O-Modularity and O-Distributivity in Semilattices

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Abstract

In this paper we prove that every convex subordered set of an ordered set can be written as an intersection of a down-set and an up-set. We characterize o-modular and o-distributive semilattices in terms of ideals of the semilattices. The notion of o-modular, o-distributive and o-standard elements has been developed. We characterize the relation among the elements.

Key words: Ordered set, convex subordered set, semilattice, o-distributive semilattice, distributive semilattice.

1. Introduction

The study of semilattices has become very important in the study of general algebra. The class of semilattices has an equivalent pictorial subclass of ordered sets. A non-empty set P together with an order relation $\leq$ is said to be an **ordered set**. It is denoted by **P** = $\langle P; \leq \rangle$. The dual order of $\leq$ is denoted by $\geq$. That is, $x \leq y$ if and only if $y \geq x$.

Let **P** be an ordered set and $Q \subseteq P$. Define

$$L(Q) := \{x \in P \mid x \leq a \text{ for all } a \in Q\},$$

$$U(Q) := \{x \in P \mid x \geq a \text{ for all } a \in Q\}.$$

Then L(Q) is said to be the **lower bound** of Q and U(Q) is said to be the **uper bound** of Q. An element $y \in L(Q)$ is said to be the **greatest lower bound** of Q if $x \leq y$ for all $x \in L(Q)$. Dually, an element $y \in U(Q)$ is said to be the **least upper bound** of Q if $y \leq x$ for all $x \in U(Q)$. If the least upper bound of {x, y} exists for each $x, y \in P$, then we say that the ordered set **P** is a join-semilattice as an ordered set. An algebra **S** = $\langle S; \vee \rangle$ is said to be a join-semilattice as an algebra if the binary operation $\vee$ is reflexive, commutative and associative. In this paper, by a semilattice we mean join-semilattice. It is a natural question: whether we can generalize the results of semilattices (or lattices) to ordered sets. A Convex sublattice play an important role in the study of lattice theory (see [2]). In Section 2 we generalize a result of a convex sublattice to a convex subordered set.

The classes of modular and distributive semilattices are very suitable subclasses of semilattices. A semilattice **S** is called a **modular semilattice** if for all a, b, $c \in S$ with $c \leq a \leq b \vee c$ implies the existence of $b_1 \leq b$ such that $a = b_1 \vee c$.

A semilattice **S** is called a **distributive semilattice** if for all a, b, $c \in S$ with $a \leq b \vee c$ implies the existence of $b_1 \leq b$ and $c_1 \leq c$ such that $a = b_1 \vee c_1$. A semilattice

S is directed below if any pair of element of *S* has a common lower bound. It is well known that all modular and distributive semilattices are directed below. Larmerov′a and Rachuneck [4](see also [1]) introduced the modularity and distributivity for an ordered set using only set-theoretical concepts. Rachuneck [5, 6] introduces the notion of (o-modular) o-distributive semi-lattices which are a proper superclass of (modular) distributive semilattices. In Section 3 we discuss the o-modular and o-distributive semilattices.

Let **S** be a semilattice. A non-empty subset I of S is said to be an **ideal** of **S** if

(i) $i \vee j \in I$ for all $i, j \in I$ and

(ii) $i \in I$, $x \in S$ with $x \leq i$ implies $x \in I$.

The set of all ideals of S is denoted by I(S). It is well known that a semilattice **S** is modular (distributive) if and only if I**(S)** is a modular (distributive) lattice. In Section 4 we give some characterizations of o-modular and o-distributive semilattices in terms of ideals.

Modular elements, distributive elements and standard elements in a lattice have been studied by several authors (see [3, 2, 7]). Let **L** be a lattice. An element $m \in L$ is said to be a **modular** element of L if for all $a, b \in L$ with $a \leq b$ implies $a \vee (m \wedge b) = (a \vee m) \wedge b$. An element $d \in L$ is said to be a **distributive** element of L if for all $a, b \in L$ implies $(a \wedge b) \vee d = (a \vee d) \wedge (b \vee d)$. An element $s \in L$ is to be a **standard** element of L if for all $a, b \in L$ implies $a \wedge (s \vee b) = (a \wedge s) \vee (a \wedge b)$. In Section 5 we generalize the idea of modular, distributive and standard elements in a lattice to o-modular, o-distributive and o-standard elements in a join-semilattice.

2. Convex subordered sets

Let **P** be an ordered set. A subset Q of P is said to be a **subordered set** of **P** if **Q** is itself an ordered set where the order in Q is induced by the order of P. A subordered set Q of P is said to be convex if $x, y \in Q$ with $x \leq z \leq y$ implies $z \in Q$. A subordered set Q of P is said to be a **down-set** if $x \in Q$, $y \leq x$ implies $y \in Q$. Dually, Q is said to be **up-set** if $x \in Q$, $y \geq x$ implies $y \in Q$. For $Q \subseteq P$, define:

$\downarrow Q := \{x \in P \mid x \leq y \text{ for some } y \in Q\}$,

$\uparrow Q := \{x \in P \mid y \leq x \text{ for some } y \in Q\}$.

Then $\downarrow Q$ is the down-set and $\uparrow Q$ is the up-set.

The following Theorem is the generalization of the result in case of lattices (see [2]) and as well as semilattices.

Theorem 2.1 *Let P be an ordered set, I be a down-set and D be an up-set such that $I \cap D \neq \phi$. Then $I \cap D$ is a convex subordered set of P. Moreover, every convex subordered set of* **P** *can be written as an intersection of a down-set and an up-set.*

Proof. Let $C = I \cap D$ and let $x, y \in C$ and $z \in P$ such that $x \leq z \leq y$. Then clearly $z \in C$ as I is a down-set and D is an up-set. Hence C is convex. Since both I and D are subordered sets. So, C is a subordered set. Therefore C is a convex subordered set of **P**.

Suppose **C** is a convex subordered set of **P**. We show that $C = \downarrow C \cap \uparrow C$. Clearly, $C \subseteq \downarrow C \cap \uparrow C$. Let $x \in \downarrow C \cap \uparrow C$, then $c_1 \leq x \leq c_2$ for some $c_1, c_2 \in C$. Now since C is convex, we have $x \in C$. Therefore $C = \downarrow C \cap \uparrow C$.

Observe that the intersection of the above theorem is not uniquely determined as the result of lattices. For example, consider the ordered set given in Figure 1. Let C = {a, b}, A = {a, b, 1}, B = {b, a, 0} and D = {c, b, a, 0}. Then $C = A \cap B = A \cap D$.

3. O-modular and O-distributive semilattices

An ordered set **P** is called **modular ordered set** if for all a, b, c$\in$ P with a $\le$ c implies:
L(U(a, L(b, c))) = L(U(a, b), c).

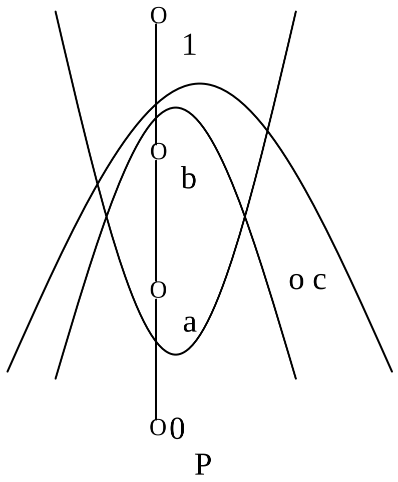

Figure 1

For any ordered set ***P***, it is easy to verify that for all $a, b, c \in P$ with $a \le c$ implies:

$$L(U(a, L(b, c))) \subseteq L(U(a, b), c).$$

So, an ordered set ***P*** is modular if for all $a, b, c \in P$ with $a \le c$ implies :

$$L(U(a, b), c) \subseteq L(U(a, L(b, c))).$$

A semilattice $S = \langle S; \vee \rangle$ is said to be **o-modular** if it is modular as an ordered set. That is, if for all $a, b, c \in S$ with $a \le c$ implies:

$$L(a \vee b, c) \subseteq L(U(a, L(b, c))).$$

An ordered set P is said to be **distributive ordered set** if for all $a, b, c \in P$

$$L(U(L(a, c), L(b, c))) = L(U(a, b), c).$$

For any ordered set P, we have:

$$L(U(L(a, c), L(b, c))) \subseteq L(U(a, b), c),$$

for all $a, b, c \in P$. So, an ordered set ***P*** is distributive if:

$$L(U(a, b), c) \subseteq L(U(L(a, c), L(b, c))),$$

for all a, b, c $\in$ P. A semilattice S = $\langle$S; $\vee\rangle$ is said to be **o-distributive** if it is distributive as an ordered set. That is, if for all a, b, c $\in$ S,

$$L(a \vee b, c) \subseteq L(U(L(a, c), L(b, c))).$$

Clearly, every o-distributive semilattice is o-modular. The converse is not true. For example, the semilattices $\mathbf{M_4}$ and $\mathbf{M_5}$ given in Figure 3 are o-modular but not o-distributive. In the case of lattices, the notion of modularity (distributivity) and o-modularity (o-distributivity) are the same (see [4]). Every modular (distributive) semilattice is o-modular (o-distributive), but the converse is not true. For example, consider the semilattice $\mathbf{M_3}$ given in Figure 2. The semilattice $\mathbf{M_3}$ is not modular (distributive), as it is not directed below. But it can be easily seen that $\mathbf{M_3}$ is o-modular (o-distributive).

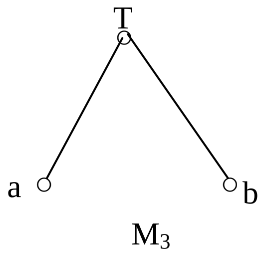

Figure 2

It is easy to show, a subsemilattice of an o-modular (o-distributive) semilattice is o-modular (o-distributive). Let **A** be a subsemilattice of a semilattice **S**. For a, b $\in$A, define $L_A(a, b) = \{x\in A | x \le a, b\}$. If A = S, then we write L(a, b) instead of $L_S(a, b)$. A subsemilattice **A** is said to be an **LU-subsemilattice** of **S** if for all a, b $\in$A, $L_A(a, b) = \phi \Leftrightarrow L(a, b) = \phi$ and **A** is said to be a **strong subsemilattice** of **S** if $U(L_A(a, b)) = U(L(a, b))$ for all a, b$\in$A.

The following results (Theorem 3.1 and Theorem 3.2) are due to Rachunek [5, 6].

Theorem 3.1 Let **S** be a semilattice.
(a) *If **S** is not o-modular, then it contains an LU-subsemilattice isomorphic to one of the ordered sets $\mathbf{P_4}$, $\mathbf{P_5}$ given in Figure 3.*

(b) *If **S** is not o-distributive, then it contains an LU-subsemilattice isomorphic to one of the ordered sets $\mathbf{P_4}$, $\mathbf{P_5}$, $\mathbf{M_4}$, $\mathbf{M_5}$ given in Figure 3.*

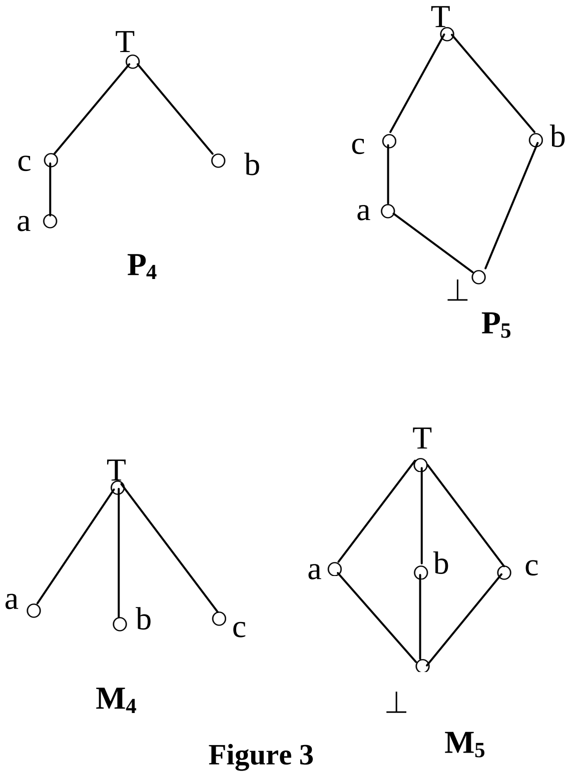

Figure 3

The following theorem is the converse of the above theorem.

Theorem 3.2 Let **S** be a semilattice.
(a) *If **S** contains an LU-subsemilattice isomorphic to the ordered set $\mathbf{P_4}$, or it contains a strong subsemilattice isomorphic to the ordered set $\mathbf{P_5}$, then **S** is not o-modular.*
(b) *If **S** contains an LU-subsemilattice isomorphic to the ordered sets $\mathbf{P_4}$ or $\mathbf{M_4}$, or it contains a strong subsemilattice isomorphic to the ordered set $\mathbf{P_5}$ or $\mathbf{M_5}$, then S is not o-distributive.*

4. Ideals of o-modular and o-distributive semilattices

The semilattice I(**S**) of all ideals of a semilattice **S** is not necessarily a lattice. Define $I_0(S) = I(S) \cup \{\phi\}$. If $I_0(S)$ is ordered by set inclusion, then $I_0(S)$ is a lattice where the supremum and infimum are set-theoretic union and intersection, respectively. Moreover, if **S** is modular (distributive), then, $I_0(\mathbf{S})$ is a modular (distributive) lattice. Rachunek [6] has proved the following result.

Theorem 4.1 Let **S** be a semilattice.
(a) If $I_0(\mathbf{S})$ is modular, then **S** is an o-modular semilattice.
(b) If $I_0(\mathbf{S})$ is distributive, then **S** is an o-distributive semilattice.
We have the following result.

Theorem 4.2 Let **S** be a semilattice.
(a) If I(**S**) is o-modular, then $I_0(\mathbf{S})$ is modular.
(b) If I(**S**) is o-distributive, then $I_0(\mathbf{S})$ is distributive.

Proof.
(a) Let $I_0(\mathbf{S})$ not be modular, then it has a sublattice isomorphic to the pentagon lattice. Thus I(**S**) contains either a LU-subsemilattice isomorphic to $\mathbf{P_4}$, or a strong subsemilattice isomorphic to $\mathbf{P_5}$. Hence by Theorem 3.2, we have I(**S**) is not o-modular. Therefore if I(**S**) is o-modular, then $I_0(\mathbf{S})$ is modular.
(b) Let $I_0(\mathbf{S})$ not be distributive, then it has a sublattice isomorphic to the diamond lattice or pentagon lattice. Thus I(S) contains either a LU-subsemilattice isomorphic to $\mathbf{P_4}$ or $\mathbf{M_4}$ or an strong subsemilattice isomorphic to $\mathbf{P_5}$ or $\mathbf{M_5}$. Hence by Theorem 3.2, we have I(**S**) is not o-distributive. Therefore if I(**S**) is o-distributive, then $I_0(\mathbf{S})$ is distributive.

By Theorem 4.1 and Theorem 4.2 we have the following result.

Corollary 4.3 Let **S** be a semilattice.
(a) If I(**S**) is o-modular, then **S** is an o-modular semilattice.
(b) If I(**S**) is o-distributive, then **S** is an o-distributive semilattice.

By $\mathbf{S_0}$ we mean $\mathbf{S}\oplus\{0\}$, the linear sum of S with a bottom element 0.

Remark 4.1 Let **S** be a semilattice. If **S** is o-modular (o-distributive), then it is not necessary that $\mathbf{S_0}$ is modular (distributive). But if **S** is o-modular (o-distributive) such that $\mathbf{S}_0$ is a lattice, then $\mathbf{S}_0$ is modular (o-distributive).

The converse of the above Corollary 4.3 is not necessarily true. For example, consider the semilattice **M** (**N**) given in the Figure 4. It can be easily seen that the ideal lattice I(**M**) (I(**N**)) is not modular (distributive), and hence is not o-modular (o-distributive).

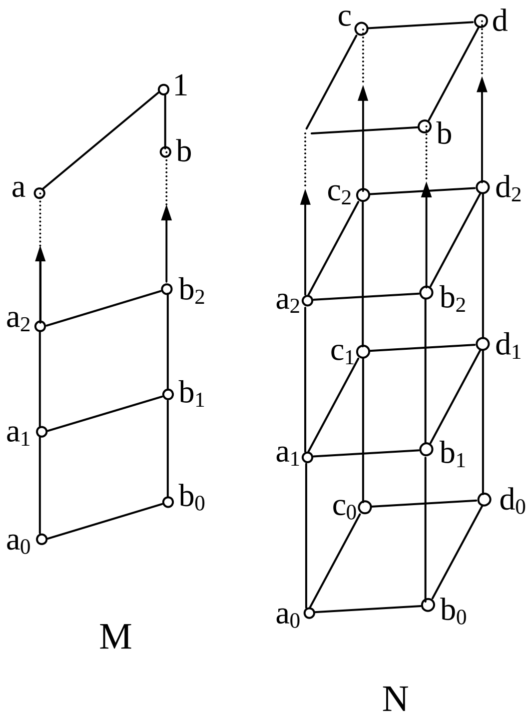

Figure 4 a o-modular join semilattice and its ideal lattice

It is well known that if **S** is a distributive semilattice and ***I, J*** $\in$***I(S),*** then each $x\in I\vee J$. We have $x = i\vee j$ for some $i\in I$ and $j\in J$. This is not true for a o-distributive semilattice. For example, in the o-distributive semilattice N given in the above Figure 4, observe that $c\in B\vee D_0$ but there is no $b\in B$ and $d\in D_0$ such that $c = b\vee d$. Now we have the following important result. By sup$\{A, B\}$ we mean the least upper bound of $A\cup B$.

Theorem 4.4 Let S be a semilattice. Then the followings are equivalent :

(a) S is o-distributive;

(b) for I, J $\in$ I(S) we have

$$I\vee J = \{x \mid x = \sup\{L(i, x), L(j, x)\} \text{ for some } i\in I \text{ and } j\in J\};$$

(c) for any principal ideals I, J of S we have:

$$I\vee J = \{x \mid x = \sup\{L(i, x), L(j,x)\} \text{ for some } i\in I \text{ and } j\in J\}.$$

Proof. (a) $\Rightarrow$(b). Let $x\in I\vee J$. Then $x\le i\vee j$ for some $i\in I$ and $j\in J$. Hence $x\in L(i\vee j, x) = L(U(L(\mathrm{i}, x),L(j, x)))$. This implies $x\le y$ for all $y\in U(L(\mathrm{i}, x), L(j, x))$. Suppose $A = L(\mathrm{i}, x)$ and $B = L(j, x)$. Then clearly $A\subseteq I$ and $B\subseteq J$ and y is an upper bound of $\{A, B\}$. Moreover, x be an upper bound of $\{A, B\}$. Since $x\le y$ for all $y\in U(A,B)$, we have $x = \sup\{A, B\}$.

(b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (a). Assume (c) holds. Let $x\in L(a\vee b, c)$. Then $x\le a\vee b$ and $x\le c$. Thus $x\in (a\vee b] = (a]\vee(b]$ and hence $x = \sup\{L(a_1, x), L(b_1, x)\}$ for some $a_1\le a$ and $b_1\le b$. Therefore, $x\le U(L(a_1, x), L(b_1, x))$. Hence $x\in L(U(L(a_1, x), L(b_1, x)))\subseteq L(U(L(a, c), L(b, c)))$.

5. O-modular, o-distributive, o-standard elements in semilattices

Let S be a semilattice. An element $m\in S$ is said to be an **o-modular** element of S if for all $a, b\in S$ with $a\le b$ implies $L(U(a, L(m, b))) = L(a\vee m, b)$. An element $d\in S$ is said to be a o-distributive element of S if for all $a, b\in S$ implies $U(d,L(a, b)) = U(L(d\vee a, d\vee b))$. An element $s\in S$ is said to be a o-standard element of S if for all $a, b\in S$ implies $L(a, s\vee b) = L(U(L(a, s),L(a, b)))$. Clearly, a semilattice S is o-modular (o-distributive) if and only if its every element is o-modular (o-distributive).

Theorem 5.1 In a lattice the notion of modular element, distributive element and standard element coincide with the notion of o-modular element, o-distributive element and o-standard element, respectively.

Proof. Let **L** be a lattice and let $m \in \boldsymbol{L}$. Then

m is o-modular $\Leftrightarrow$ for all $a, b \in L$ with $a \leq b$, $L(U(a, L(m, b))) = L(a \vee m, b)$
$\Leftrightarrow L(U(a, m \wedge b)) = L(a \vee m, b)$ for all $a,b \in L$
$\Leftrightarrow L(a \vee (m \wedge b)) = L(a \vee m, b)$ for all $a,b \in L$
$\Leftrightarrow (a \vee (m \wedge b)] = ((a \vee m) \wedge b]$ for all $a,b \in L$
$\Leftrightarrow a \vee (m \wedge b) = (a \vee m) \wedge b$ for all $a, b \in L$ with $a \leq b$,
$\Leftrightarrow$ m is modular.

m is o-distributive $\Leftrightarrow$ for all $a, b \in L$;
$L(a \vee b, m) = L(U(L(a, m), L(b, m)))$
$\Leftrightarrow ((a \vee b) \wedge m)] = ((a \wedge m) \vee (b \wedge m)]$ for all $a,b \in L$
$\Leftrightarrow (a \vee b) \wedge m = (a \wedge m) \vee (b \wedge m)$ for all $a, b \in L$,
$\Leftrightarrow$ m is distributive.

m is o-standard $\Leftrightarrow$ for all $a, b \in L$,
$L(a, m \vee b) = L(U(L(a, m), L(a, b)))$
$\Leftrightarrow (a \wedge (m \vee b)] = ((a \wedge m) \vee (a \wedge b)]$
$\Leftrightarrow a \wedge (m \vee b) = (a \wedge m) \vee (a \wedge b)$ for all $a, b \in L$,
$\Leftrightarrow$ m is standard.

Let **S** be a semilattice. An element $m \in S$ is said to be an **modular element of** S if for all $a, b \in S$ with $b \leq a \leq m \vee b$ implies the existence of $m_1 \leq m$ such that $a = m_1 \vee b$. An element $d \in S$ is said to be a **distributive element of** S if for all $x, a, b \in S$ with $x \leq d \vee a$ and $x \leq d \vee b$ implies the existence of $c \leq a, b$ such that $x \leq d \vee c$. An element $s \in S$ is said to be a **standard element** of S if for all $a, b \in S$ with $a \leq s \vee b$ implies the existence of $s_1 \leq s$, $b_1 \leq b$ such that $a = s_1 \vee b_1$.

Theorem 5.2 In a semilattice **S**,

(a) every modular element of S is o-modular,

(b) every distributive element of S is o-distributive,

(c) every standard element of S is o-standard.

The converse of (a), (b) and (c) need not be true.

Proof. (a) Let m be a modular element of S and let $a, b \in S$ with $a \leq b$. We have to show that $L(a \vee m, b) = L(U(a, L(m, b)))$. Let $x \in L(a \vee m, b)$. Then $x \leq a \vee m$ and $x \leq b$. Thus $a \leq a \vee x \leq a \vee m$. Since m is modular, there exists $m_1 \leq m$ such that $a \vee x = a \vee m_1$. If $y \in U(a, L(m, b))$, then $y \geq a \vee r$ for all $r \leq m, b$. Now $m_1 \leq a \vee x \leq b \vee x = b$ and $m_1 \leq m$. Thus $x \leq a \vee x = a \vee m_1 \leq y$. Hence $x \in L(U(a, L(m, b)))$. Therefore $L(a \vee m, b) \subseteq L(U(a, L(m, b)))$. The reverse inclusion is trivial for any semilattice.

(b) Let d be a distributive element of S. We have to prove that for all $a, b \in S$ implies $U(d, L(a, b)) = U(L(d \vee\_a, d \vee b))$.
Let $x \in (U(d, L(a, b))$. Then $x \geq d \vee c$ for all $c \leq a, b$. If $y \in L(d \vee a, d \vee b)$, then $y \leq d \vee a$ and $y \leq d \vee b$. Since d is distributive, there exists $r \leq a, b$ such that $y \leq d \vee r$. Hence $y \leq x$. Thus $x \in U(L(d \vee a, d \vee b))$. Hence $U(d, L(a, b)) \subseteq U(L(d \vee a, d \vee b))$. The reverse inclusion is trivial for any semilattice.

(c) Let s be a standard element of S and let $a, b \in S$. We have to show that $L(a, s \vee b) = L(U(L(a, s), L(a, b)))$. Let $x \in L(a, s \vee b)$. Then $x \leq a$ and $x \leq s \vee b$. Since s is standard, there exists $s_1 \leq s$ and $b_1 \leq b$ such that $x = s_1 \vee b_1$. If $y \in U(L(a, s), L(a, b))$,

then $y \geq p \vee q$ for all p≤ a, s and q ≤ a, b. Since $s_1 \leq x \leq a$, $s_1 \leq s$ and $b_1 \leq x \leq a$, $b_1 \leq b$,
we have $y \geq s_1 \vee b_1 = x$. Hence $x \in L(U(L(a, s), L(a, b)))$. Therefore, $L(a, s \vee b) \subseteq L(U(L(a, s), L(a, b)))$. The reverse inclusion is trivial for any semilattice.

Consider the o-modular semilattice M in the Figure 4. Here a is o-modular, as every element of o-modular semilattice is o-modular, but a is not modular as $b_2 \leq b \leq a \vee b_2$, but there is no $a_1 \leq a$ such that $b = a_1 \vee b_2$. Hence the converse of (a) is not true. Now consider the o-distributive semilattice N in the Figure 4. Here, d_0 is o-modular, as every element of o-distributive semilattice is o-distributive, but d_0 is not distributive as $c \leq d_0 \vee b$ and $c \leq d_0 \vee c$, but there is no $a \leq b, c$ such that $c \leq d_0 \vee a$. Hence, the converse of (b) is not true. By a similar argument we can show that b is o-standard, but not standard.

Theorem 5.3 Let S be a semilattice and let s ∈ S. Then the following are equivalent:
(a) s is o-standard ,
(b) s is o-modular and o-distributive,

Proof. Suppose s is o-standard. Let $a, b \in S$ with $a \leq b$. Then

$$L(a \vee s, b) = L(U(L(a, b), L(s, b))) \quad \text{as } s \text{ is o-standard}$$
$$= L(U(a, L(s, b))).$$

Therefore, s is o-modular. To prove s is o-distributive, we have to show that for all $a, b \in S$ implies

$$U(s, L(a, b)) = U(L(a \vee s, b \vee s)).$$

We have $U(L(a \vee s, b \vee s))$

$$= U(L(U(L(a \vee s, b), L(a \vee s, s)))) \quad \text{as } s \text{ is o-standard}$$
$$\supseteq U(L(a \vee s, b), L(a \vee s, s)) \quad \text{as } U(L(A)) \supseteq A$$
$$= U(L(s), L(U(L(a, b), L(s, b)))) \quad \text{as } s \text{ is o-standard}$$
$$\supseteq ULU(L(a, b), s)$$
$$\supseteq U(L(a, b), s).$$

The reverse inclusion is trivial for any semilattice. Hence s is o-distributive. Conversely, let (b) hold. To prove (a), it is enough to show that $L(a \vee s, b) \subseteq L(U(L(a, b), L(s, b)))$ for all $a, b \in S$. We have:

$$L(a \vee s, b) \subseteq LU(L(a \vee s, b), s)$$
$$\subseteq LUL(a \vee s, b \vee s) \text{ (trivial)}$$
$$= L(a \vee s, b \vee s)$$
$$= LU(L(a, b), s), \text{ as } s \text{ is o-distributive}$$

Taking the intersection with $L(b)$ on both sides, we have $L(a \vee s, b) \subseteq L(U(L(a, b), s), b)$.
Now we shall show that $U(L(a, b), L(b, s)) \subseteq UL(U(L(a, b), s), b)$. To prove this, let:
$x \in U(L(a, b), L(b, s))$
$\Rightarrow x \geq p, q$ for all $p \leq a, b$ and $q \in L(b, s)$
$\Rightarrow x \in U(L(b, s), p)$ for all $p \leq a, b$
$\Rightarrow x \in ULU(L(b, s), p)$
$\Rightarrow x \in UL(U(p, s), b)$ as s is o-modular
$\Rightarrow x \geq y$ for all $y \leq b, z$ for all $z \geq p, s$
$\Rightarrow x \geq y$ for all $y \leq b, z$ for all $z \in U(L(a, b), s)$
$\Rightarrow x \geq y$ for all $y \in L(U(L(a, b), s), b)$
$\Rightarrow x \in UL(U(L(a, b), s), b)$.
Hence $L(U(L(a, b), s), b) = LUL(U(L(a, b), s), b) \subseteq LU(L(a, b), L(b, s))$.
Thus, $L(a \vee s, b) \subseteq L(U(L(a, b), L(b, s)))$. The reverse inclusion is trivial for any ordered set. Hence the theorem.

6. References

[1] Chajda I. and Rachunek J., Forbidden Configurations of Modular and Distributive Ordered Sets, Order 5, pp. 407-425, 1989.

[2] Grätzer G., General Lattice Theory, 2^{nd} edn, BirkhÄauser, 1998.

[3] Grätzer G. and Schmidt E. Standard Ideals in Lattices, Acta Math. Acad. Sci. Hungar 12 (191), 17{8.

[4] Larmerova J. and Rachunek J. Translations of Distributive and Modular Ordered sets, Acta Univ. Palack. Olomouc. Fac. Rerum Natur. Math., 91, 13{23, 1988.

[5] Rachunek J., A Characterization of O-Distributive Semilattices, Acta Sci. Math. (szeged), 54, 241{24,1990.

[6] Rachunek J., On O-modular and O-distributive Semilattices, Math. Slovaca, 42, 3{13, 1992.

[7] Talukder M. R. and Noor A. S. A., Standard Ideals in a Join-Semilattice Directed Below, Southeast Asian Bull. of Mathematics, 4, 435{438, 1997.