Symmetric implicit multiderivative numerical integrators for direct solution of fifth-order differential equations

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Abstract

A direct method of solution of fifth order ordinary differential equations (odes) is proposed in this paper. Collocation of the differential system is taken at selected grid points to reduce the number of functions to be evaluated per iteration. A number of predictors of the same order of accuracy with the main method for the estimation of y-functions and their derivatives in the main method are generated. The symmetric implicit multiderivative algorithm [SIMA] is suitable for numerical integration of non-stiff and mildly-stiff fifth order equations. Test examples are solved with the method to confirm its efficiency.

Keywords: Symmetric; multiderivative; implicit; interval of absolute stability; error constant.

1. Introduction

In this article, a direct solution of fifth order ordinary differential equations (odes) of the form (1)

$$y^{(5)} = f(x, y, y', ..., y^4), \quad x \in [a,b] \qquad (1)$$

together with initial conditions

$$y^{(i)}(a) = y_i, \; i = 0(1)4,$$

is explored. This class of problems has a lot of applications in applied sciences, mechanical, civil and aerospace engineering especially where vibration of structures due to passage of moving loads under damping forces are of paramount importance.

Akbar and Siddique [1] investigated the analytical solution of fifth order weakly nonlinear oscillatory systems of type (1). In practice, the numerical integration process of problem (1) involves a reduction to systems of first order equations which may then be solved with any known methods for first order equations. This approach has its inherent setbacks, Awoyemi [2]. Different authors have proposed methods for direct solution of higher order problems. Awoyemi [2], Parand and Hojjati[3], Saravai and Mirrajei [4], Golbabai and Arabshahi [5], Kayode[6], [7], all considered second order equations using different step-lengths and step-numbers to obtain methods of various order of accuracy. Kayode and Adeyeye [8] and Kayode and Obarhua [9] also developed numerical hybrid methods for direct solution of general second order differential equations by using different basis functions to generate collocation and interpolation equations,. In their work, Awoyemi and Idowu [10] considered direct numerical solution of third order equations. Awoyemi [11] and Kayode [12] investigated the direct solution of fourth order differential equations. However, all these methods are not suitable to solve fifth order problems of type (1) without reduction to lower order problems. Kayode and Awoyemi [13] proposed a 5-step

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multiderivative collocation method in a predictor-corrector mode for problem (1). The method is non-symmetric and may not be suitable for oscillatory problems. Kayode [14] proposed an explicit method for solving (1) directly. The aim of this article is to propose a symmetric implicit multiderivative algorithm [SIMA] with increased stepnumber as well as increased order of accuracy (p) for the approximate solution of (1) directly.

2. Methodology

The basis function for the approximate solution of problem (1) is taken to be the partial sum of power series of a single variable x in the form

$$y(x)=\sum_{j=0}^{k+2}\lambda_j x^j \qquad (2)$$

where

$$\lambda_j \in R, y\in C^m(a,b)\subset P(x).$$

The fifth derivative of (2) substituted in (1) is given as

$$\sum_{j=0}^{k+2} j(j-1)(j-2)(j-3)(j-4)\lambda_j x^{j-5} = f(x,y,y^{(1)},y^{(2)},y^{(3)},y^{(4)}) \qquad (3)$$

Collocating (3) at $x=x_{n+j},\ j=0(1)k-1$ and interpolating (2) at $x=x_{n+j},\ j=0(3)k$ yield the following collocation and interpolation matrix

$$AX=B \qquad (4)$$

where

$$A=\begin{bmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8\\
1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8\\
1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 & x_{n+2}^8\\
1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 & x_{n+3}^8\\
1 & x_{n+4} & x_{n+4}^2 & x_{n+4}^3 & x_{n+4}^4 & x_{n+4}^5 & x_{n+4}^6 & x_{n+4}^7 & x_{n+4}^8\\
1 & x_{n+5} & x_{n+5}^2 & x_{n+5}^3 & x_{n+5}^4 & x_{n+5}^5 & x_{n+5}^6 & x_{n+5}^7 & x_{n+5}^8\\
0 & 0 & 0 & 0 & 0 & 120 & 720x_n & 2520x_n^2 & 6720x_n^3\\
0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+3} & 2520x_{n+3}^2 & 6720x_{n+3}^3\\
0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+6} & 2520x_{n+6}^2 & 6720x_{n+6}^3
\end{bmatrix}$$

$$X=\left[\lambda_0 \quad \lambda_1 \quad \lambda_2 \quad \lambda_3 \quad \lambda_4 \quad \lambda_5 \quad \lambda_6 \quad \lambda_7 \quad \lambda_8\right]^T,$$

$$B=\left[f_n \quad f_{n+3} \quad f_{n+6} \quad y_n \quad y_{n+1} \quad y_{n+2} \quad y_{n+3} \quad y_{n+4} \quad y_{n+5}\right]^T,$$

T is the matrix transpose,

$$f_{n+r}=f(x_{n+r},y_{n+r},y'_{n+r},y''_{n+r},y'''_{n+r},y^{iv}_{n+r}),\ r=0,1,2,\ldots,$$
$$y_{n+i}\approx y(x_{n+i}).$$

Solving the matrix (4) for $\lambda_j^{'}s$ and using $t=\dfrac{x-x_{n+5}}{h}$ in (3) yielded, after simplification, the continuous method

$$y_k(t)=\sum_{j=0}^{k-1}\alpha_j(t)y_{n+j}+\sum_{j=0}^{k-4}\beta_{3j}(t)f_{n+3j}. \qquad (5)$$

The continuous coefficients $\alpha_j(t)$ and $\beta_j(t)$ in (5), when k is taken to be 6, are found to be:

$$\alpha_0(t)=\frac{1}{25200}[3600t+9108t^2+8456t^3+3521t^4+560t^5-28t^6-16t^7-t^8],$$

$$\alpha_1(t)=\frac{-1}{5040}[2340t+6798t^2+7196t^3+3311t^4+560t^5-28t^6-16t^7-t^8],$$

$$\alpha_2(t)=\frac{1}{2520}[240t+3228t^2+5516t^3+3101t^4+560t^5-28t^6-16t^7-t^8],$$

$$\alpha_3(t)=\frac{1}{2520}[3960t+2862t^2-3416t^3-2891t^4-560t^5+28t^6+16t^7+t^8],$$

$$\alpha_4(t)=\frac{-1}{5040}[16560t+12732t^2-896t^3-2681t^4-560t^5+28t^6+16t^7+t^8],$$

$$\alpha_5(t)=\frac{1}{25200}[25200+48900t+27642t^2+2044t^3-2471t^4-560t^5+28t^6+16t^7+t^8],$$

$$\beta_0(t) = \frac{-1}{2721600}[29520t + 73044t^2 + 66668t^3 + 27923t^4 + 4760t^5 - 154t^6 - 148t^7 - 13t^8],$$

$$\beta_3(t) = \frac{1}{136080}[43920t + 98964t^2 + 78736t^3 + 27265t^4 + 3430t^5 - 224t^6 - 86t^7 - 5t^8],$$

$$\beta_6(t) = \frac{1}{544320}[5040t + 13068t^2 + 13132t^3 + 6769t^4 + 1960t^5 + 322t^6 + 28t^7 + t^8].$$

A discrete implicit multiderivative form and its derivatives arising from (5), when $t = 1$, are obtained to be

$$y_{n+6} = 4y_{n+5} - 5y_{n+4} + 5y_{n+2} - 4y_{n+1} + y_n + \frac{2h^5}{27}[f_{n+6} + 25f_{n+3} + f_n] \tag{6}$$

$p = 7$ and c_{p+2} is 0.40277;

$$y'_{n+6} = \frac{1}{420h}[1632y_{n+5} - 2175y_{n+4} - 2440y_{n+3} + 6360y_{n+2} - 4440y_{n+1} + 1063y_n] + \frac{h^4}{3780}[761f_{n+6} + 16730f_{n+3} + 707f_n], \tag{7}$$

$p = 7,\ c_{p+2} = -0.400297619$;

$$y''_{n+6} = \frac{1}{h^2}[\frac{452}{403}y_{n+5} + \frac{5429}{1260}y_{n+4} - \frac{4526}{193}y_{n+3} + \frac{9291}{271}y_{n+2} - \frac{13267}{630}y_{n+1} + \frac{585}{122}y_n] + \frac{h^3}{68040}[29531f_{n+6} + 527756f_{n+3} + \frac{120601}{5}f_n], \tag{8}$$

$p = 7,\ c_{p+2} = 0.442890211$;

$$y'''_{n+6} = \frac{1}{50h^3}[-146y_{n+5} + 955y_{n+4} - 2310y_{n+3} + 2660y_{n+2} - 1480y_{n+1} + 321y_n] + \frac{h^3}{120}[89f_{n+6} + 1134f_{n+3} + \frac{287}{5}f_n], \tag{9}$$

$p = 7, c_{p+2} = -0.09583333$;

$$y^{iv}_{n+6} = \frac{1}{50h^4}[-201y_{n+5} + 1055y_{n+4} - 2210y_{n+3} + 2310y_{n+2} - 1205y_{n+1} + 251y_n] + \frac{h}{1080}[1069f_{n+6} + 7180f_{n+3} + \frac{2063}{5}f_n], \tag{10}$$

$p = 7,\ c_{p+2} = 0.35972222$.

Definitions: [Lambert [15]**]**

(a) Order and error constant

Let L be the linear difference operator associated with (5) given by

$$\mathscr{L}[y(x);h] = \sum_{j=0}^{k}[\alpha_j y(x+jh) - h^5\beta_j y^{v}(x+jh)] \tag{11}$$

where $y(x)$ is an arbitrary function, continuously differentiable on an interval [a, b]. Assuming that $y(x)$ has as many higher derivatives as may be required, then expanding (11) by Taylor series gives

$$\mathscr{L}[y(x);h] = C_0y(x) + C_1y^{(1)}(x) + C_2y^{(2)}(x) + \cdots + C_py^{(p)}(x) + \cdots \tag{12}$$

where

$$\begin{aligned}
C_0 &= \alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_p,\\
C_1 &= \alpha_1 + 2\alpha_2 + \cdots + k\alpha_k,\\
C_2 &= \frac{1}{2!}(\alpha_1 + 2^2\alpha_2 + \cdots + k^2\alpha_k) - (\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k),\\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots\\
C_p &= \frac{1}{p!}(\alpha_1 + 2^p\alpha_2 + \cdots + k^p\alpha_k) - \frac{1}{(p-2)!}(\beta_1 + 2^{p-2}\beta_2 + \cdots + k^{p-2}\beta_k),\ p = 3, 4, \cdots.
\end{aligned}$$

The order of accuracy of (5) is p if $C_0 = C_1 = C_2 = \cdots = C_p = C_{p+1} = 0,\ C_{p+2} \neq 0$ and its error constant is C_{p+2}. Applying the definitions of order and error constant above to (6), (7), (8), (9) and (10) produces order of accuracy p = 7 for each of them and error constants C_{p+2} to be 0.40277, -0.400297619, 0.442890211, -0.09583333, 0.35972222, respectively.

(b) Symmetry, Consistency zero stability and convergence

Consider the characteristic equation associated with (5) given by

$$\prod(r,\bar{h}) = \rho(r) - \sigma(r),\ \bar{h} = \lambda h^5, \tag{13}$$

where $\lambda = \frac{\partial f}{\partial y}$ is the eigenvalue(s) of the Jacobian of (1). ρ and σ are the first and second characteristic polynomials of (5) respectively. They are given by

$$\rho(r)=\sum_{j=0}^{k}\alpha_j r^j = r^6-4r^5+5r^4-5r^2+4r-1 \qquad (14)$$

$$\sigma(r)=\sum_{j=0}^{k}\beta_j r^j = \frac{2}{27}(r^6+25r^3+1) \qquad (15)$$

The linear multistep method (5) is consistent if the order of accuracy $p \ge 1$,

$$\sum_{j=0}^{k}\alpha_j r^j = 0,$$

$$\rho'(r)=\rho''(r)=\rho'''(r)=\cdots\rho^{n-1}(r)=0$$

and $\rho^n(r)=n!\sigma(r)$ for the principal root r = 1.

Zero stability property of a linear multistep method (5) requires that the roots of (14) must satisfy $|r| \le 1$, and every root with $|r|=1$ must have multiplicity 6.

The necessary and sufficient conditions for a linear multistep method to be convergent are consistency and zero stability, [see Henrici [16] for the proof].

Applying these definitions to (6) shows that the method is symmetric, consistent, zero-stable and convergent.

3. Generating the Starting Values

Implementation of the implicit discrete method (6) requires some starting values for the evaluation of $y_{n+j}^{(r)}$, $j=1(1)6;\ r=0(1)4$ in the $f-$function. In this paper, the explicit method of order seven of Kayode [14] is adopted as the main predictor to implement y_{n+6} and its required derivatives. These are reproduced as follows:

$$y_{n+6}=4y_{n+5}-5y_{n+4}+5y_{n+2}-4y_{n+1}+y_n+\frac{h^5}{6}[f_{n+5}+10f_{n+3}+f_{n+1}] \qquad (11)$$

p = 7 and $c_{p+2}=-0.125$,

$$y'_{n+6}=\frac{1}{420h}[1704y_{n+5}-2535y_{n+4}-1720y_{n+3}+5640y_{n+2}+991y_{n+1}-4080y_n] + \frac{h^4}{420}[191f_{n+5}+164f_{n+3}+1595f_{n+1}], \qquad (12)$$

p = 7, $c_{p+2}=0.49156746$,

$$y''_{n+6}=\frac{1}{600h^2}[13078y_{n+5}-2915y_{n+4}-87620y_{n+3}+155870y_{n+2}+24197y_{n+1}-102610y_n] + \frac{h^3}{75600}[74519f_{n+5}+47525f_{n+3}+451760f_n], \qquad (13)$$

p = 7, $c_{p+2}=0.800165343$,

$$y'''_{n+6}=\frac{1}{50h^3}[12y_{n+5}+165y_{n+4}-730y_{n+3}+1080y_{n+2}+163y_{n+1}-690y_n] + \frac{h^2}{600}[1021f_{n+5}+310f_{n+3}+3175f_{n+1}], \qquad (14)$$

p = 7, $c_{p+2}=0.69716666$,

$$y^{iv}{}_{n+6}=\frac{1}{h^4}[491y_{n+5}-2305y_{n+4}+4310y_{n+3}-4010y_{n+2}-341y_{n+1}+1855y_n] + \frac{h}{3600}[8291f_{n+5}-1556f_{n+3}-4120f_{n+1}], \qquad (15)$$

p = 7, c_{p+2}=-0.47361111.

The starting values for $y_{n+5}^{(r)}, r=0,1,...,4$ in Kayode and Awoyemi [9] are adopted for f_{n+5} in (11) – (15).

4. Numerical Examples

Two non-linear numerical examples are solved to demonstrate the accuracy and usability of the new method.

Problem 1

$$y^{(5)}=2y^{(1)}y^{(2)}-yy^{(4)}-y^{(1)}y^{(3)}-8x+(x^2-2x-3)e^x,$$
$$0 \le x \le 1,$$
$$y(0)=1,\ y^{(1)}(0)=1,\ y^{(2)}(0)=3,\ y^{(3)}(0)=1,\ y^{(4)}(0)=1,\ h=0.01.$$

Theoretical solution is y(x) = e^x+x^2.

Table1. Results of problem 1.

x-value	y-exact	y-computed	Absolute Errors	Errors in [14]
0.1000	1.115170918075647	1.115170918074188	1.459721e-012	1.210563e-008
0.1500	1.184334242728283	1.184334242718375	9.908074e-012	
0.2000	1.26140275816017	1.261402758118294	4.187584e-011	1.927066e-008
0.2500	1.346525416687741	1.346525416558994	1.287477e-010	
0.3000	1.439858807576003	1.439858807253826	3.221776e-010	3.025973e-008
0.3500	1.541567548593257	1.541567547894400	6.988572e-010	
0.4000	1.651824697641271	1.651824696276096	1.365175e-009	4.697311e-008
0.4500	1.770812185490169	1.770812183028581	2.461589e-009	
0.5000	1.898721270700129	1.898721266533392	4.166737e-009	7.208283e-008
0.5500	2.035753017867396	2.035753011165992	6.701404e-009	
0.6000	2.182118800390510	2.182118790058871	1.033164e-008	1.092800e-007
0.6500	2.338040829013897	2.338040813641658	1.537224e-008	
0.7000	2.503752707470478	2.503752685281417	2.218906e-008	1.635841e-007
0.7500	2.679500016612677	2.679499985410939	3.120174e-008	
0.8000	2.865540928492470	2.865540885606931	4.288554e-008	2.417328e-007
0.8500	3.062146851925993	3.062146794152898	5.777310e-008	
0.9000	3.269603111156952	3.269603034701117	7.645583e-008	3.526740e-007
0.9500	3.488209659315849	3.488209559730279	9.958557e-008	
1.0000	3.718281828459047	3.718281700584095	1.278750e-007	5.081954e-007

Problem 2

$$y^{(5)} = 6\{2(y^{(1)})^3 + 6yy^{(1)}y^{(2)} + y^2y^{(3)}\}, \quad 1 \le x \le 2,$$

$$y(1) = 1,\ y^{(1)}(1) = -1,\ y^{(2)}(1) = 2,\ y^{(3)}(1) = -6,\ y^{(4)}(1) = 24,\ h = 0.1.$$

Theoretical solution is $y(x) = \dfrac{1}{x}.$

Table2. Results of problem 2.

x-value	y-exact	y-computed	Absolute Errors	Errors in [14]
1.10	0.909090909090908	0.909090909705684	6.147755e-010	6.380986e-009
1.15	0.869565217391303	0.869565221664023	4.272720e-009	
1.20	0.833333333333331	0.833333350448592	1.711526e-008	1.702314e-007
1.25	0.799999999999999	0.800000049850512	4.985051e-008	
1.30	0.769230769230767	0.769230887857545	1.186268e-007	1.196389e-006
1.35	0.740740740740738	0.740740986399939	2.456592e-007	
1.40	0.714285714285712	0.714286174015168	4.597295e-007	4.670389e-006
1.45	0.689655172413790	0.689655969010327	7.965965e-007	
1.50	0.666666666666664	0.666667966010523	1.299344e-006	1.325687e-005
1.55	0.645161290322578	0.645163309002528	2.018680e-006	
1.60	0.624999999999997	0.625003013203557	3.013204e-006	3.082943e-005
1.65	0.606060606060603	0.606064955702371	4.349642e-006	
1.70	0.588235294117644	0.588241397183878	6.103066e-006	6.256635e-005
1.75	0.571428571428568	0.571436928520323	8.357092e-006	
1.80	0.555555555555552	0.55556675961580	1.120406e-005	1.150269e-004
1.85	0.540540540540537	0.54055528575267	1.474521e-005	
1.90	0.526315789473681	0.526334880320716	1.909085e-005	1.962155e-004
1.95	0.512820512820509	0.512844873295431	2.436047e-005	
2.00	0.499999999999997	0.500030682958982	3.068296e-005	3.156371e-004

5. Conclusion

A six-step collocation method has been proposed to directly solve fifth order odes of the form (1) without the burden of reduction to system of lower order problems. The method and its associated derivatives are of order seven. The starting values and their associated derivatives for the implementation of the method are obtained to be same order seven. Two non-linear fifth order problems were solved with the new method as test problems. The accuracy of the results is compared with Kayode [14] as shown in Tables 1 and 2 above. The results obtained showed the usability of the method for solving fifth order odes especially nonlinear problems.
However, stiff problems have not been addressed in this work. In future research work, efforts will be directed at proposing numerical methods capable of solving general stiff fifth order odes directly.

6. References

[1] Akbar, M.A. and Siddique, T.A., Perturbation solutions of fifth order oscillatory nonlinear system. *Nonlinear Anal. Mod. Control*, Vol. 16, No. 2, pp. 123–134, 2011.

[2] Awoyemi, D.O., A P-stable linear multistep method for solving general third order ordinary differential equations. *Int. J. Comp. Math.* Vol. 80, No. 8, pp. 987–993, 2003.

[3] Parand, K. and Hojjati, G., Solving volterra's population model using new second derivative multistep methods. *Am. J. Appl. Sci.* Vol. 5, pp. 1019–1022, 2008.

[4] Saravai, M.F. and Mirrajei, S.R., Numerical solution of linear ordinary differential equations in quantum chemistry by Clenshaw method. *World Acad. Sci. Eng. Technol.*, Vol. 49, pp. 1051–1054, 2009.

[5] Golbabai and Arabshahi, M. M., Getting started with Numerov's method. *Comp. Phys.*, Vol. 10, No. 5, pp. 514–515, 2010.

[6] Kayode, S.J., A zero-stable optimal order method for direct solution of second-order differential equations. *J. Math. Stat*., Vol. 6, No. 3, pp. 367–371, 2010.

[7] Kayode, S.J., A class of on-point zero-stable continuous hybrid methods for direct solution of second-order differential equations. *African J. Math. Comp. Sci. Res*. Vol. 4, No. 3, pp. 93–99, 2011.

[8] Kayode, S.J. and Adeyeye, O., A 3-step hybrid method for direct solution of second order initial value problems. *Australian J. Basic Appl. Sci.*, Vol. 5, No. 12, pp. 2121–2126, 2011.

[9] Kayode S.J. and Obarhua, F.O., Continuous $y-function$ hybrid methods for direct solution of differential equations. *Int. J. Diff. Eqns. Appl.*, Vol. 12, No. 1, pp. 37-48, 2013.

[10] Awoyemi, D. O. and Idowu, O. M., A class of hybrid collocation methods for third order ordinary differential equations. *Int. J. Comp. Math.*, Vol. 82, No. 10, pp. 1287-1293, 2005.

[11] Awoyemi, D.O., Algorithmic collocation approach for direct solution of fourth–order initial value problems of ordinary differential

equations. *Int. J. Comp. Math.* Vol. 82, No. 3, pp. 321–329, 2005.

[12] Kayode S.J., An order six zero-stable method for direct solution of fourth order ordinary differential equations. *American J. Appl. Sci.,* Vol. 5 No. 11, pp. 1461-1466, 2008.

[13] Kayode, S.J. and Awoyemi, D.O., A multiderivative collocation method for 5^{th} order ordinary differential equations. *J. Math. Stat.,* Vol. 6, No. 1, pp. 60 – 63, 2010.

[14] Kayode S.J., An explicit continuous algorithm for direct solution of general fifth order ordinary differential equations. *Int. J. Diff. Eqns. Appl.,* ISSN 1311-2872 (In Print)

[15] Lambert, J. D., *Computational methods in ordinary differential equation*. John Willey & Sons Inc., New York, 1973.

[16] Henrici, P., Discrete variable methods in Odes. John Wiley and Sons, N. York, 1962.