On Prime and Quasi Prime ideals in AG-Groupoids

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Abstract

In this paper, we study ideals, prime and quasi prime ideals in AG-groupoids. Some characterizations of prime ideals and quasi prime ideals are obtained. Moreover, we investigate relationships between prime and quasi prime ideals in AG-groupoids. Finally, we obtain necessary and sufficient conditions of a prime ideal to be a quasi prime ideal in AG-groupoids.

Keywords :   AG-groupoid, LA-semigroup, AG-3-band, quasi prime   ideal, prime ideal

1. Introduction

A groupoid S is called an Abel-Grassmann's groupoid, abbreviated as an AG-groupoid, if its elements satisfy the left invertive law [1, 2], that is: $(ab)c = (cb)a$ for all $a,b,c \in S$. Several examples and interesting properties of AG-groupoids can be found in [3], [4], [5] and [6]. It has been shown in [3] that if an AG-groupoid contains a left identity then it is unique. It has been proved also that an AG-groupoid with right identity is a commutative monoid, that is, a semigroup with identity element. It is also known [2] that in an AG-groupoid S, the medial law, that is,

$$(ab)(cd) = (ac)(bd)$$

for all $a,b,c,d \in S$ holds. An AG-groupoid S is called AG-3-band [7] if its every element satisfies $a(aa) = (aa)a = a.$

Now we define the concepts that we will used. Let S be an AG-groupoid. By an AG-subgroupoid of S [8], we means a non-empty subset A of S such that $A^2 \subseteq A$. A non-empty subset A of an AG-groupoid S is called a left (right) ideal of S [7] if $SA \subseteq A$ ($AS \subseteq A$). By two-sided ideal or simply ideal, we mean a non-empty subset of an AG- groupoid S which is both a left and a right ideal of S. A proper ideal P of an AG-groupoid S is called prime [8] if $AB \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq P$, for all ideals A and B in S. A proper left ideal P of an AG-groupoid S is called quasi prime [8] if $AB \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq P$, for all left ideals A and B in S. It is easy to see that every quasi prime ideal is prime.

In this paper we characterize the AG-groupoid. We investigate relationships between prime and quasi prime ideals in AG-groupoids. Finally, we obtain necessary and sufficient conditions of a prime ideal to be a quasi prime ideal in AG-groupoids

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2. Basic results

In this section we refer to [7, 8] for some elementary aspects and quote few definitions and examples which are essential to step up this study. For more details we refer to the papers in the references.

Lemma 2.1   [8] If S is an AG-groupoid with left identity, then every right ideal is an ideal.

Lemma 2.2   [8] If A is a left ideal of an AG-groupoid S with left identity, then aA is a left ideal in S, where $a \in S$.

Lemma 2.3   [8] If A is a right ideal of an AG-groupoid S with left identity, then A^2 is an ideal in S.

Lemma 2.4   [8] An ideal A of an AG-groupoid S is prime if and only if it is semiprime and strongly irreducible.

Lemma 2.5   [7] A subset of an AG-3-band is a right ideal if and only if it is left.

Proof.   Same as [7].

Lemma 2.6   [8] If S is an AG-groupoid with left identity, then a left ideal P of S is quasi prime if and only if $a(Sb) \subseteq P$ implies that either $a \in P$ or $b \in P$, where $a, b \in S$.

Lemma 2.7   [8] If A is a proper left (right) ideal of an AG-groupoid S with left e identity, then $e \notin A$.

2. Ideals in AG-groupoids

The results of the following lemmas seem to be at the heart of the theory of AG-groupoids; these facts will be used so frequently that normally we shall make no reference to this lemma.

Lemma 3.1   Let S be an AG-groupoid with left identity, and let B be a left ideal of S. Then $AB = \{ab : a \in A, b \in B\}$ is a left ideal in S, where $\varnothing \neq A \subseteq S$.

Proof.   Suppose that S is an AG-groupoid with left identity. Let B be a left

ideal of S. Then $S(AB) = A(SB) \subseteq AB$. By Definition of left ideal, we get AB is a left ideal in S.

Lemma 3.2   Let S be an AG-groupoid with left identity and let $a \in S$. Then a^2S is an ideal in S.

Proof.   By Lemma 2.2, we have a^2S is a left ideal of S. Now consider

$$
\begin{aligned}
(a^2r)s &= ((aa)r)s \\
&= ((ra)a)s \\
&= [e((ra)a)]s \\
&= [s((ra)a)]e \\
&= [(ra)(sa)]e \\
&= [((sa)a)r]e \\
&= [((aa)s)r]e \\
&= [(rs)(aa)]e \\
&= [ea^2](rs) \\
&= a^2(rs) \in a^2S
\end{aligned}
$$

for all $r, s \in S$. Therefore a^2S is an ideal in S.

Lemma 3.3   Let S be an AG-groupoid with left identity, and let A, B be left ideals of S. Then $(A : B)$ is a left ideal in S, where $(A : B) = \{r \in S : Br \subseteq A\}$.

Proof.   Suppose that S is an AG-groupoid. Let $s \in S$ and let $a \in (A : B)$. Then $Ba \subseteq A$ so that

$$B(sa) \subseteq s(Ba) \subseteq sA \subseteq A.$$

Therefore $sa \in (A : B)$ so that $S(A : B) \subseteq (A : B)$. Hence $(A : B)$ is a left ideal in S.

Lemma 3.4   Let S be an AG-groupoid with left identity, and let A be a left ideal of S. Then $(A : r)$ is a left ideal in S, where $(A : r) = \{a \in S : ra \in A\}$ and $r \in S$.

Proof.   By Lemma 3.3, we have $(A : r)$ is a left ideal in S.

Corollary 3.5 Let S be an AG-3-band with left identity, and let A be a left ideal

of S. Then $(A:r)$ is an ideal in S, where $r \in S$.

Proof. By Lemma 3.4, we have $(A:r)$ is a left ideal in S. By Lemma 2.5, it follows that $(A:r)$ is a right ideal in S. By Lemma 2.1, we have $(A:r)$ is an ideal in S.

Remark 1. Let S be an AG-groupoid and let A be a left ideal of S. It is easy to verify that $A \subseteq (A:r)$.

2. Let S be an AG-groupoid with left identity e, and let A be a proper left (right) ideal of S. By Lemma 2.7, we have $e \notin (A:r)$, where $r \in S - A.$

3. Let S be an AG-groupoid and let A, B, C be left ideals of S. It is easy to verify that $(A:C) \subseteq (A:B)$, where $B \subseteq C$.

Corollary 3.6 Let S be an AG-3-band with left identity, and let A, B be left ideals of S. Then $(A:B)$ is an ideal in S.

Proof. This follows from Corollary 3.5.

4. Properties of quasi prime ideals in AG-groupoids

We start with the following theorem that gives a relation between prime and quasi prime ideal in AG-groupoid. Our starting points is the following lemma:

Lemma 4.1 If S is an AG-groupoid with left identity , then a left ideal P of S is quasi prime if and only if $(Sa)(Sb) \subseteq P$ implies that either $a \in P$ or $b \in P$, where $a, b \in S$.

Proof. Let P be a quasi prime ideal of an AG-groupoid S with left identity. Now suppose that $(Sa)(Sb) \subseteq P$. Then by the definition of left ideal, we get

$$
\begin{aligned}
(Sa)(Sb) &= (SS)(ab) \\
&= S(ab) \\
&= a(Sb)
\end{aligned}
$$

that is $a(Sb) \subseteq (Sa)(Sb) \subseteq P$. By Lemma 2.6, we have either $a \in P$ or $b \in P$ Conversely, assume that if $(Sa)(Sb) \subseteq P$, then either $a \in P$ or $b \in P$ for all $a, b \in S$. Let $a(Sb) \subseteq P$. Now consider

$$
(Sa)(Sb) = a(Sb) \subseteq P.
$$

By using given assumption, if $a(Sb) \subseteq P$, then either $a \in P$ or $b \in P$. Then by Lemma 2.6, we have P is a quasi prime ideal in S.

Theorem 4.2 If S is an AG-groupoid with left identity, then a left ideal P of S is quasi prime if and only if $ab \in P$ implies that either $a \in P$ or $b \in P$, where $a, b \in S$.

Proof. Let P be a left ideal of an AG-groupoid S with left identity. Now suppose that $ab \in P$, where $a, b \in S$. Then by the definition of left ideal, we get

$$
\begin{aligned}
(Sa)(Sb) &= (SS)(ab) \\
&= S(ab) \\
&\subseteq SP \\
&\subseteq P
\end{aligned}
$$

By Lemma 4.1, we have either $a \in P$ or $b \in P$.

Conversely, the proof is easy.

Theorem 4.3 Let S be an AG-groupoid with left identity, and let A be a quasi prime ideal of S. Then $(A:r^2)$ is a quasi prime ideal in S, where $r \in S$.

Proof. Assume that A is a quasi prime ideal of S. By Lemma 3.4, we have $(A:r^2)$ is an left ideal in S. Let $ab \in (A:r^2)$. Then $r^2(ab) \in A$. We have

$$a\left(r^2 b\right) = r^2\left(ab\right) \in A.$$

By the definition of quasi prime ideal, we have either

$$r^2 a \in r^2 A \subseteq A \text{ or } r^2 b \in A.$$

Therefore either $a \in (A:r^2)$ or $b \in (A:r^2)$ and hence $(A:r^2)$ is a quasi prime ideal in S.

Corollary 4.4 Let S be an AG-3-band with left identity, and let A be a prime ideal of S. Then $(A:r^2)$ is a prime ideal in S.

Proof.   By Theorem 4.3, it is obvious that $(A:r^2)$ is a prime ideal in S

Theorem 4.5 Let S be an AG-3-band with left identity. Every quasi prime ideal in S is a prime ideal in S.

Proof.   This follows directly from Lemma 2.5.

Theorem 4.6 Let S be an AG-groupoid with left identity, and let B be a left ideal of S. Then $(A:B^2)$ is a quasi prime ideal in S, where A is a quasi prime ideal in S.

Proof. Assume that S be an AG-groupoid with left identity. By Lemma 3.3, we have $(A:B^2)$ is a left ideal in S. Let $ab \in (A:B^2)$, where $a,b \in S$. Then $B^2 ab \subseteq A$ so that

$$(Ba)(Bb) = (BB)ab = B^2 ab \subseteq A.$$

Therefore either $a \in (A:B^2)$ or $b \in (A:B^2)$ and hence $(A:B^2)$ is a quasi prime ideal in S.

Theorem 4.7 Let S be an AG-groupoid with left identity, and let A be a quasi prime ideal of S. Then $(A:r)$ is a quasi prime ideal in S, where $r \in S$.

Proof. Assume that A is a quasi prime ideal of S. By Lemma 3.4, we have $(A:r)$ is a left ideal in S. Let $ab \in (A:r)$, where $a,b \in S$. Then $r(ab) \in A$. We have $a(rb) = r(ab) \in A$. By the definition of

quasi prime ideal, we have either $a \in A$ or $rb \in A$. since $ra \in rA \subseteq A$, we have either $a \in (A:r)$ or $b \in (A:r)$. Hence $(A:r)$ is a quasi prime ideal in S.

Theorem 4.8 Let S be an AG-groupoid with left identity, and let P be a prime ideal of S. If $\left(Sa^2\right)\left(Sb^2\right) \subseteq P$, then either $a^2 \in P$ or $b^2 \in P$, where $a,b \in S$.

Proof. Let P be a prime ideal of an AG-groupoid S with left identity. Now suppose that $\left(Sa^2\right)\left(Sb^2\right) \subseteq P$, where $a,b \in S$. Then by the definition of left ideal, we get

$$
\begin{aligned}
\left(Sa^2\right)\left(Sb^2\right) &= \left(\left(Sb^2\right)a^2\right)S \\
&= \left(\left(a^2 b^2\right)S\right)S \\
&= \left(SS\right)\left(a^2 b^2\right) \\
&= a^2\left(\left(SS\right)b^2\right) \\
&= a^2\left(\left(b^2 S\right)S\right) \\
&= \left(b^2 S\right)\left(a^2 S\right)
\end{aligned}
$$

that is $\left(b^2 S\right)\left(a^2 S\right) \subseteq P$. By Lemma 3.2, we have $a^2 S$ and $b^2 S$ are ideals in S. Therefore either

$$b^2 = (eb)b = b^2 e \in b^2 S \subseteq P$$

or $\quad a^2 = (ea)a = a^2 e \in a^2 S \subseteq P.$

5. Conclusions

Many new classes of AG-groupoids have been discovered recently. All this has attracted researchers of the field to investigate these newly discovered classes in detail. This current article investigates the ideals, prime and quasi prime ideals in AG-groupoids. Some characterizations of prime ideal and quasi prime ideals are obtained. Moreover, we investigate relationships between prime and quasi prime ideals in AG-groupoids. Finally, we obtain necessary and sufficient conditions of a prime ideal to be a quasi prime ideal in AG-groupoids.

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