Determination of Free Surface in Steady-State Seepage through a Dam with Toe Drain by the Boundary Element Method

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Abstract

A numerical method is required for the determination of free surface in seepage through a dam having an arbitrary geometry. The boundary element method has advantages over domain-type methods such as the finite difference method and the finite element method, and is used to solve seepage problems in this paper. Previously, an efficient algorithm has been proposed for solving a seepage problem in a dam with tail water. This paper proposes an algorithm to solve a seepage problem in a dam with toe drain. In such a problem, the dam is transformed into a dam with seepage surface. The width of the dam is then determined iteratively until the height of the seepage surface attains the preset minimum value. Numerical tests on two test problems by the proposed algorithm reveal the agreement between numerical and analytical solutions.

Keywords: porous medium, moving boundary, seepage surface

1. Introduction

Different water levels across the width of a permeable dam result in free-surface flow through the dam. The steady-state free-surface seepage problem is governed by the Laplace equation. The difficulty of such a problem lies in the fact that the free surface is unknown although two boundary conditions are specified at the free surface. Analytical solutions are possible for dams of simple geometry [1, 2]. In order to determine the free surface of the seepage through a dam having arbitrary geometry, a numerical method is required. Although domain-type numerical methods such as the finite difference method [3-5] and the finite element method [6-9] have been successfully used to solve this problem, it is more efficient to use boundary-type methods such as the boundary element method.

The boundary element method has been used by Liggett [10], Chen [11], Leontiev and Huacasi [12], and Rafiezadeh and Ataie-Ashtiani [13] to solve the seepage problem for rectangular dams with tail water. In addition, the method of fundamental solutions, which is another boundary-type method, has also been used by Chaiyo et al. [14] to solve a similar problem. Liggett [10] has presented an efficient algorithm for determining the free surface in this problem. However, there is difficulty in using this algorithm to determine the free surface of seepage through a dam with toe drain. It is therefore interesting to investigate the modification of this algorithm to extend its applicability.

In this paper, the modification of Liggett's algorithm for solving the seepage problem for dams with toe drain is proposed. It is then shown that the modified algorithm can provide accurate solutions for two test problems. The following sections present the seepage theory, the iterative algorithm for determining the free surface of seepage through a dam with toe drain, and a comparison of numerical results for test problems with analytical results in order to demonstrate the effectiveness of the proposed algorithm.

2. Seepage Problems

It is assumed that a dam is filled with saturated porous medium that is both isotropic and homogeneous. Furthermore, the dam length is sufficiently large that the two-dimensional assumption is valid. As a result, the governing equation of the seepage problem is the Laplace equation:

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} = 0 \quad (1)$$

where ϕ is piezometric head, defined as

$$\phi = y\frac{p}{\gamma} \quad (2)$$

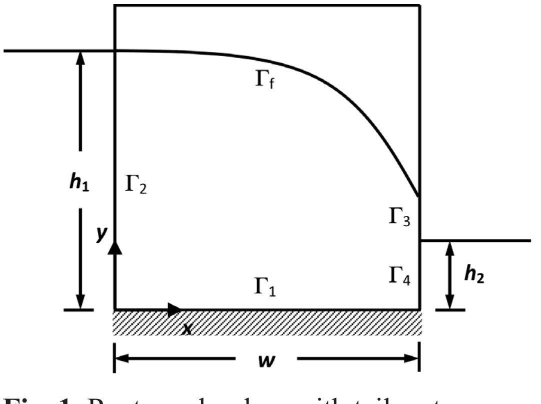

Fig. 1. Rectangular dam with tail water.

Fig. 1. illustrates the seepage problem involving a rectangular dam having an impervious foundation and different water levels across its width. Boundary condition on Γ_1 is the no-flow condition:

$$\frac{\partial\phi}{\partial n} = 0 \quad (3)$$

where n is the coordinate normal to the bound-ary. Since Γ_2 is subjected to hydrostatic pressure, its boundary condition is

$$\phi = h_1 \quad (4)$$

Γ_3 is the seepage surface, on which

$$\phi = y \quad (5)$$

The boundary condition of Γ_4 is similar to that of Γ_2.

$$\phi = h_2 \quad (6)$$

Γ_f is the free surface, on which both Eqs. (3) and (5) apply. Both Γ_f and the length of the seepage surface Γ_3 are unknown, and must be determined during the solution process.

Fig. 2 shows a different type of dam, which has a toe drain instead of tail water. Therefore, boundaries Γ_3 and Γ_4 are replaced by Γ_t, of which the boundary condition is

$$\phi = 0 \quad (7)$$

In this problem, the free surface Γ_f and the width of the toe drain Γ_t are determined during the solution process.

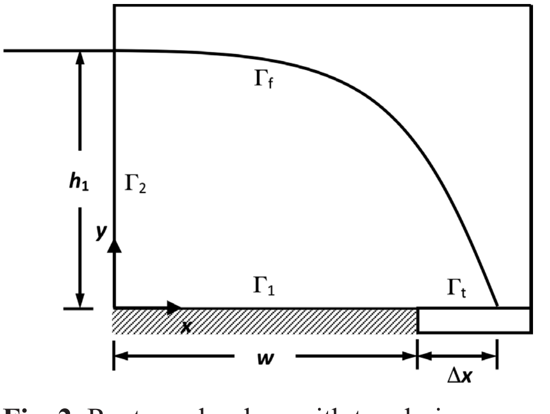

Fig. 2. Rectangular dam with toe drain.

3. Iterative Algorithm

The boundary integral equation corresponding to Eq. (1) is

$$c(x, y)\phi(x, y)= \int \left[\phi(\xi,\eta) \frac{\partial G}{\partial n} (x,y;\xi,\eta) - \frac{\partial \phi}{\partial n} (\xi,\eta) G(x,y;\xi,\eta) \right] d\Gamma \quad (8)$$

where (ξ, η) are coordinates on Γ, and G is the fundamental solution of the Laplace equation:

$$G(x, y; \xi, \eta) = \frac{1}{2\pi} \ln \left[(x-\xi)^2 + (y-\eta)^2 \right] \quad (9)$$

The standard procedure of transforming Eq. (8) into the system of algebraic equations is well-known and described in detail elsewhere [15]. This system of equations may be written as the following matrix equation:

$$\mathbf{A} \frac{\partial \phi}{\partial n} - \mathbf{B}\phi = 0 \quad (10)$$

Equation (10) may be rearranged so that known ϕ and $\partial\phi/\partial n$ are moved to the right hand side. The resulting equation is then solved for unknown ϕ and $\partial\phi/\partial n$.

The seepage problem in Fig. 1 may be solved by the iterative algorithm proposed by Liggett [10]. Because the free surface Γ_f is unknown, an initial guess is needed. Assume that coordinates of a free-boundary node on Γ_f are (x_i, y_i). Impose boundary conditions in Eqs. (3) - (6), respectively, on Γ_1 - Γ_4. In addition, let the boundary condition imposed on Γ_f be Eq. (1). Eq. (10) can now be solved for ϕ on Γ_f, and Eq. (5) is then used to update y_i on Γ_f with x_i unchanged. Note that updating the right end of Γ_f will change the length of the seepage surface Γ_3. In order to stabilize the iterative process, node distribution on Γ_3 should be uniform; therefore, it is advisable to redistribute nodes on Γ_3 after each iteration. The flow chart of this algorithm is shown in Fig. 3.

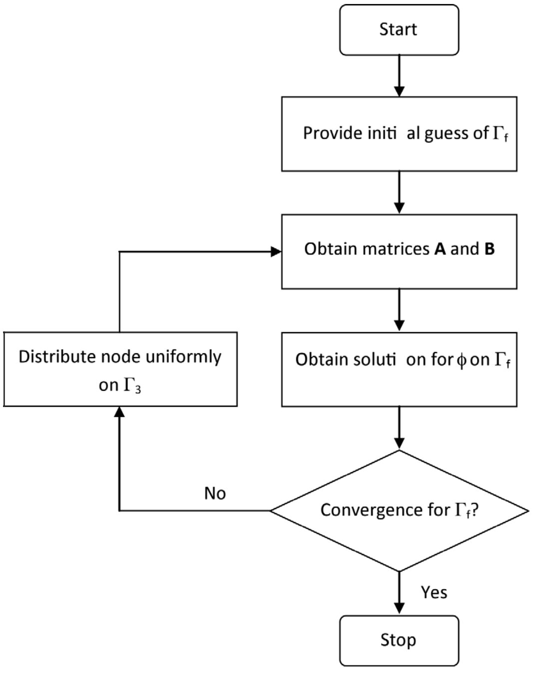

Fig. 3. Algorithm for solving the seepage problem in Fig. 1 according to Liggett [10].

It may be expected that the seepage problem in Fig. 2 can also be solved by the above algorithm. However, an important difference between this problem and the seepage problem in Fig. 1 is that the width of Γ_f in this problem is unknown, and must be found iteratively. It turns out that the converged solution cannot be found in this case. Demetracopoulos and Hadjitheodorou [16] have proposed an alternative algorithm, in which coordinates of a free-boundary node are updated from (x_i, y_i) to (ψ_i, ϕ_i) with ϕ_i equal to the value of ϕ at (x_i, y_i) from the solution of Eq. (10) and

$$\psi_i = x_i \left(\frac{\phi_i - y_s}{y_i - y_s} \right) \quad (11)$$

where y_s is an arbitrarily chosen negative y-coordinate. It should be noted that, with $y_s = -\infty$, this algorithm is identical with Liggett's algorithm. This algorithm was successfully used to solve problems of seepage from surface canals [16]. Unfortunately, it is found that this algorithm cannot yield a satisfactory solution to the see-page problem in Fig. 2.

Instead of solving the seepage problem in Fig. 2. directly, the seepage problem in Fig. 4 is considered. This is the problem of seepage through a dam with seepage surface and toe drain. This problem is similar to the seepage problem in Fig. 2 except for the fact that $\Delta x' < \Delta x$, which results in the existence of the seepage surface Γ_3. The flow chart of the algorithm for solving this problem is shown in Fig. 5. The iteration process consists of inner iteration and outer iteration. At the beginning of the outer iteration, the initial guess of Γ'_t and Γ_f is required. During inner iteration, the width of Γ'_f is fixed, and Liggett's algorithm is used to find converged Γ'_f and Γ_3. The outer iteration then continues with Γ'_t being adjusted so that the length of Γ_3 is equal to a preset value h_{min}. Ideally, h_{min} should be set to zero, which will turn the seepage problem in Fig. 4 into the seepage problem in Fig. 2. In practice, however, there is a minimum value of h_{min}, below which outer iteration may be unstable. It should be noted that Γ_f is normal to Γ_t for the seepage problem in Fig. 2 [17]. This means that a smaller value of hmin requires a finer grid spacing in order to capture the profile of Γ_f near Γ_t accurately.

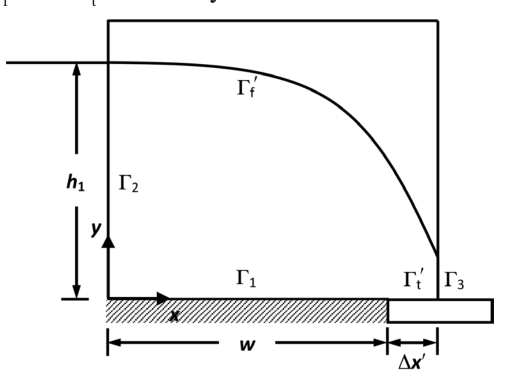

Fig. 4. Rectangular dam with seepage surface and toe drain.

The proposed algorithm shown in Fig. 5 is found to be effective and stable if the initial guess of Γ'_t is close to Γ_t. In practice, however, the initial guess of Γ'_t may be quite different from Γ_t as it is usually convenient to start the iteration process with $\Delta x' = 0$. This may cause the iteration process to be unstable. An effective method to remedy this problem is use under-relaxation in updating Γ'_t. In addition, instability may be caused by starting iteration with $\Delta x' > \Delta x$, which may result in y-coordinates of some nodes on Γ'_f falling below zero. In order to avoid such instability, inner iteration should be terminated if the y-coordinate of any node on Γ'_f turns out to be negative.

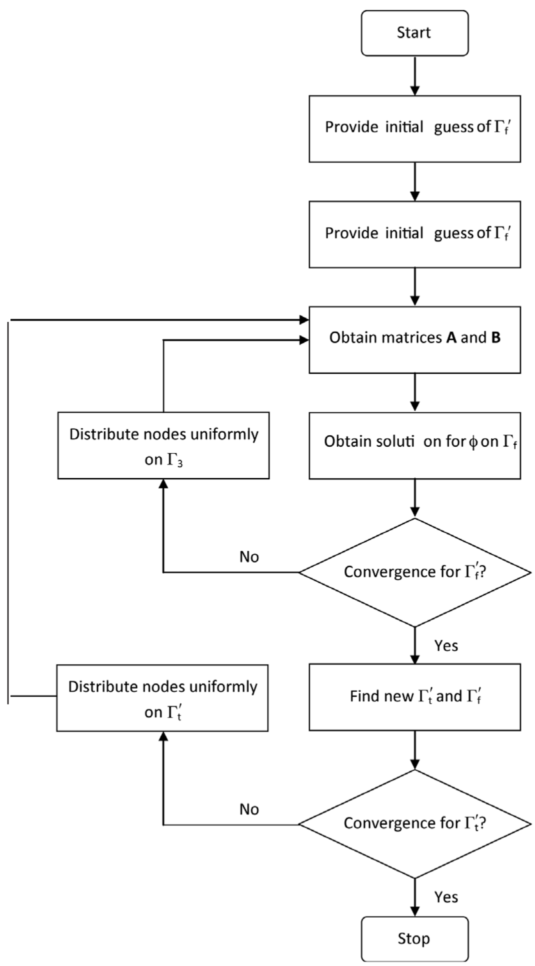

Fig. 5. Algorithm for solving the seepage problem in Fig. 4.

4. Results and Discussion

Liggett's algorithm shown in Fig. 3 is used to solve the seepage problem in Fig. 1 in which h_1 = 1, w = 2/3, and h_2 = 1/6. Initially, it is assumed that Γ_f is the horizontal line at y = 1. Converged solution of Γ_f is obtained by using 144 boundary nodes and 72 quadratic boundary elements. Quadratic elements are also used produce subsequent results. The numerical solution of Γ_f is shown in Fig. 6 and compared with the analytical solution [2]. It can be seen that the solution is quite accurate.

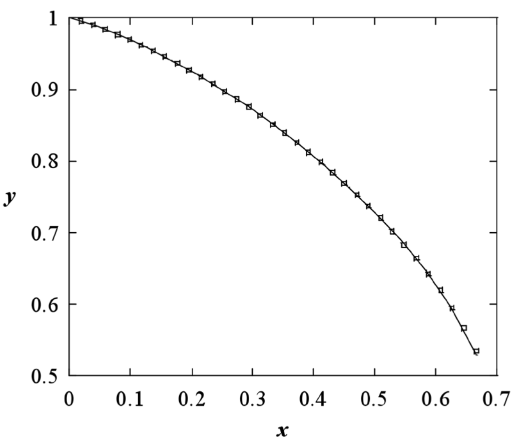

Fig. 6. Comparison of solutions of free surface of the seepage problem in Fig. 1 from the boundary element method (square) and the analytical method (solid line) [2].

The proposed algorithm shown in Fig. 5 is used to solve the seepage problem in Fig. 4 in which h_1 = 10 and w = 10. Initially, it is assumed that Γ_f is the horizontal line at y = 10, and $\Delta x'$ = 0. Converged solutions of Γ'_f are obtained by using 176 and 350 boundary nodes. Figure 7 shows good agreement between Γ'_f of the seepage problem in Fig. 6 determined by the proposed algorithm and Γ_f of the seepage problem in Fig. 2 determined by the analytical method [1].

Fig. 8 illustrates the seepage problem in a trapezoidal dam with toe drain. The analytical solution to this problem was given by Polubarinova-Kochina [2]. Like the seepage problem in Fig. 2, this problem can be solved numerically by transforming the dam in Fig. 8 into a dam with seepage surface. Initially, it is assumed that Γ_f is the horizontal line at y = 1, with a length of 2.16. Converged solutions of the free surface are obtained by using 150 boundary nodes and 294 boundary nodes. It can be seen from Fig. 9 that numerical solutions are very close to the analytical solution.

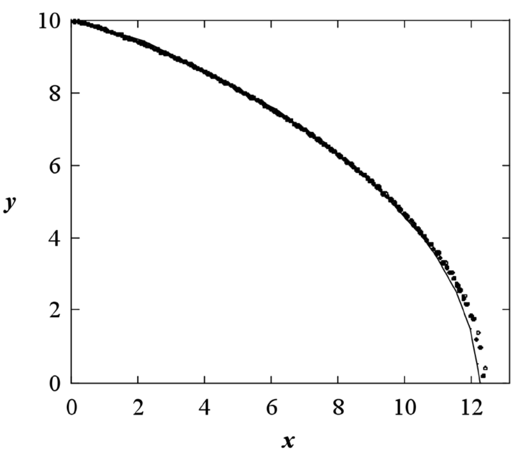

Fig. 7. Comparison of solutions of free surface of the seepage problem in Fig. 4 from the boundary element method using 176 nodes (white circle) and 350 nodes (black circle) and free surface of the seepage problem in Fig. 2 from the analytical method (solid line) [1].

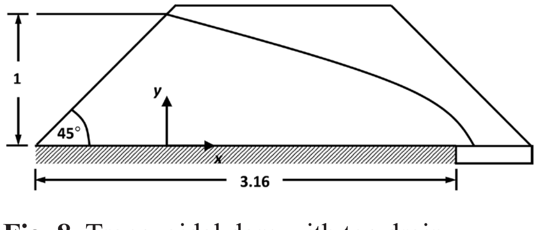

Fig. 8. Trapezoidal dam with toe drain.

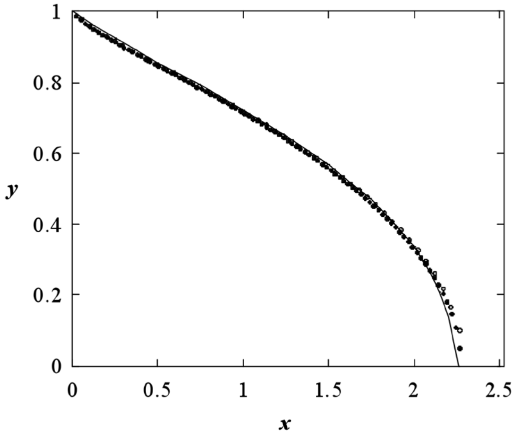

Fig. 9. Comparison of solutions of free surface of the seepage problem in Fig. 8 from the boundary element method using 150 nodes (white circle) and 294 nodes (black circle) and the analytical method (solid line) [2].

5. Conclusion

A steady-state free-surface of seepage through a homogeneous permeable dam with tail water can be efficiently determined by using the boundary element method and the algorithm proposed by Liggett [10]. A problem involving a dam with toe drain and without seepage surface, however, requires an improvement of Liggett's algorithm. In this paper, an iterative algorithm is proposed for determining a steady-state free-surface seepage through a dam with toe drain and seepage surface. For a given toe drain width, Liggett's algorithm is used to find the corresponding seepage surface. The toe drain width is then adjusted iteratively until the seepage surface attains a given small value. The free surface of such a problem is found to be close to the free surface of the corresponding dam with toe drain and without seepage surface. Two test problems are solved by the proposed algorithm to demonstrate its effectiveness.

6. References

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