0.001	-0.095	0.000	-0.130	-0.001	-0.174	0.003	0.004	0.003	0.004	0.002	0.004
		0.07	-0.002	-0.096	-0.001	-0.130	-0.002	-0.175	0.003	0.005	0.003	0.005	0.002	0.005
		0.09	-0.002	-0.098	-0.002	-0.131	-0.002	-0.175	0.003	0.006	0.002	0.006	0.002	0.006
	5	MLE	0.479	-0.032	0.482	0.167	0.482	0.415	1.044	0.012	1.048	0.050	1.049	0.098
		0.01	0.004	-0.092	0.003	-0.128	0.003	-0.174	0.012	0.002	0.014	0.002	0.010	0.002
		0.03	0.000	-0.093	0.000	-0.128	0.000	-0.174	0.005	0.003	0.007	0.003	0.005	0.003
		0.05	-0.001	-0.095	-0.001	-0.129	-0.001	-0.174	0.004	0.004	0.005	0.004	0.003	0.004
		0.07	-0.002	-0.096	-0.003	-0.129	-0.002	-0.174	0.003	0.005	0.003	0.005	0.003	0.005
		0.09	-0.003	-0.097	-0.003	-0.130	-0.003	-0.174	0.003	0.006	0.003	0.006	0.002	0.006
50	1	MLE	0.042	-0.072	0.042	-0.033	0.042	0.017	0.006	0.001	0.006	0.002	0.006	0.003
		0.01	0.003	-0.091	0.003	-0.124	0.003	-0.164	0.001	0.001	0.001	0.001	0.001	0.001
		0.03	0.002	-0.092	0.002	-0.126	0.002	-0.167	0.001	0.002	0.001	0.002	0.001	0.002
		0.05	0.001	-0.093	0.002	-0.127	0.001	-0.169	0.001	0.002	0.001	0.002	0.001	0.002
		0.07	0.001	-0.094	0.001	-0.128	0.001	-0.170	0.001	0.003	0.001	0.002	0.001	0.002
		0.09	0.000	-0.095	0.001	-0.129	0.001	-0.171	0.001	0.003	0.001	0.003	0.001	0.003
	3	MLE	0.341	-0.052	0.345	0.067	0.336	0.216	0.305	0.003	0.309	0.009	0.299	0.018
		0.01	0.002	-0.091	0.001	-0.128	0.001	-0.174	0.002	0.001	0.001	0.001	0.001	0.001
		0.03	0.001	-0.092	0.001	-0.128	0.000	-0.174	0.001	0.002	0.001	0.002	0.001	0.002
		0.05	0.000	-0.093	0.000	-0.129	0.000	-0.174	0.001	0.002	0.001	0.002	0.001	0.002
		0.07	0.000	-0.094	0.000	-0.129	0.000	-0.174	0.001	0.003	0.001	0.003	0.001	0.003
		0.09	0.000	-0.095	0.000	-0.129	-0.001	-0.174	0.001	0.003	0.001	0.003	0.001	0.003
	5	MLE	0.697	-0.032	0.717	0.167	0.720	0.415	1.273	0.006	1.311	0.025	1.317	0.049
		0.01	0.000	-0.091	0.001	-0.128	0.001	-0.174	0.001	0.001	0.001	0.001	0.002	0.001
		0.03	0.000	-0.092	0.000	-0.128	0.001	-0.174	0.001	0.002	0.001	0.002	0.002	0.002
		0.05	0.000	-0.093	0.000	-0.128	0.000	-0.174	0.001	0.002	0.001	0.002	0.001	0.002
		0.07	0.000	-0.094	0.000	-0.128	0.000	-0.174	0.001	0.003	0.001	0.003	0.001	0.003
		0.09	-0.001	-0.094	-0.001	-0.129	-0.001	-0.174	0.001	0.003	0.001	0.003	0.001	0.003
100	1	MLE	0.047	-0.072	0.046	-0.033	0.047	0.017	0.005	0.001	0.005	0.001	0.005	0.001
		0.01	0.004	-0.090	0.004	-0.123	0.004	-0.162	0.000	0.001	0.000	0.001	0.000	0.001
		0.03	0.003	-0.091	0.003	-0.124	0.003	-0.165	0.000	0.001	0.000	0.001	0.000	0.001
		0.05	0.002	-0.092	0.002	-0.125	0.002	-0.167	0.000	0.001	0.000	0.001	0.000	0.001
		0.07	0.002	-0.092	0.002	-0.126	0.002	-0.167	0.000	0.001	0.000	0.001	0.000	0.001
		0.09	0.002	-0.093	0.002	-0.126	0.002	-0.168	0.000	0.001	0.000	0.001	0.000	0.001
	3	MLE	0.442	-0.052	0.440	0.067	0.448	0.216	0.327	0.001	0.325	0.005	0.330	0.009
		0.01	0.002	-0.091	0.002	-0.128	0.001	-0.174	0.001	0.001	0.001	0.001	0.001	0.001
		0.03	0.001	-0.092	0.001	-0.128	0.001	-0.174	0.000	0.001	0.000	0.001	0.000	0.001
		0.05	0.001	-0.092	0.001	-0.128	0.001	-0.174	0.000	0.001	0.000	0.001	0.000	0.001
		0.07	0.001	-0.093	0.001	-0.128	0.001	-0.174	0.000	0.001	0.000	0.001	0.000	0.001
		0.09	0.001	-0.093	0.001	-0.129	0.001	-0.174	0.000	0.002	0.000	0.002	0.000	0.001
	5	MLE	0.972	-0.032	0.997	0.167	0.973	0.415	1.529	0.003	1.566	0.012	1.529	0.024
		0.01	0.002	-0.091	0.001	-0.128	0.001	-0.174	0.001	0.001	0.000	0.001	0.001	0.001
		0.03	0.002	-0.092	0.001	-0.128	0.001	-0.174	0.001	0.001	0.000	0.001	0.001	0.001
		0.05	0.001	-0.092	0.001	-0.128	0.001	-0.174	0.000	0.001	0.000	0.001	0.000	0.001
		0.07	0.001	-0.092	0.001	-0.128	0.001	-0.174	0.000	0.001	0.000	0.001	0.000	0.001
		0.09	0.001	-0.093	0.001	-0.128	0.001	-0.174	0.000	0.002	0.000	0.002	0.000	0.001

Table3. The simulated(sim) and the asymptotic(asy) relative bias and quadratic risk of the MLE and WLE for $W(1,10)+\varepsilon W(1(1+\Delta),10(1+\Delta))$.

n	Δ	α	relative bias						quadratic risk					
			ε =0.01		ε =0.05		ε =0.10		ε =0.01		ε =0.05		ε =0.10	
			sim	asy	sim	asy	sim	asy	sim	asy	sim	asy	sim	asy
25	1	MLE	0.093	-0.039	0.097	0.001	0.093	-0.039	0.043	0.001	0.044	0.002	0.043	0.001
		0.01	0.003	-0.059	0.003	-0.097	0.003	-0.059	0.002	0.001	0.002	0.001	0.002	0.001
		0.03	0.001	-0.060	0.002	-0.098	0.001	-0.060	0.001	0.002	0.001	0.002	0.001	0.002
		0.05	0.001	-0.061	0.001	-0.099	0.001	-0.061	0.001	0.003	0.001	0.003	0.001	0.003
		0.07	0.000	-0.062	0.000	-0.099	0.000	-0.062	0.001	0.004	0.001	0.004	0.001	0.004
		0.09	0.000	-0.063	0.000	-0.100	0.000	-0.063	0.001	0.005	0.001	0.005	0.001	0.005
	3	MLE	0.413	-0.019	0.427	0.101	0.413	-0.019	0.782	0.004	0.810	0.018	0.782	0.004
		0.01	0.006	-0.059	0.006	-0.096	0.006	-0.059	0.012	0.001	0.013	0.001	0.012	0.001
		0.03	0.002	-0.060	0.003	-0.097	0.002	-0.060	0.006	0.002	0.007	0.002	0.006	0.002
		0.05	0.001	-0.061	0.001	-0.097	0.001	-0.061	0.003	0.003	0.004	0.003	0.003	0.003
		0.07	0.000	-0.062	0.000	-0.098	0.000	-0.062	0.003	0.004	0.003	0.004	0.003	0.004
		0.09	-0.001	-0.063	0.000	-0.099	-0.001	-0.063	0.002	0.005	0.003	0.005	0.002	0.005
	5	MLE	0.735	0.001	0.729	0.201	0.735	0.001	2.467	0.010	2.447	0.050	2.467	0.010
		0.01	0.005	-0.059	0.004	-0.096	0.005	-0.059	0.020	0.001	0.018	0.001	0.020	0.001
		0.03	0.001	-0.060	0.001	-0.097	0.001	-0.060	0.008	0.002	0.006	0.002	0.008	0.002
		0.05	0.000	-0.061	0.000	-0.097	0.000	-0.061	0.004	0.003	0.004	0.003	0.004	0.003
		0.07	-0.001	-0.061	-0.001	-0.097	-0.001	-0.061	0.002	0.004	0.003	0.004	0.002	0.004
		0.09	-0.001	-0.062	-0.001	-0.098	-0.001	-0.062	0.002	0.005	0.002	0.005	0.002	0.005
50	1	MLE	0.140	-0.039	0.138	0.001	0.140	-0.039	0.052	0.000	0.051	0.001	0.052	0.000
		0.01	0.003	-0.059	0.003	-0.096	0.003	-0.059	0.001	0.001	0.001	0.001	0.001	0.001
		0.03	0.002	-0.059	0.002	-0.097	0.002	-0.059	0.001	0.001	0.001	0.001	0.001	0.001
		0.05	0.001	-0.060	0.002	-0.097	0.001	-0.060	0.000	0.001	0.000	0.001	0.000	0.001
		0.07	0.001	-0.060	0.001	-0.098	0.001	-0.060	0.000	0.002	0.000	0.002	0.000	0.002
		0.09	0.001	-0.061	0.001	-0.098	0.001	-0.061	0.000	0.002	0.000	0.002	0.000	0.002
	3	MLE	0.668	-0.019	0.680	0.101	0.668	-0.019	1.152	0.002	1.171	0.009	1.152	0.002
		0.01	0.003	-0.058	0.002	-0.096	0.003	-0.058	0.004	0.001	0.004	0.001	0.004	0.001
		0.03	0.001	-0.059	0.001	-0.097	0.001	-0.059	0.002	0.001	0.002	0.001	0.002	0.001
		0.05	0.000	-0.060	0.001	-0.097	0.000	-0.060	0.001	0.001	0.001	0.001	0.001	0.001
		0.07	0.000	-0.060	0.000	-0.097	0.000	-0.060	0.001	0.002	0.001	0.002	0.001	0.002
		0.09	0.000	-0.061	0.000	-0.097	0.000	-0.061	0.001	0.002	0.001	0.002	0.001	0.002
	5	MLE	1.226	0.001	1.230	0.201	1.226	0.001	3.801	0.005	3.817	0.025	3.801	0.005
		0.01	0.003	-0.058	0.001	-0.096	0.003	-0.058	0.007	0.001	0.004	0.001	0.007	0.001
		0.03	0.001	-0.059	0.000	-0.096	0.001	-0.059	0.004	0.001	0.000	0.001	0.004	0.001
		0.05	0.000	-0.059	0.000	-0.097	0.000	-0.059	0.000	0.001	0.000	0.001	0.000	0.001
		0.07	0.000	-0.060	0.000	-0.097	0.000	-0.060	0.000	0.002	0.000	0.002	0.000	0.002
		0.09	0.000	-0.060	0.000	-0.097	0.000	-0.060	0.000	0.002	0.000	0.002	0.000	0.002
100	1	MLE	0.185	-0.039	0.183	0.001	0.185	-0.039	0.058	0.000	0.057	0.001	0.058	0.000
		0.01	0.002	-0.058	0.002	-0.096	0.002	-0.058	0.000	0.000	0.000	0.000	0.000	0.000
		0.03	0.002	-0.059	0.002	-0.097	0.002	-0.059	0.000	0.000	0.000	0.000	0.000	0.000
		0.05	0.002	-0.059	0.001	-0.097	0.002	-0.059	0.000	0.001	0.000	0.001	0.000	0.001
		0.07	0.001	-0.059	0.001	-0.097	0.001	-0.059	0.000	0.001	0.000	0.001	0.000	0.001
		0.09	0.001	-0.060	0.001	-0.097	0.001	-0.060	0.000	0.001	0.000	0.001	0.000	0.001
	3	MLE	1.002	-0.019	1.005	0.101	1.002	-0.019	1.596	0.001	1.604	0.004	1.596	0.001
		0.01	0.001	-0.058	0.002	-0.096	0.001	-0.058	0.001	0.000	0.002	0.000	0.001	0.000
		0.03	0.001	-0.059	0.002	-0.096	0.001	-0.059	0.001	0.000	0.002	0.000	0.001	0.000
		0.05	0.001	-0.059	0.002	-0.096	0.001	-0.059	0.000	0.001	0.001	0.001	0.000	0.001
		0.07	0.001	-0.059	0.002	-0.097	0.001	-0.059	0.000	0.001	0.001	0.001	0.000	0.001
		0.09	0.001	-0.059	0.001	-0.097	0.001	-0.059	0.000	0.001	0.001	0.001	0.000	0.001
	5	MLE	1.828	0.001	1.842	0.201	1.828	0.001	5.305	0.003	5.343	0.012	5.305	0.003
		0.01	0.001	-0.058	0.001	-0.096	0.001	-0.058	0.002	0.000	0.002	0.000	0.002	0.000
		0.03	0.001	-0.059	0.001	-0.096	0.001	-0.059	0.001	0.000	0.001	0.000	0.001	0.000
		0.05	0.001	-0.059	0.001	-0.096	0.001	-0.059	0.001	0.001	0.001	0.001	0.001	0.001
		0.07	0.001	-0.059	0.001	-0.096	0.001	-0.059	0.001	0.001	0.001	0.001	0.001	0.001
		0.09	0.000	-0.059	0.001	-0.096	0.000	-0.059	0.000	0.001	0.000	0.001	0.000	0.001

5. Conclusion

The Weibull distribution plays a central role in lifetime models. When data are contaminated with outliers, the MLE can be very unreliable. In this study, the WLE is applied to the Weibull distribution for the more robust estimation of the scale parameter, assuming that the shape parameter is known, and when the data set shows the Weibull contamination that is the presence of the outliers. To examine the performance of the WLE in comparison with the MLE, we found that the WLE outperforms the MLE based on the relative bias and that quadratic risk values. In most of the cases, the relative bias and the quadratic risk of the WLE decrease as the sample size increases. This is expected because most estimators in statistical theory perform better when the sample size increases. The gain in terms of the relative bias and the quadratic risk of the WLE decreases as α increases.

For the future work, we note that the WLE can be extended to some further modifications of the Weibull distribution. In addition, in this study, we considered only an estimator of the scale parameter of the Weibull distribution and assumed that the shape parameter was known. The WLE can be extended to estimate the two-parameter Weibull distribution when we assume both of the scale and shape parameter are unknown.

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8. Appendixes

Proof of Result 1

Proof. Prove of the asymptotic bias for the MLE, by definition the relative bias is

$$\frac{bias(\delta)}{\delta} = \frac{E[\hat{\delta}]-\delta}{\delta}$$
$$= (1-\varepsilon)\Gamma\left(1+\frac{1}{\beta}\right)+\varepsilon(1+\Delta)\Gamma\left(1+\frac{1}{\beta_1}\right)-1.$$

For the asymptotic bias of estimator $\tilde{\delta}$, we have

$$E[\tilde{\delta}] \approx E[Y]$$
$$= \int_0^A x f_A(x;,\delta,\beta)dx$$
$$= \frac{\delta}{G_\alpha}\left((1-\varepsilon)IG_1+\varepsilon(1+\Delta)IG_2\right),$$

where $IG_1 = \Gamma\left(\frac{1}{\beta}+1\right)P\left(\frac{1}{\beta}+1, A_u\right)$,

and $IG_2 = \Gamma\left(\frac{1}{\beta_1}+1\right)P\left(\frac{1}{\beta_1}+1, A_v\right)$.

So we have relative bias as

$$\frac{bias(\delta)}{\delta} = \frac{E[\tilde{\delta}]-\delta}{\delta}$$
$$= \frac{1}{G_\alpha}\left((1-\varepsilon)IG_1+\varepsilon(1+\Delta)IG_2\right)-1.$$

Proof of Result 2

Proof. By the definition of the quadratic risk of MLE, we have

$$R(\hat{\delta}) = \frac{E[(X-\hat{\delta})^2]}{n}$$
$$= \frac{\delta^2}{n}\left(\frac{1}{\delta^2}\left(E[X^2]-E[X])^2\right)+\left(\frac{\delta-\delta_\varepsilon}{\delta}\right)^2\right).$$

From the obstructing model

$$G_\varepsilon(x) = (1-\varepsilon)\left(1-e^{-\left(\frac{x}{\delta}\right)^\beta}\right)+\varepsilon\left(1-e^{-\left(\frac{x}{\delta_1}\right)^{\beta_1}}\right),$$

we get

$$E[X] = (1-\varepsilon)\delta\Gamma\left(1+\frac{1}{\beta}\right)+\varepsilon\delta_1\Gamma\left(1+\frac{1}{\beta_1}\right) = \delta E_1,$$

where

$$E_1 = \left((1-\varepsilon)\Gamma\left(1+\frac{1}{\beta}\right)+\varepsilon(1+\Delta)\Gamma\left(1+\frac{1}{\beta(1+\Delta)}\right)\right),$$

$$E[X^2] = (1-\varepsilon)\delta^2\Gamma\left(1+\frac{2}{\beta}\right)+\varepsilon\delta_1^2\Gamma\left(1+\frac{2}{\beta_1}\right) = \delta^2 E_2,$$

where

$$E_2 = \left((1-\varepsilon)\Gamma\left(1+\frac{2}{\beta}\right)+\varepsilon(1+\Delta)^2\Gamma\left(1+\frac{2}{\beta(1+\Delta)}\right)\right).$$

Therefore, the quadratic risk is

$$R(\hat{\delta}) = \frac{\delta^2}{n}\left(\frac{1}{\delta^2}\left(E[X^2]-(E[x])^2\right)+\left(\frac{\delta-\delta_\varepsilon}{\delta}\right)^2\right)$$
$$= \frac{\delta^2}{n}\left(E_2-E_1^2+\left(1-E_1\right)^2\right).$$

Additionally, the asymptotic distribution of $\tilde{\delta}$ equals the distribution of the $\alpha-$generalized trimmed sample mean where we defined

$$Y = \left(\frac{1}{n(1-\alpha)}\sum_{k=1}^{n(1-\alpha)} Y_k^\beta\right)^{\frac{1}{\beta}}.$$

The asymptotic quadratic risk of $\tilde{\delta}$ is

$$R(\tilde{\delta}) = \frac{E_\alpha[(Y-\delta)^2]}{n(1-\alpha)}$$
$$= \frac{1}{n(1-\alpha)}\left(Var_\alpha(Y) + E_\alpha[(\delta-\mu)^2]\right).$$

The asymptotic distribution of $\tilde{\delta}$ equals the distribution of the α-generalized trimmed sample mean of the random sample of size $n(1-\alpha)$ from the distribution concentrated on the interval $(0, A)$. The probability density of this distribution is positive and has the form

$$f_A(x;\delta,\beta) = \frac{(1-\varepsilon)f(x;\delta,\beta)+\varepsilon f(x;\delta_1,\beta_1)}{G_\alpha}.$$

Then,

$$\mu = E_\alpha[Y] = \int_0^A x f_A(x;,\delta,\beta)dx$$
$$= \frac{\delta}{G_\alpha}\left((1-\varepsilon)IG_1 + \varepsilon(1+\Delta)IG_2\right),$$

where $IG_1 = \Gamma\left(\frac{1}{\beta}+1\right)P\left(\frac{1}{\beta}+1, A_u\right)$,

and $IG_2 = \Gamma\left(\frac{1}{\beta_1}+1\right)P\left(\frac{1}{\beta_1}+1, A_v\right)$.

Also,

$$E_\alpha[Y^2] = \int_0^A x^2 f_A(x;,\delta,\beta)dx$$
$$= \frac{\delta^2}{G_\alpha}\left((1-\varepsilon)IG_3 + \varepsilon(1+\Delta)^2 IG_4)\right),$$

where $IG_3 = \Gamma\left(\frac{2}{\beta}+1\right)P\left(\frac{2}{\beta}+1, A_u\right)$, and

$P\left(\frac{2}{\beta}+1, A_u\right) = \dfrac{\gamma\left(\frac{2}{\beta}+1, A_u\right)}{\Gamma\left(\frac{2}{\beta}+1\right)}$ is a normalized function of a lower incomplete gamma function $\gamma\left(\frac{2}{\beta}+1, A_u\right)$,

$IG_4 = \Gamma\left(\frac{2}{\beta_1}+1\right)P\left(\frac{2}{\beta_1}+1, A_v\right)$, and

$P\left(\frac{2}{\beta_1}+1, A_v\right) = \dfrac{\gamma\left(\frac{2}{\beta_1}+1, A_v\right)}{\Gamma\left(\frac{2}{\beta_1}+1\right)}$ is a normalized function of a lower incomplete gamma function $\gamma\left(\frac{2}{\beta_1}+1, A_v\right)$.

Therefore, the quadratic risk is

$$R(\tilde{\delta}) = \frac{1}{n(1-\alpha)}\left(E_\alpha[Y^2] - (E_\alpha[Y])^2 + E_\alpha[(\delta-\mu)^2]\right)$$

$$= \frac{E_\alpha[Y^2]}{n(1-\alpha)} + \frac{\delta^2}{n(1-\alpha)} - \frac{2\delta\mu}{n(1-\alpha)}$$
$$= \frac{\delta^2((1-\varepsilon)IG_3 + \varepsilon(1+\Delta)^2 IG_4)}{n(1-\alpha)G_\alpha} + \frac{\delta^2}{n(1-\alpha)}$$
$$- \frac{2\delta^2((1-\varepsilon)IG_1 + \varepsilon(1+\Delta)IG_2)}{n(1-\alpha)G_\alpha}.$$