Modeling of an RC Circuit Using a Stochastic Differential Equation

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Abstract

Two types of noises can exist in an electrical circuit: *external noise* and *internal noise*. The effects of both types of noises in electrical circuits are ignored when using a deterministic differential equation for their modeling. The deterministic model of the circuit is replaced by a stochastic model by adding a noise term in both the potential source (external noise) and the resistance (internal noise). Stochastic models are more appropriate to describe for instance the outcome of repeated experiments. The noise added in the potential source is assumed to be a white noise and that added in the resistance is assumed to be a correlated process. A first-order ordinary differential equation and its stochastic analogues is used for the DC response analysis of an RC circuit. The first-order differential equation, which describes the concentration of charge in an the capacitor, is solved explicitly in both the deterministic and stochastic case. This method gives an explicit relation expressing the sensitivity of the performance with respect to input parameters and their tolerances so it can be used for design optimization. Numerical simulations in MATLAB are obtained using the Euler-Maruyama method.

Keywords: RC electrical circuit, ordinary differential equation, stochastic differential equation, Brownian motion process, Euler-Maruyama method.

1. Introduction

Two types of noises can exist in an electrical circuit: *external noise* and *internal noise*. External noise denotes fluctuations in an otherwise deterministic system to which external forces are applied. An example that has wide applications in engineering consists of adding a noise term to the right side of a deterministic equation. Langevin formulated the first such model in 1908 to describe the velocity of a particle moving in a random force field. A well known example of internal noise in an electrical circuit is thermal noise, caused by the discreteness of electric charges [1][2]. Many other types of internal noise in electrical circuits are: shot noise, low frequency noise, burst noise etc. [3]. Obviously, at a finite temperature, there is no dissipative electrical circuit without thermal noise.

The effects of both types of noises in electrical circuits are ignored when using a deterministic differential equation for their modeling. Random effects due to both external and internal noise can be included by replacing the input and internal parameters in the deterministic model by random processes. Random differential equations of this type can be interpreted as ,stochastic differential equations, following Ito's basic work in the early 1940s [4][5]. Solutions of such equations represent Markov diffusion processes, the prototype of which is the Brownian motion process alternatively called the Wiener process [6]-[9].

The literature mainly distinguishes between sampling and non-sampling methods for the purpose of stochastic analysis. Sampling is done with the Monte Carlo method [10] [11]. Stochastic information on circuits without sampling is obtained with Hermite-Polynomial chaos [12]. Our method falls in the later category.

In this paper, the ordinary differential equation and its stochastic analogous, which describes the concentration of charge in the capacitor of a RC circuit, is solved explicitly for both the deterministic and stochastic cases. This method is based on results from the theory of stochastic differential equations (SDE) [5]. The method is general in the following sense. Any electrical circuit which consists of resistor, inductor and capacitor can be modeled by an ordinary differential equation in which the parameters of the differential operators are functions of circuit elements. The deterministic ordinary differential equation can be converted into a stochastic differential equation by adding noise to the input potential source and to the circuit elements. The noise added in the potential source is assumed to be a white noise and that added in the parameters is assumed to be a correlated process because these parameters change very slowly with time and hence must be modeled as a correlated process (Appendix A). The resulting SDE is solved by using a non-Monte Carlo method. MATLAB simulations are used to verify analytical results.

In the next section, the stochastic calculus theory is reviewed. Sections 3 and 4 describe the modeling of an RC circuit using an ordinary differential equation (ODE) and an SDE, respectively, and their analytical solution is also presented. In Section 5, the Euler-Maruyama scheme is reviewed for the numerical solution of SDE. Finally, the analytical results obtained in Sections 3 & 4 are verified with MATLAB simulations in Section 6 followed by the conclusions in Section 7.

2. Stochastic Differential Equation and Stochastic Calculus

Consider a SDE;

$$d\mathbf{X}(t) = \mathbf{f}(\mathbf{X}(t),t)dt + \mathbf{G}(\mathbf{X}(t),t)d\mathbf{B}(t) \qquad (1)$$

where $\mathbf{f}$ is an $n-$vector valued function, $\mathbf{G}$ is an $n \times m$ matrix valued function, $\mathbf{B}(t)$ is an $m-$dimensional Brownian motion process or Wiener process, and the solution $\mathbf{X}(t)$ is an $n-$vector process. By a solution $\mathbf{X}(t)$ of the stochastic differential equation (1), is meant a process $\mathbf{X}(t)$, for all t, in some interval $[t_0,T]$ must satisfy the integral equation:

$$\mathbf{X}(t) = \mathbf{X}(t_0) + \int_{t_0}^{T} \mathbf{f}(s,\mathbf{X}(s))ds + \int_{t_0}^{T} \mathbf{G}(s,\mathbf{X}(s))d\mathbf{B}(s) \qquad (2)$$

where $\mathbf{X}(t_0)$ is a specified initial value, the first integral in (2) is an ordinary integral, and the second integral in (2) is the stochastic integral.

Stochastic integrals and differential equations were introduced by Ito [4] and are being widely used and, hence, are called the Ito integral. Stratonovich [13] proposed another representation for stochastic integrals and differential equations under rather restrictive conditions, which has a number of advantages in computational techniques. Using this representation, we can work with stochastic integrals in the same way as with the ordinary integrals of smooth functions, such as integration by parts and changing of variables. The Ito integral, however, has a mathematical expectation which can be written more concisely. Simple formulas for the transition from one integral to the other allow at all times for the selection of the representation which is most convenient for any particular purpose. In this section, we only explicitly summarize the difference and the conversion between Ito and Stratonovich integrals. More rigorously mathematical treatments of this issue are given in [14]-[16].

A. Stochastic Integrals

Let $t_0 \le t_1 \le \cdots \le t_n = T$ be a partition of the interval $[t_0,T]$ and $\delta_n = \max(t_i - t_{i-1})$.

Definition 1: The Ito integral $\int_{t_0}^{T} \mathbf{G}(s,\mathbf{X}(s))d\mathbf{B}(s)$ is defined as the limit in the quadratic mean (qm)

$$Y_I(t) = \int_{t_0}^{T} \mathbf{G}(s,\mathbf{X}(s))d\mathbf{B}(s) \overset{\text{def}}{=} \underset{\boldsymbol{\delta}_n \to 0}{\text{qm-lim}} \sum_{i=1}^{n} \mathbf{G}(\mathbf{X}_{t_{i-1}}, t_{i-1})(\mathbf{B}_{t_i} - \mathbf{B}_{t_{i-1}}) \qquad (3)$$

If the integrand $\mathbf{G}$ is jointly measurable and

$$\int_{t_0}^{T} E(|\mathbf{G}(s,\mathbf{X}(s))|^2)ds \prec \infty \qquad (4)$$

then the limit in (3) exists [14][15]. If $\mathbf{G}$ satisfies, instead of (4), the weaker condition

$$\int_{t_0}^{T} |\mathbf{G}(s,\mathbf{X}(s))|^2 \, ds \prec \infty, \ \textit{almost surely} \qquad (5)$$

the stochastic integral in (3) is defined as the limit *in probability*.

Definition 2: The Stratonovich integral is defined by :

$$Y_S(t) = \int_{t_0}^{T} \mathbf{G}(s, \mathbf{X}(s)) d\mathbf{B}(s)$$

$$\overset{\text{def}}{=} \underset{\boldsymbol{\delta}_n \to 0}{\text{qm-lim}} \sum_{i=1}^{n} \mathbf{G}\left(\frac{\mathbf{X}_{t_{i-1}} + \mathbf{X}_{t_i}}{2}, \mathbf{t}_{i-1}\right)(\mathbf{B}_{t_i} - \mathbf{B}_{t_{i-1}}) \quad (6)$$

In addition to the conditions on the existence of the Ito integral, it is required for the existence of the Stratonovich integral in (6) that the $\mathbf{G}(\mathbf{X}(t),t)$ function be continuous in t and have continuous partial derivatives with respect to $\mathbf{X}_t$ [13]-[16].

Under the existence of (6), Stratonovich [13] provided the connection between these two stochastic integrals by :

$$\mathbf{Y}_S(t) = \mathbf{Y}_I(t) + \frac{1}{2}\int_{t_0}^{T}\sum_{j=1}^{m}\sum_{k=1}^{m}\left(\mathbf{G}_{x_k}(\mathbf{X}_s,s)\right)_j \mathbf{G}_{kj}(\mathbf{X}_s,s)ds \quad (7)$$

where the n-vector $(\mathbf{G}_{x_k})_j$ is the jth column of $n \times m$ matrix $\mathbf{G}_{x_k} = \left(\partial \mathbf{G}_{ij} / \partial x_k\right)$. To clearly illustrate these integration concepts, the simple example in [13] using a one-dimensional stochastic integral $\int_{t_0}^{T} w(s)dw_s$, is considered.

Example 1: By using (3), we have [9]:

$$y_I(t) = \int_{t_0}^{T} w(s)dw_s = (w_T^2 - w_{t_0}^2)/2 - (T - t_0)/2 \quad (8)$$

By using (6), we have [9]:

$$y_s(t) = \int_{t_0}^{T} w(s)dw_s = (w_T^2 - w_{t_0}^2)/2 \quad (9)$$

which can be also obtained by a direct integration by parts just as for ordinary integrals.

Since stochastic integrals can be computed in two distinct ways, there are two different solutions, Ito and Stratonovich solutions to (2), and hence to (1). However these two different solutions can be converted into each other by using simple formulas [13] of :

$$\mathbf{X}_I(T) = \mathbf{X}_I(t_0) + \int_0^T \Big[\mathbf{f}(\mathbf{X}_I(\tau),\tau) - \frac{1}{2}\sum_{j=1}^{m}\sum_{k=1}^{n} .(\mathbf{G}_{x_k}(\mathbf{X}_I(\tau),\tau))_j \mathbf{G}_{kj}(\mathbf{X}_I(\tau),\tau)\Big]d\tau + (S)\int_0^T \mathbf{G}(\mathbf{X}_I(\tau),\tau)d\mathbf{B}_\tau \quad (10)$$

or

$$\mathbf{X}_s(T) = \mathbf{X}_s(t_0) + \int_0^T \Big[\mathbf{f}(\mathbf{X}_s(\tau),\tau) + \frac{1}{2}\sum_{j=1}^{m}\sum_{k=1}^{n} (\mathbf{G}_{x_k}(\mathbf{X}_s(\tau),\tau))_j \mathbf{G}_{kj}(\mathbf{X}_s(\tau),\tau)\Big]d\tau + (I)\int_0^T \mathbf{G}(\mathbf{X}_s(\tau),\tau)d\mathbf{B}_\tau \quad (11)$$

where the last integrals of right side of (10) and (11) are Stratonovich and Ito integrals, respectively. From (10) and (11) the Staratonovich differential equation:

$$d\mathbf{X}_t = \left(\mathbf{f} - \frac{1}{2}\sum_{j=1}^{m}\sum_{k=1}^{n}(\mathbf{G}_{x_k})_j \mathbf{G}_{kj}\right)dt + \mathbf{G}d\mathbf{B}_t \quad (12)$$

corresponds to the Ito differential equation

$$d\mathbf{X}_t = \mathbf{f}dt + \mathbf{G}d\mathbf{B}_t \quad (13)$$

where $\sum_{j=1}^{m}\sum_{k=1}^{n}(\mathbf{G}_{x_k})_j \mathbf{G}_{kj}$ is the so-called correction term.

B. Interpretation

The paradox of obtaining two different stochastic processes as solutions to the same stochastic differential equation arises from the pathological nature of white noise. Since all sample functions of a Wiener process are nowhere differentiable and of unbounded variation, and hence, are not smooth, we can not interpret the Ito integral as an ordinary Riemann-Stieltjes integral. Therefore, Ito calculus does not conform to the rules of ordinary calculus, which can be easily seen in *example 1*. The correction term is due to the fact that dB_t is always independent of B_t, and dB_t^2 is approximately dt.

In contrast to the above Ito description, in (1) the white noise driving term is considered as an approximation of a very wide band but smooth colored process. The differential equation can be solved exactly for each sample function of the smooth process using classical calculus. As driving noise approaches white, the solved process does converge to the solution in the Stratonovich sense. Thus, the rule of ordinary calculus can be applied to a Stratonovich equation. However, if the noise in SDE is not state multiplicative, i.e., $\mathbf{G}$ in (1) is independent of states, and only a nonrandom function of time t, the correction term disappears, which can be seen from (10)-(13). That is, Ito interpretation and Stratonovich interpretation coincides.

The process defined by the Ito integral in (3) is a *Martingale*, i.e., the conditional expectation $E[Y_t \mid Y_\tau, \tau \le s \prec t] = Y_s$, while the Stratonovich integral $Y_s(t)$ is not. This property provides the expectation computation advantage for an Ito equation, which is very convenient for theoretic moment stability analysis. Moreover, because of more restrictive conditions on the existence of Stratonovich integrals, an Ito equation is suited for more general applications than a Stratonovich equation.

3. Deterministic Modeling of an RC Circuit

Let $Q(t)$ be the charge on the capacitor and $V(t)$ be the potential source applied to the input of a RC circuit. Using Kirchoff's second law,

$$V(t) = I(t)R + \frac{Q(t)}{C} \qquad (14)$$

and since $I(t) = dQ(t)/dt$, the following equation holds:

$$RQ'(t) + C^{-1}Q(t) = V(t) \qquad (15)$$

or

$$Q'(t) + (RC)^{-1}Q(t) = R^{-1}V(t) \qquad (16)$$

If $V(t)$ is a piecewise continuous function, the solution of the first order linear differential equation (16) is:

$$Q(t) = Q(0)\exp\left(-\frac{t}{RC}\right) + \frac{1}{R}\int_0^t \exp\left(-\frac{(t-s)}{RC}\right)V(s)ds \qquad (17)$$

where $Q(0)$ is the initial charge stored in the capacitor.

4. Stochastic Modeling of RC Circuit

The resistance and potential source may not be deterministic but of the form :

$$R^* = R + ``noise" = R + \alpha w(t) \qquad (18)$$

and

$$V^*(t) = V(t) + ``noise" \; = V(t) + \beta N_2(t) \qquad (19)$$

where $w(t)$ is a zero mean, exponentially correlated stationary process, and $N_2(t)$ is a white noise process of mean zero and variance one, and α, β are nonnegative constants, known as the intensity of noise. Their magnitudes determine the deviation of the stochastic case from the deterministic one. The correlated process $w(t)$, sometimes called colored noise, can be generated by a linear stochastic differential equation forced by white noise, which is given as:

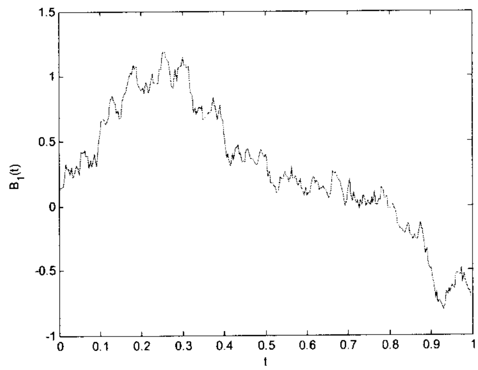

Fig. 1 Brownian motion process $B_1(t)$

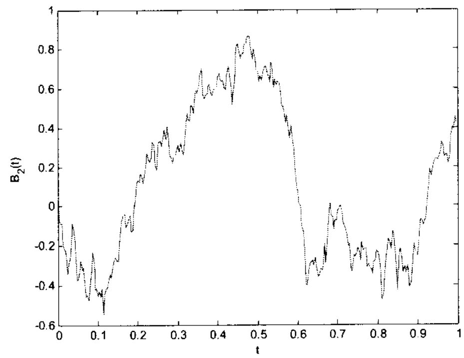

Fig. 2 Brownian motion process $B_2(t)$

$$dw(t) = -\rho w(t)dt + \sigma\rho dB_1(t) \qquad (20)$$

where $B_1(t)$ is a Brownian motion process with unit variance parameter, $\rho, \sigma \succ 0$ are fixed constants and $w_0 \sim N(0, \sigma^2\rho/2)$ is independent of $B_1(t)$. As we explained in section 2, (20) is the same in the Ito and Stratonovich sense (since the coefficient of $dB_1(t)$ is nonrandom), and can be manipulated by formal rules as if $B_1(t)$ were continuously differentiable. Therefore equation (20) can be rewritten as:

$$\frac{dw(t)}{dt} = -\rho w(t) + \sigma\rho N_1(t) \quad (21)$$

where $N_1(t)$ is a zero mean, white Gaussian noise with $E[N_1(t)N_1(\tau)] = \delta(t-\tau)$. The correlation function of $w(t)$ is $R_w(\tau) = \sigma^2 \frac{\rho}{2} e^{-\rho|\tau|}$. For large values of $\rho, w(t)$ is a wide-band process that approximates white noise. Substituting (18) and (19) in (15), we get :

$$(R + \alpha w(t))Q'(t) + C^{-1}Q(t) = V(t) + \beta N_2(t) \quad (22)$$

or

$$\frac{dQ(t)}{dt} + \frac{Q(t)}{C(R+\alpha w(t))} = \frac{V(t)}{R+\alpha w(t)} + \frac{\beta N_2(t)}{R+\alpha w(t)} \quad (23)$$

or

$$dQ(t) = -\frac{Q(t)}{C(R+\alpha w(t))}dt + \frac{V(t)}{R+\alpha w(t)}dt + \frac{\beta dB_2(t)}{R+\alpha w(t)} \quad (24)$$

where $dB_2(t) = N_2(t)dt$ and $B_2(t)$ is the Brownian motion process, independent of $B_1(t)$. Writing equations (20) and (24) in matrix-vector form, we obtain :

$$d\mathbf{X}(t) = \mathbf{A}(t)\mathbf{X}(t)dt + \mathbf{Z}(t)dt + \mathbf{K}(t)d\mathbf{B}(t) \quad (25)$$

where

$$\mathbf{X}(t) = \begin{pmatrix} w(t) \\ Q(t) \end{pmatrix} \quad (26)$$

$$\mathbf{A}(t) = \begin{pmatrix} -\rho w(t) & 0 \\ 0 & -\frac{1}{C(R+\alpha w(t))} \end{pmatrix} \quad (27)$$

$$\mathbf{Z}(t) = \begin{pmatrix} 0 \\ \frac{V(t)}{R+\alpha w(t)} \end{pmatrix} \quad (28)$$

$$\mathbf{K}(t) = \begin{pmatrix} \sigma\rho & 0 \\ 0 & \frac{\beta}{R+\alpha w(t)} \end{pmatrix} \quad (29)$$

and

$$\mathbf{B}(t) = \begin{pmatrix} B_1(t) \\ B_2(t) \end{pmatrix} \quad (30)$$

The analytical solution of (25) is a random process.

$$\mathbf{X}(t) = \exp(\mathbf{A}t)\{\mathbf{X}(0) + \int_0^t \exp(-\mathbf{A}s)[\mathbf{Z}(s)ds + \mathbf{K}(s)d\mathbf{B}(s)]\} \quad (31)$$

If $E[\mathbf{X}(0)\mathbf{X}^T(0)] \prec \infty$, the expectation $E[\mathbf{X}(t)] = \mathbf{M}(t)$ is the solution of the ordinary differential equation:

$$\frac{d\mathbf{M}(t)}{dt} = \mathbf{A}\mathbf{M}(t) + E[\mathbf{Z}(t)] \quad (32)$$

Taking the expectation of (31), we get:

$$E[\mathbf{X}(t)] = E[\mathbf{X}(0)]\exp(\mathbf{A}t) + \int_0^t \exp(\mathbf{A}(t-s))E[\mathbf{Z}(s)]ds \quad (33)$$

for every $t \succ 0$. If the random variable $\mathbf{X}(0)$ is constant then the expectation of the stochastic solution is equal to the deterministic solution of the circuit. The function $\mathbf{M}(t) = E[\mathbf{X}(t)]$ is independent of the fluctuational part of the SDE.

5. Numerical Solutions of SDEs

The Euler-Maruyama numerical method is used for the simulation of $\mathbf{X}(t)$ [5]. The Euler-Maruyama scheme is based on numerical methods for ordinary differential equations.

Let the stochastic process $\mathbf{X}_t, t_0 \le t \le T$ be the solution of the scalar SDE with M Brownian motion processes,

$$dX_t = f(X_t,t)dt + \sum_{j=1}^{M} g^j(X_t,t)dB_t^j \quad (34)$$

with an initial value $X_{t_0} = X_0$. Let us consider an equidistant discretization of the time interval $t_n = t_0 + nh$, where $h = (T-t_0)/n = t_{n+1} - t_n = \int_{t_n}^{t_{n+1}} dt$ and the corresponding discretization of the Brownian motion process $\Delta B_n^j = B_{t_{n+1}}^j - B_{t_n}^j = \int_{t_n}^{t_{n+1}} dB_s^j$. To be able to apply any stochastic numerical scheme, first one has to generate, for all j, the random increments of the Brownian motion process B^j

as independent Gaussian random variables with mean $E[\Delta B_n^j]=0$ and $E[(\Delta B_n^j)^2]=h$.
The Euler-Maruyama scheme for this SDE has the form:

$$X_{n+1}=X_n+f(X_n,t_n)h+\sum_{j=1}^{M}g^j(X_n,t_n)\Delta B_n^j \tag{35}$$

For measuring the accuracy of a numerical solution to an SDE, we use the strong order of convergence. A stochastic numerical scheme converges with strong order γ, if there exist real constants $K\succ 0$ and $\delta\succ 0$ [5], so that:

$$E[|X_T-X_T^h|]\le Kh^{\gamma}\qquad h\in(0,\delta) \tag{36}$$

where the numerical solution is denoted by X_T^h.
The Euler-Maruyama scheme converges with strong order $\gamma=\frac{1}{2}$.

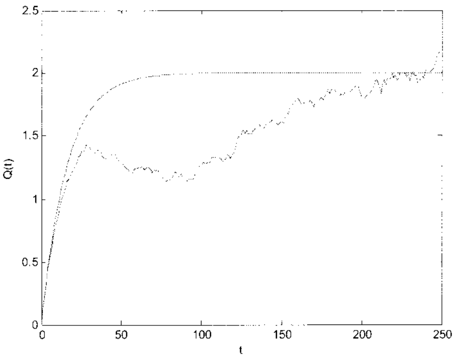

Fig. 3 The deterministic solution and sample path of the stochastic solution with stochastic resistance and stochastic potential source (noise intensity $\alpha=1,\beta=1$).

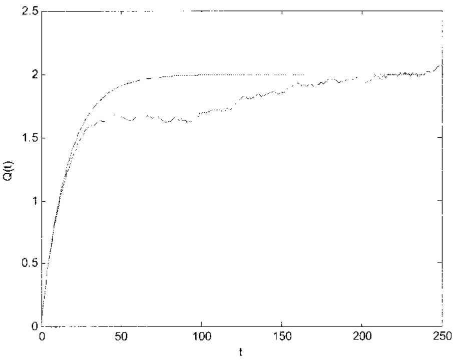

Fig. 4 The deterministic solution and sample path of the stochastic solution with stochastic resistance (noise intensity $\alpha=1$).

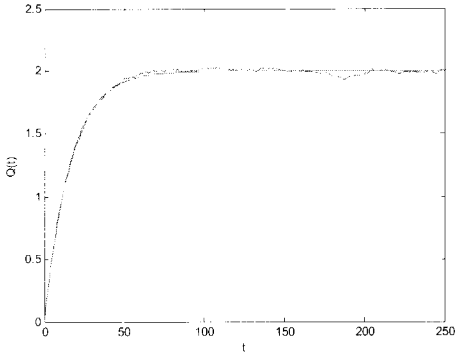

Fig. 5 The deterministic solution and sample path of the stochastic solution with stochastic potential source (noise intensity $\beta=1$).

6. Simulation Results

The Brownian motion processes that we use for the simulation of the stochastic solution are shown in Fig. 1 and Fig. 2. Assume $R=10\Omega$, $C=0.1F$, $V(t)=V=20V$, and $\mathbf{X}(0)=0$. The results of the stochastic solution of the circuit with both stochastic resistance and stochastic potential source $(\alpha=1,\beta=1)$ is shown in Fig. 3. A random behavior which depends on the intensity of noise is observed in Fig. 3. If the resistance is modeled as a superposition of mean value plus a zero mean correlated process, then an impact on the DC response is observed in Fig. 4. A random behavior is observed in Fig. 5, if the potential source is modeled as a superposition of mean value plus a noise, where the noise is modeled as a white Gaussian noise.

7. Conclusion

We have seen that deterministic models are always accompanied by stochastic ones, which arise whenever we plug in random variables. These stochastic models are more appropriate to describe, for instance, the outcome of repeated experiments. This paper shows an application of the Ito stochastic calculus to the problem of modeling a series RC electrical circuit, including both analytical and numerical solutions. The stochastic model is obtained from the deterministic model by adding noise in both the potential source and the resistance. The noise added in the potential source is assumed to be a white noise and that added in the resistance is assumed to be a correlated process. The resulting SDE is solved using a non-Monte Carlo method. A monte Carlo method does not give explicit relation an expressing the sensitivity of the performance with respect to input parameters and their tolerances. So, our method can be used for design optimization.

Appendix

A. Generalization of the Method

Consider a linear circuit that is built out of resistors, capacitors and inductors. $v(t)$ is the input process and $x(t)$ the output process. $x(t)$ satisfies the differential equation :

$$\frac{d^n x(t)}{dt^n} + a_1 \frac{d^{n-1}x(t)}{dt^{n-1}} + \cdots + a_{n-1}\frac{dx(t)}{dt} + a_n x(t) = v(t) \quad (37)$$

The parameters $a_1, a_2, \cdots, a_n$ are functions of the circuit elements. These parameters become random variables if they are influenced by random parameters such as surrounding temperature. These parameters, with the random variations can be written as :

$$a_i^* = a_i + \alpha_i w_i(t) \quad (38)$$

where $w_i(t)$s are zero mean exponentially correlated processes and α_is are nonnegative constants known as the intensity of noise. Their magnitude determines the deviation of the stochastic case from the deterministic one. The correlated process $w_i(t)$, sometimes called colored noise, can be generated by a linear SDE forced by white noise [9].

$$dw_i(t) = -\rho_i w_i(t)dt + \sigma_i \rho_i dB_i(t) \quad 1 \le i \le n \quad (39)$$

where $B_i(t)$ is a Brownian motion process with unit variance parameter, $\rho_i, \sigma_i \succ 0$ are fixed constants and $w_i(0) \sim N(0, \sigma_i^2 \rho_i/2)$ is independent of $B_i(t)$. As we explained in section 2, (39) is the same in the Ito and Stratonovich sense (since the coefficient of $dB_i(t)$ is nonrandom), and can be manipulated by formal rules as if $B_i(t)$ were continuously differentiable. Therefore equation (39) can be rewritten as:

$$\frac{dw_i(t)}{dt} = -\rho_i w_i(t) + \sigma_i \rho_i N_i(t) \quad (40)$$

where $N_i(t)$ is a zero mean, white Gaussian noise with $E[N_i(t)N_i(\tau)] = \delta(t-\tau)$. The correlation function of $w_i(t)$ is $R_{w_i}(\tau) = \sigma_i^2 \frac{\rho_i}{2} e^{-\rho_i|\tau|}$. For large values of ρ_i, $w(t)$ is a wide-band process that approximates white noise. Let $\mathbf{X}(t)$ be the state vector, which is defined as :

$$\mathbf{X}(t) = [w_1(t) \quad w_2(t) \cdots w_n(t) \quad x_1(t) \cdots x_n(t)]^T \quad (41)$$

If the potential source is modeled as a superposition of mean value plus a white noise, then this state vector satisfies the following SDEs :

$$\begin{aligned}
dw_1(t) &= -\rho_1 w_1(t)dt + \sigma_1 \rho_1 dB_1(t) \\
&\vdots \\
dw_n(t) &= -\rho_n w_n(t)dt + \sigma_n \rho_n dB_n(t) \\
dx_1(t) &= x_2(t)dt \\
dx_2(t) &= x_3(t)dt \\
&\vdots \\
dx_{n-1}(t) &= x_n(t)dt \\
dx_n(t) &= -\left(a_1^* x_n(t) + a_2^* x_{n-1}(t) + \cdots + a_n^* x_1(t)\right)dt \\
&\quad + v(t)dt + \beta dB(t)
\end{aligned}$$

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