Parameter Estimation for Re-Parametrized Inverse Gaussian Distribution

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Abstract

In this article, we consider a new parametrization of the two-parameter Inverse Gaussian distribution. We find the estimators for parameters of the Inverse Gaussian distribution by the method of moments and the method of maximum likelihood. Then, we compare the efficiency of the estimators for the two methods based on their bias and mean square error (MSE). For this we fix values of parameters, run simulations, and report MSE and bias for estimates obtained by both methods. The conclusion is that when sample sizes are 10, the method of moments tends to be more efficient than the maximum likelihood method for estimates of both parameters (lambda and theta). When sample sizes 50, are the estimates of the method of moments tend to be efficient, for Theta The maximum likelihood the method tends to be more efficient than the method of moments for Lambda. When sample sizes are 100, the estimates of the method of moments tend to be more efficient for Theta. For the estimate, of parameter Lambda, the maximum likelihood method tends to be more efficient.

Keywords: parametrization, method of moments, method of maximum likelihood, point estimation, bias, MSE

1. Introduction

The Inverse Gaussian distribution became known in statistics only in the late 40s of the twentieth century. Many contemporary statistical methods for data analysis involve and are derived with the extensive use of the Inverse Gaussian distribution. As it is mentioned in the monograph by Chhikara and Folks (1989) [1], the Inverse Gaussian distribution has several properties analogous to a Gaussian distribution, but the name can be misleading. It is an "Inverse" only in that, while the Gaussian distribution describes a Brownian Motion's level at a fixed time, the Inverse Gaussian describes the distribution of time a Brownian Motion with positive drift takes to reach a fixed positive level. An inverse Gaussian distribution in the classical parametrization contains two parameters, shape parameter μ and scale parameter β.

Chhikara and Folks (1989)[1] reviewed some analogies between the statistical properties of the Inverse Gaussian and the Gaussian distributions and the summarized statistical properties are done by Johnson et al. (1995).[2]

Before that, Tweedie (1957) [3] extended some results of Schrodinger in theoretical physics and noticed the inverse relationship between the cumulant-generating function of the time per unit distance and the cumulant-generating function of the distance covered in unit time. The detailed study of the corresponding distribution was published in Tweedie (1957) [3] with the new name, Inverse Gaussian, for the first passage-time distribution of the Brownian Motion with drift. Even before this, Wald (1947) [4] derived an approximation of the sample size distribution in a sequential probability ratio test as a special case of the Inverse Gaussian distribution.

The Inverse Gaussian distribution is a useful statistical tool for engineering, biology, physics, finance, and many other applications. For instance, in tracer dynamics, emptiness of a dam, a purchase incidence model, the distribution of strike duration, a word frequency distribution, and many others. Moreover, the Inverse Gaussian is the most appropriate statistical distribution when skewed data analysis is needed.

The Inverse Gaussian distribution is closely related to the Birnbaum-Saunders distribution. Cysneiros, et al. (2008) [5] mentioned that the Birnbaum-Saunders distribution, also known as the fatigue-life distribution, is frequently used in reliability studies. Furthermore, Leiva et al. (2008) [6] provided a lifetime analysis based on the generalized Birnbaum-Saunders distribution. The estimation method is examined by means of Monte Carlo simulations.

In Ahmed et al (2008) [7] a new parametrization for a related Birnbaum-Saunders distribution is proposed. This re-parametrization fits the study of materials since the proposed parameters characterize or specify the thickness of the sample and the nominal treatment loading on the sample. The usual shape and scale parameters of the distribution do not offer this physical interpretation. In Ahmed et al (2008) [7] the statistical properties of the direct application of the standard methods of point estimation to the new parameters are investigated. In an effort to appraise the performance of proposed estimators in a practical setting, Monte-Carlo simulations are conducted for small, moderate and large sample sizes.

In this article, we will estimate new parameters of the Inverse Gaussian distribution by the method of moments and the method of maximum likelihood. Also we compare bias and mean square error (MSE) for both methods.

2. The Density Functions of the Inverse Gaussian (IG) Distribution with the New Parameters and Classical Parameters.
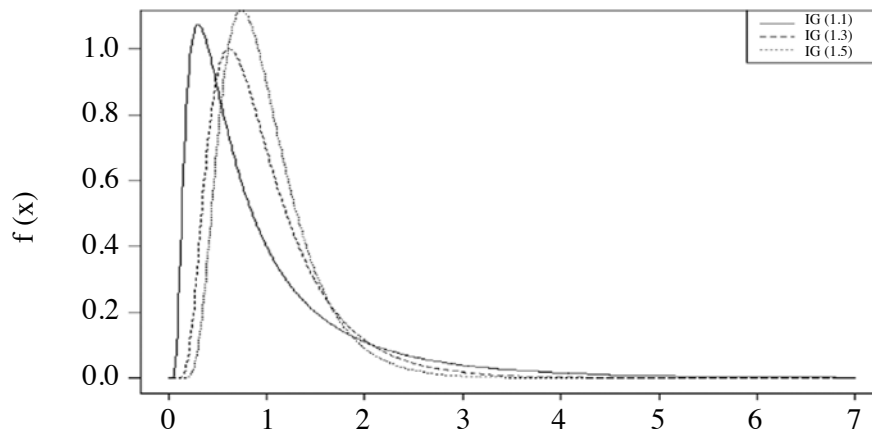
The density function of the Inverse Gaussian distribution with classical parametrization can be written as

$$f_{IG}(x; \mu, \beta) = \sqrt{\frac{\beta}{2\pi}} \, x^{-3/2} \, exp\left\{ -\frac{\beta(x-\mu)^2}{2\mu^2 x} \right\}, x > 0. \tag{1}$$

while with the new parametrization that we consider in this paper, it is:

$$f_{IGnew}(x; \lambda, \theta) = \frac{\lambda}{\theta\sqrt{2\pi}} \left(\frac{\theta}{x}\right)^{3/2} exp\left\{ -\frac{1}{2}\left(\lambda\sqrt{\frac{\theta}{x}} - \sqrt{\frac{\theta}{x}}\right)^2 \right\}, x > 0. \tag{2}$$

In the following we consider the Inverse Gaussian distribution only with the new parametrization and we denote it as IG (λ, θ). As we mentioned above, this re-parametrization fits the study of materials and new parameters have direct physical interpretation. Interrelations between the usual parameters μ, β and the new parameters θ and λ are as follows:

Fig.1. Probability density function for classical Inverse Gaussian distribution
when $\lambda = 1$ and $\theta = 1, 3, 5$

$$\lambda = \frac{\beta}{\mu}, \qquad \mu = \lambda\,\theta, \tag{3}$$

$$\theta = \frac{\mu^2}{\beta}, \qquad \beta = \lambda^2\,\theta. \tag{4}$$

In Fig.1. we present graphs of a few densities functions of the Inverse Gaussian distribution for various values of parameters.

3. The Estimation Method.

3.1 Method of Moments. (Berger and Casella, 1990) [8]

Let $X_1,...,X_n$ be a sample from a population with pdf. $f(x:\theta_1...,\theta_k)$. Method of moments estimators are found by equating the first k sample moments to the correspond*i*ng k population, and solving the resulting system of simultaneous equations. More precisely, define

$$m_1 = \frac{1}{n}\sum_{i=1}^{n} X_i \;,\; \mu_1 = E(X)$$

$$m_1 = \frac{1}{n}\sum_{i=1}^{n} X^2_i \;,\; \mu_2 = E(X^2)$$

$$.$$
$$.$$
$$.$$

$$m_k = \frac{1}{n}\sum_{i=1}^{n} X^k_i \;,\; \mu_k = E(X^k) \tag{5}$$

The population moment μ_j will typically be a function of $\theta_1,...,\theta_k$, say $\mu_j(\theta_1,...,\theta_k)$. The method of moments estimator $(\hat{\theta}_1,...,\hat{\theta}_k)$ of $(\theta_1,...,\theta_k)$ is obtained by solving the following system of equation for $(\theta_1,...,\theta_k)$ in terms of $(m_1,...,m_k)$:

$$m_1 = \mu_1(\theta_1,...,\theta_k)$$
$$m_1 = \mu_2(\theta_1,...,\theta_k).$$
$$.$$
$$.$$
$$.$$
$$m_1 = \mu_k(\theta_1,...,\theta_k). \tag{6}$$

For each integer n, the nth moment of X (or $F_x(x)$), μ'_n, is $\mu'_n = E(X^n)$ The nth central moment of X, μ_n, is $\mu_n = E(X-\mu)^n$, where $\mu = \mu'_1 = E(X)$

3.2 Maximum Likelihood Method (Hogg and Tenis, 2001, p.342-343) [9]

Let $X_1, X_2,...,X_n$ be a random sample in a distribution that depends on one or more unknown parameters $\theta_1, \theta_2,...,\theta_m$ with probability density function denoted by $f(x : \theta_1, \theta_2,...,\theta_m)$. Suppose that ($\theta_1$, $\theta_2,...\theta_m$) is restricted to a given parameter space Ω. Then the joint probability density function of $X_1, X_2,...,X_n$ is:

$$L(\theta_1,\theta_2,...,\theta_m) = f(x_1 : \theta_1,\theta_2,...,\theta_m) f(x_2 : \theta_1,\theta_2,...,\theta_m)$$
$$... f(x_n : \theta_1,\theta_2,...,\theta_m), \theta_1,\theta_2,...,\theta_m \in \Omega \tag{7}$$

When regarded as a function of $\theta_1, \theta_2,...,\theta_m$, is called the likelihood function. Say $[u_1(X_1,X_2,...,X_n), u_2(X_1,X_2,...,X_n),...,u_m(X_1,X_2,...,X_n)]$ is that m –tuple in Ω that maximizes $L(\theta_1,\theta_2,...,\theta_m)$.

$$\text{Then } \hat{\theta}_1 = u_1(X_1,X_2,...,X_n)$$
$$\hat{\theta}_2 = u_2(X_1, X_2,...,X_n)$$
$$.$$
$$.$$
$$.$$
$$\hat{\theta}_m = u_m(X_1,X_2,...,X_n) \tag{8}$$

are maximum likelihood estimators of $\theta_1, \theta_2,...,\theta_m$, respectively; and the corresponding observation of these statistics, namely $u_1(x_1,x_2,...,x_n), u_2(x_1,x_2,...,x_n),...,u_m(x_1,x_2,...,x_n)$ are called maximum likelihood estimates. In many practical cases, these estimators (and estimates) are unique.

4. Methodology

We study the characteristic function of the Inverse Gaussian distribution. Next, we calculate the first two moments for the new parametrization and derive the maximum likelihood for the Inverse Gaussian distribution. Moreover, we derive the solution of the method of moments and the maximum likelihood equations. Finally, we compare bias and MSE of both parameters and both estimation methods for the Inverse Gaussian distribution.

4.1 Calculation of the First Two Moments for the Inverse Gaussian (IG) Distribution.

The characteristic function of a continuous random variable is defined as

$$\phi_x(t) = E(e^{itx}) = \int_{-\infty}^{\infty} e^{itx} f(x,\theta)dx. \tag{9}$$

We found the characteristic of the IG (λ,θ) distribution

$$\phi_{IG}(t;\theta,\lambda) = \exp\{\lambda[1-(1-2i\theta t)^{\frac{1}{2}}]\} \tag{10}$$

Next, we calculated $\ln \phi_{IG}$ (t; λ,θ) and usual series expansion to find first two cummulants This expansion gives the first population moment $\mu'_1 = \lambda\theta$, and gives the second central population moment $\mu'_2 = \theta^2$ of the Inverse Gaussian (IG) distribution.

4.2 The Solution of the Method of Moments Equations

We will estimate equations for two parameters of the Inverse Gaussian (IG) distribution when

$$\begin{aligned} m_1 &= \mu'_1 = \overline{X} \\ m_2 &= \mu'_2 = \frac{1}{n}\sum(X-\overline{X})^2 \end{aligned} \tag{11}$$

where

m_1 is first sample moment

m_2 is second central sample moment

Then, we find the first central population moment (k_1) equal to the first central sample moment (m_1) and the second central population moment (k_2) equal to the second central sample moment (m_2),so we solve the parameters λ, θ for the estimator.

The method of moments estimates for two parameters of Inverse-Gaussian distribution are

$$\hat{\lambda} MME = \sqrt{\frac{(\sum_{i=1}^{n} X_i)^2}{n\sum(X-\overline{X})^2}} \qquad \hat{\theta} MME = \sqrt{\frac{(\sum_{i=1}^{n} X_i - \overline{X})^2}{n}} \tag{12}$$

4.3 The Solution of Maximum Likelihood Equations

We derive the maximum likelihood function of the Inverse Gaussian (IG) distribution is denoted as $f_{IG}(X : \lambda, \theta)$, and the likelihood function:

$L(x : \lambda, \theta = f_{IG}(X_1; \lambda, \theta) f_{IG}(X_2; \lambda\theta...f_{IG}(X_n; \lambda, \theta)$ Then, maximize $L(x: \lambda,\theta)$ by setting $\lambda = 0$ and minimize $L(x: \lambda,\theta)$ by setting $\theta = 0$.

After that we find θ and λ for the estimators. Moreover, we find the second derivative for checking the estimators that provide maximum of the likelihood function. The estimates for two parameters of IG- distribution by the maximum likelihood method are

$$\widetilde{\lambda}_{MLE} = \frac{n}{T\overline{X}-n} \quad , \quad \widetilde{\theta}_{MLE} = \frac{T\overline{X}^2 - \sum_{i=1}^{n} X_i}{n}, \tag{13}$$

$$\text{where} \quad T = \sum_{i=1}^{n} \frac{1}{X_i}.$$

5. Findings and Results

After that, we generate an Inverse Gaussian distribution and compute the estimates of parameters θ, λ. We use the following values of parameters for simulations $\theta = 0.5, 1, 5, 10,$ and $50, \lambda = 0.5, 1, 5, 10,$ and 50 and sample sizes n = 10, 50, and 100 by using the R program version 2.11.1. Then, we calculate bias and MSE for both parameters.

Table 1. A comparison percentage of absolute relative bias and MSE of the estimator θ ($\theta = 0.5$) and λ
($\lambda = 0.5, 1, 5, 10, 50$) by the method of moments and maximum likelihood method for n = 10.

Parameter		$\dfrac{\mid bias \mid}{parameter}$ X 100 of $\hat{\theta}$		$\dfrac{\mid bias \mid}{parameter}$ X 100 of $\hat{\lambda}$		MSE of θ		MSE of λ	
θ	λ	MME	MLE	MME	MLE	MME	MLE	MME	MLE
0.5	0.5	0.99970·	0.99998	0.99990	0.99960·	0.000022·	0.000025	0.001765	0.000001·
	1	0.99990·	0.99994	0.99970·	0.99983	0.000002	0.000002	0.000001·	0.000011
	5	0.99970·	0.99978	0.99993·	0.99996	0.000001·	0.000025	0.000296·	300.6734
	10	0.99980·	0.99998	0.99994	0.99923·	0.000159	0.000025·	0.005171·	310.6352
	50	0.99950·	0.99998	0.99997	0.99879·	0.002569	0.000143·	0.206127	0.115932·

Note:* represents the minimum values.
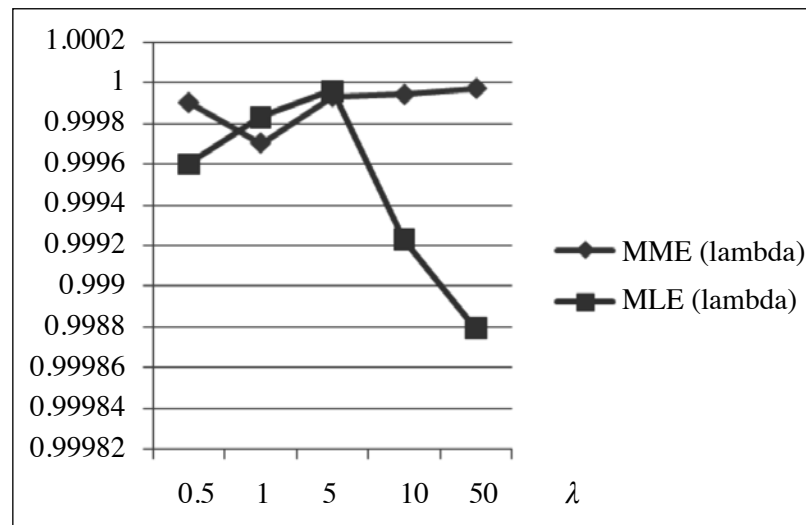
absolute relative bias of the parameter θ

where n = 10, $\theta = 0.5, 1, 5, 10, 50$ and $\lambda = 0.5, 1, 5, 10, 50$ in Tables 1-5. In Tables 6-10 we provide a comparison of bias and MSE of estimates by method of moments and maximum likelihood method where n = 50, $\theta = 0.5, 1, 5, 10, 50$ and $\lambda = 0.5, 1, 5, 10, 50$. Finally, in Tables 1-3 we give a comparison of bias and MSE of estimates by method of moments and maximum likelihood method where n = 100, $\theta = 0.5, 1, 5, 10, 50$ and $\lambda = 0.5, 1, 5, 10, 50$

As can be seen from Table 1, we begin with the simulation results for the percentage of absolute relative bias parameter θ and λ. The percentages of absolute relative bias parameter θ of MME are smaller than that of MLE for all situations. In addition, the MME also provides the smallest value of percentage of absolute relative bias parameter λ when $\lambda = 1$ and 5. In summary, the method of moments (MME) is more efficient than the maximum likelihood method (MLE) in parameter θ for the percentage of absolute relative bias criterion, which is shown in Fig.2.
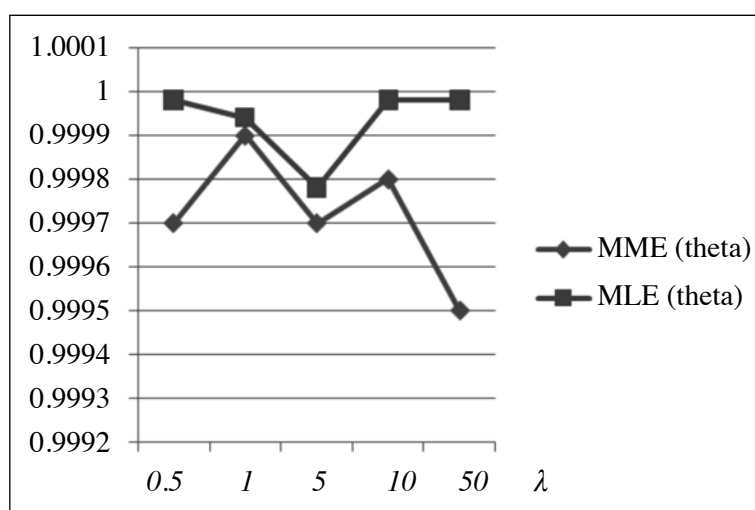
Fig.2. A comparison percentage of absolute relative bias of the parameter θ by MME and MLE when θ ($\theta = 0.5$), λ ($\lambda = 0.5, 1, 5, 10, 50$) and n = 10.

absolute relative bias of the parameter λ

Fig.3. A comparison percentage of absolute relative bias of the parameter λ by MME and MLE when θ ($\theta = 0.5$), λ ($\lambda = 0.5, 1, 5, 10, 50$) and n = 10.
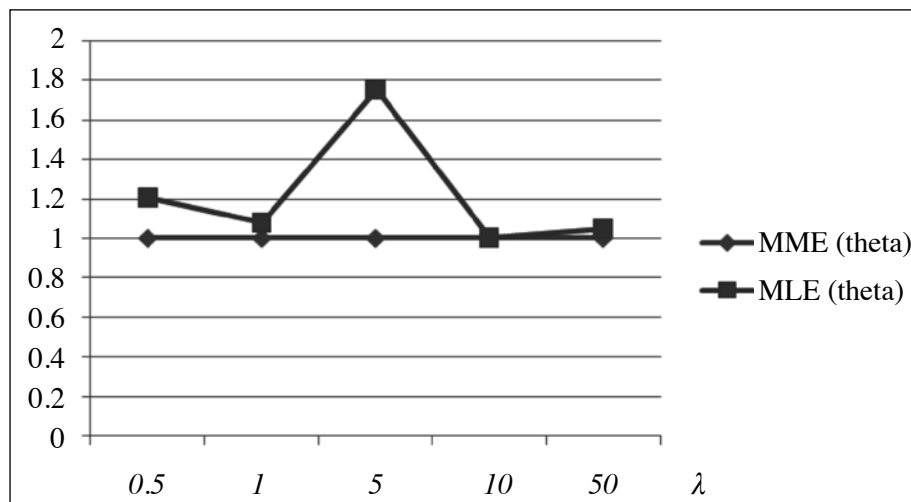
Table 2. A comparison percentage of absolute relative bias and MSE of the estimator θ (θ=5) and λ ($\lambda = 0.5, 1, 5, 10, 50$) by method of moments and maximum likelihood method where n = 50.

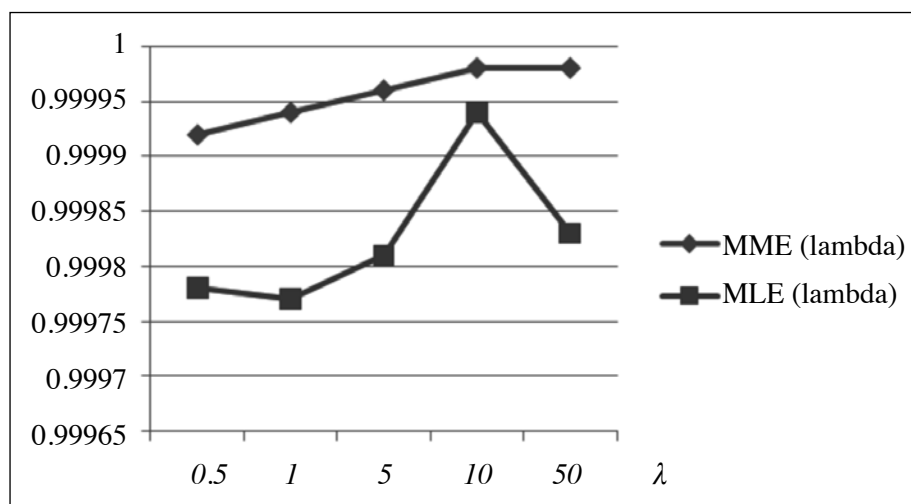| Parameter		$\dfrac{\lvert bias \rvert}{parameter}$ x 100 of $\hat{\theta}$		$\dfrac{\lvert bias \rvert}{parameter}$ x 100 of $\hat{\lambda}$		MSE of $\hat{\theta}$		MSE of $\hat{\lambda}$	
θ	λ	MME	MLE	MME	MLE	MME	MLE	MME	MLE
0.5	0.5	0.99987˙	1.20536	0.99992	0.99978˙	0.00085˙	0.000160	0.00005˙	0.00030
	1	0.99980˙	1.08254	0.99994	0.99977˙	0.00019	0.00035	0.00003	0.00001˙
	5	0.99975˙	1.75234	0.99996	0.99981˙	0.00958	0.00516˙	0.00107	0.00086˙
	10	0.99960˙	1.00356	0.99998	0.99994˙	0.00034˙	0.00236	0.00367˙	11.5498
	50	0.99944˙	1.04852	0.99998	0.99983˙	0.00169	0.00003˙	5.32695	0.26477˙

Note:* represents the minimum values.

absolute relative bias of the parameter θ

As can be seen from Table 2, we consider the percentage of absolute relative bias parameter θ and λ. The percentages of absolute relative bias parameter θ of MME are smaller than that of MLE in all cases. In addition, the MLE is smaller than that of MME value of percentage of absolute relative bias parameter λ in all cases too. In summary, the method of moments (MME) is more efficient than the maximum likelihood (MLE) method for the estimate of parameter θ but the maximum likelihood method (MLE) is more efficient than the method of moments (MME) for the estimate of parameter λ in criteria percentage of absolute relative bias as shown in Figs. 4-5.

Fig.4. A comparison percentage of absolute relative bias of the parameter θ by MME and MLE when θ (θ = 5), λ (λ = 0.5, 1, 5, 10, 50) and n = 50.

absolute relative bias of the parameter λ

Fig. 5. A comparison percentage of absolute relative bias of the parameter λ by MME and MLE when θ (θ = 5), λ (λ = 0.5, 1, 5, 10, 50) and n = 50.

Table 3. A comparison percentage of absolute relative bias and MSE of the estimator θ ($\theta = 1$) and λ ($\lambda = 0.5, 1, 5, 10, 50$) by method of moments and maximum likelihood method where n = 100.

| Parameter		$\dfrac{\lvert bias \rvert}{parameter}$ X 100 of $\hat{\theta}$		$\dfrac{\lvert bias \rvert}{parameter}$ X 100 of $\hat{\lambda}$		MSE of $\hat{\theta}$		MSE of $\hat{\lambda}$	
θ	λ	MME	MLE	MME	MLE	MME	MLE	MME	MLE
0.5	0.5	0.99988*	0.99991	0.99992	0.99988*	0.00004*	0.01328	0.00005	0.00002*
	1	0.99996	0.99976*	0.99974*	0.99996	0.00001*	0.00032	0.00001*	0.00005
	5	0.99983*	0.99993	0.99994	0.99985*	0.00009	0.00002*	0.82475	0.00029*
	10	0.99978*	0.99996	0.99996	0.99973*	0.00054*	0.00399	0.00495*	0.00721
	50	0.99914*	0.99989	0.99999	0.99990*	0.00451	0.00009*	0.19160	0.06269*

Note:* represent the minimum values.

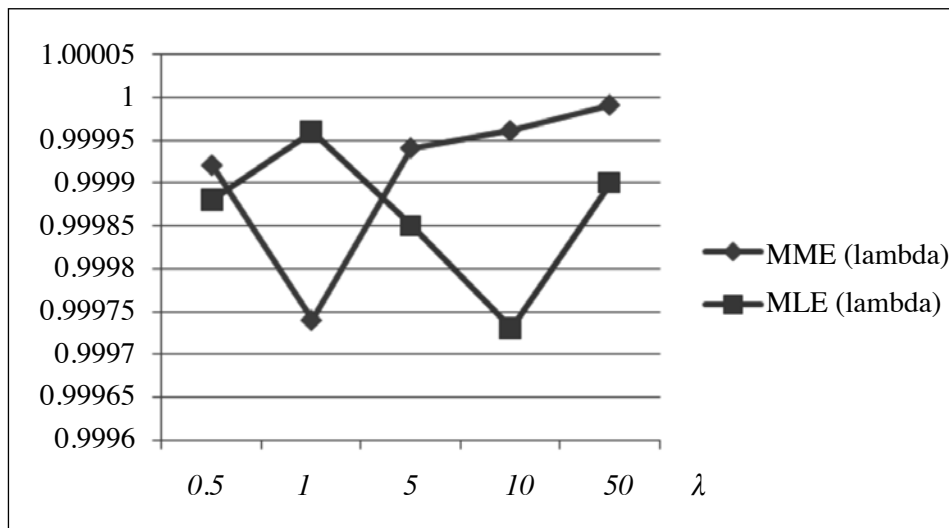absolute relative bias of the parameter θ

As can be seen from Table 3, we consider the percentage of absolute relative bias of the parameters θ and λ . The percentages of absolute relative bias of the parameter θ of MME are smaller than that of MLE when λ =, 1, 5, 10, and 50. In addition, the MLE is smaller than that of MME value of percentage of absolute relative bias parameter λ when λ = 1, 5, 10, and 50.

In summary, the percentage of absolute relative bias of parameter θ the method of moments (MME) is more efficient than the maximum likelihood method (MLE) but parameter λ the maximum likelihood method (MLE), is more efficient than the method of moments (MME) and shown in Fig.6-7.
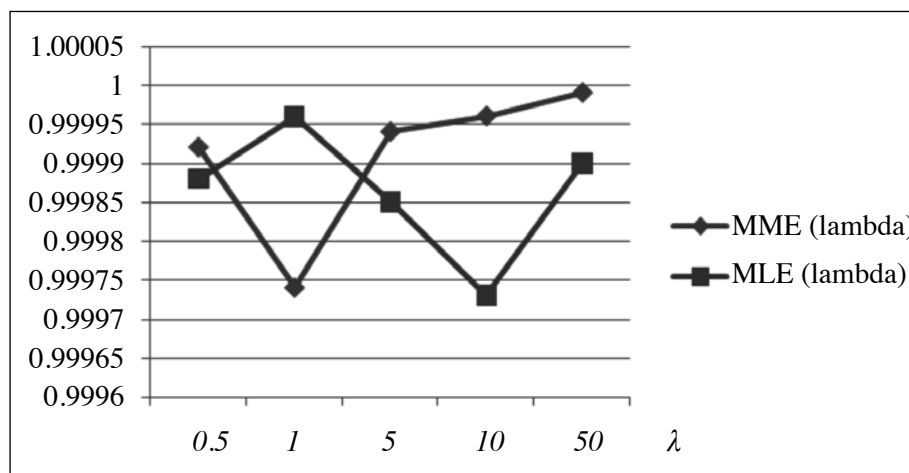
Fig.6. A comparison percentage of absolute relative bias of the parameter θ by MME and MLE when θ (θ = 1), λ ($\lambda = 0.5, 1, 5, 10, 50$) and n = 100.

absolute relative bias of the parameter λ

Fig.7. A comparison percentage of absolute relative bias of the parameter θ by MME and MLE when θ ($\theta = 1$), λ ($\lambda = 0.5, 1, 5, 10, 50$) and n = 100.

6. Conclusions

6.1 Research Conclusions

After we found the formulas for estimators, we generated the random numbers that follow the Inverse Gaussian .When sample sizes are 10, the method of moments tends to be more efficient than the maximum likelihood method for both estimates. When sample sizes are 50, for estimates of θ, the method of moments tends to be more efficient than the maximum likelihood method. For estimate of λ, the maximum likelihood method tends to be more efficient than the method of moments. When sample sizes are 100, for estimates of θ, the method of moments tends to be more efficient than the maximum likelihood method. For stimates of parameter λ, the maximum likelihood method tends to be more efficient than the method of moments.

6.2 Discussion and Future research

In this article we discussed point estimation of new parameters and for the Inverse Gaussian distribution. User should use the maximum likelihood method more than the method of moments because the maximum likelihood estimates are easier to calculate. But the method of moments is more efficient than the maximum likelihood method in some cases.

For questions of interval estimation, the test of hypothesis remains to be considered for future research. If a bias correction is applied and an appropriate distribution is the for establishing critical values, then target size and a reasonably good test can be achieved for moderate sample sizes. We suggest a more computationally intensive nested bootstrap, which calculates critical values of the test statistic from its bootstrapped distribution rather than using tests on the critical value of the *student-t* distribution. To calculate the asymptotically valid variances, covariances, and bias measures, one can use the balanced bootstrapping re-sampling methods. There are several techniques for generating confidence intervals available, for example the percentile methods and bias corrected method with acceleration.

Acknowledgement

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