Modular and Standard Filters of a Directed above Meet Semilattice

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Abstract

A meet semilattice S is called *directed above* if for each a, b $\in$ S there exist d $\in$S such that d $\geq$ a, d $\geq$ b. In this paper we have introduced the notion of modular and dual standard elements in a directed above meet semilattice and included several characterizations of these elements. We prove that an element is dual standard if and only if it is both modular and dual distributive. We have also included a number of characterizations of modular and standard filters of directed above meet semilattices.

Keywords: Directed above semilattice, Modular element, Dual distributive element, Dual standard element.

Introduction

In lattices, standard elements and ideals have been studied extensively by [2]. Ramana et al. [5] have studied them in joined semilattices directed below. Many results of [5] have been extended by Talukder & Noor in [6, 7]. In [6, 7] the authors have also introduced the notion of modular elements in joined semilattices. On the other hand, [3] have studied the modular elements very briefly in a general lattice.

In this paper we give a notion of modular and dual standard elements of a meet semilattice directed above.
We also include some characterizations of modular and standard filters of a meet semilattice.

A semilattice S is called a *directed above semilattice* if for all a, b $\in$ S there exist
d $\in$S such that d $\geq$ a, d $\geq$ b.

A meet semilattice S is called a *modular semilattice* if w $\geq$ a $\wedge$ b and w $\leq$ a, which implies that there exist y $\in$ S such that y $\geq$ b and y $\wedge$ a = w.

A meet semilattice S is called a *distributive semilattice* if w $\geq$ a $\wedge$ b, which implies that there exist x $\geq$ a, y $\geq$ b in S such that w = x $\wedge$ y.

Of course every (modular) distributive semilattice is directed above.

1. Modular elements and modular filters.

By [6], an element m of a lattice L is *modular* if for all x, y $\in$ L with y $\leq$ x, x $\wedge$ (m $\vee$ y) = (x $\wedge$ m) $\vee$ y.

We define an element m of a meet semilattice S as a *modular element* if y $\geq$ m $\wedge$ x with y $\leq$ x (x, y $\in$ S), which implies the existence of
$m_1 \geq m$ such that $y = m_1 \wedge x$.
It is very easy to see that the above two definitions are equivalent in the case of a lattice.

Thus by [6], in a lattice we have the following result:

Theorem1.1. *An element* m *of a lattice is modular if it has any of the following equivalent properties.*

(i) For x, y $\in$ L *with* y $\leq$ x, x $\wedge$ (m $\vee$ y) = (x $\wedge$ m) $\vee$ y

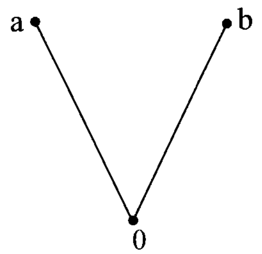
(ii) For x $\leq$ m $\vee$ y *with* y $\leq$ x (x, y $\in$ L) *implies the existence of* $m_1 \leq m$ *such that* $x = m_1 \vee y$.

(iii) For y $\geq$ m $\wedge$ x *with* y $\leq$ x (x, y $\in$ L) *implies the existence* of $m_1 \geq m$ *such that* $y = m_1 \wedge x$.

By [1], an element d of a meet semilattice S is called *dual distributive* if
X $\geq$ d $\wedge$ a, x $\geq$ d $\wedge$ b (x, a, b$\in$ S), which implies the existence of c $\in$ S such that c $\geq$ a, c $\geq$ b and

$x \geq d \wedge c$.

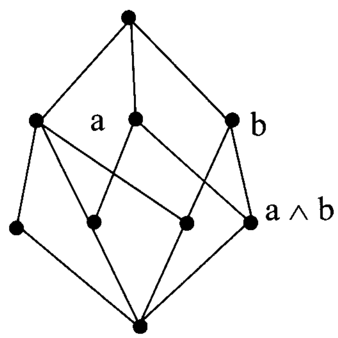
We also know by [1] that any semilattice with a dual distributive element is directed above. But this is not true for a semilattice with only a modular element. For example in the meet semilattice S of Fig 1,



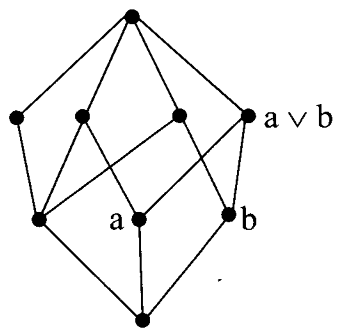
S

Figure 1

0 is modular but S is not directed above.



S_1



S_2

Figure 2

Remark. If a and b are two modular elements of a meet semilattice directed above, then $a \wedge b$ and $a \vee b$ (if they exist) are not necessarily modular.

For example, consider the lattices S_1 and S_2 given below:

A routine calculation shows that a and b are modular in both the lattices, but neither $a \wedge b$ in S_1 nor $a \vee b$ in S_2 is modular.

Clearly every element of a modular semilattice is modular. Moreover, if every element of a semilattice is modular then the semilattice is modular.

A filter M of a meet semilattice S is called *modular* if M is a modular element of the lattice of filters D(S). By [4] D(S) is a lattice if and only if S is directed above. So we will consider here only the directed above meet semilattices.

Now we prove the following result:

Theorem 1.2. *A filter* M *of a directed above semilattice* S *is modular if and only if for all principal filters* I *and* J *of* S *with* $J \subseteq I$, $I \wedge (M \vee J) = (I \wedge M) \vee J$.

Proof. If M is modular, then the condition obviously holds.

Conversely, suppose the given condition holds. Let P and Q be any two filters of S with $Q \subseteq P$. Let $x \in P \wedge (M \vee Q)$. Then $x \in P$ and $x \geq m \wedge q$ for some $m \in M$ and $q \in Q$. This implies $x \in P$ and $x \in M \vee [q)$. Thus, $x \wedge q \in M \vee [q)$ and so $x \wedge q \in [x \wedge q) \wedge (M \vee [q)) = ([x \wedge q) \wedge M) \vee [q)$. But $x \in P$ and $q \in Q \subseteq P$ implies $x \wedge q \in P$. Therefore, $x \wedge q \in (P \wedge M) \vee Q$, and so $x \in (P \wedge M) \vee Q$, i.e. $P \wedge (M \vee Q) \subseteq (P \wedge M) \vee Q$. The reverse inclusion is trivial. Hence, M is modular.

Now, we give a characterization of a modular filter.

Theorem 1.3. *A filter* M *of a directed above meet semilattice* S *is modular if and only if* $x \geq m \wedge y$ *with* $y \geq x$, *for some* $m \in M$ $(x, y \in S)$, *then there exists* $m_1 \in M$ *such that* $x = m_1 \wedge y$.

Proof. Suppose M is modular and $x \geq m \wedge y$ with $y \geq x$, $(x, y \in S)$, for some $m \in M$. Then $x \in [x) \wedge (M \vee [y)) = ([x) \wedge M) \vee [y)$. This implies $x \geq m_1 \wedge y$ for some $m_1 \geq x$, $m_1 \in M$. Since $m_1 \geq x$ and $y \geq x$, $m_1 \wedge y \geq x$ and hence, $x = m_1 \wedge y$.

Conversely, let the given condition holds. Suppose $x \in I \wedge (M \vee J)$ with $J \subseteq I$, I, J are filters of S. Then $x \in I$ and $x \geq m \wedge j$ for some $m \in M$, $j \in J$. Then $x \wedge j \geq m \wedge j$. Since $j \geq x \wedge j$

and $m \in M$ by the given condition $x \wedge j = m_1 \wedge j$ for some $m_1 \in M$. Now, $x \in I$ and $j \in J \subseteq I$ implies $x \wedge j \in I$. Also, $m_1 \geq x \wedge j$ implies $m_1 \in I$. Thus, $m_1 \in I \wedge M$. Therefore, $x \wedge j \in (I \wedge M) \vee J$ and $x \in (I \wedge M) \vee J$. Hence, $I \wedge (M \vee J) \subseteq (I \wedge M) \vee J$. The reverse inclusion is trivial. Therefore, M is modular.

The following result immediately follows from the above theorem.

Theorem 1.4. *An element* m *of a directed above semilattice* S *is modular if and only if* [m) *is a modular filter of* S.

Now we give another characterization of modular filters.

Theorem1.5. *A filter* M *of a directed above semilattice* S *is modular if and only if* $\theta = \{(x, y) \in S \times S | x \wedge m = y \wedge m = x \wedge y$ *for some* $m \in M\}$ *is a congruence.*

Proof. First suppose M is modular. It will be shown that $\theta = \psi_M$. Clearly, $\theta \subseteq \psi_M$. Now, let $x \equiv y\ (\psi_M)$. Then $x \wedge m = y \wedge m$ for some $m \in M$. This implies $x \geq y \wedge m$, and so $x \wedge y \geq y \wedge m$. Since M is modular, by Theorem 1.3, there exists $m_1 \in M$ such that $x \wedge y = y \wedge m_1$.

Similarly, $x \wedge y = y \wedge m_2$ for some $m_2 \in M$. Thus, $x \wedge y = x \wedge m_1 \wedge m_2 = y \wedge m_1 \wedge m_2$, and $m_1 \wedge m_2 \in M$. Hence, $x \equiv y(\theta)$. Therefore, $\theta = \psi_M$ and so θ is a congruence.

Conversely, let θ be a congruence. Let $x \geq m \wedge y$ with $y \geq x$, $(x, y \in S)$ for some $m \in M$. Obviously, $y \wedge m \equiv y\ (\theta)$. Also, $x \wedge (y \wedge m) = x \wedge m = y \wedge m = (y \wedge m) \wedge m$ implies that $x \equiv y \wedge m\ (\theta)$. Then by transitivity, $x \equiv y(\theta)$. So $x \wedge y = x \wedge m_1 = y \wedge m_1$ for some $m_1 \in M$, that is $x = y \wedge m_1$, which implies that M is modular by Theorem 1.3.

Thus we have the following corollary:

Corollary 1.6. *An element* m *of a directed above semilattice* S *is modular if and only if* $\theta = \{(x, y) \in S \times S | x \wedge m_1 = y \wedge m_1 = x \wedge y$ *for some* $m_1 \geq m\}$ *is a congruence.*

According to [3], an element m of a lattice L is modular if $a \leq b$, $a \wedge m = b \wedge m$ and $a \vee m = b \vee m$ imply that $a = b$. It is easy to show that this definition of modularity is equivalent to our definition. With the help of this definition we prove the following result which concludes the section.

Theorem 1.7. *Let* F *be an arbitrary filter and* M *be a modular filter of a directed above semilattice* S. *If* $F \vee M$ *and* $F \wedge M$ *are both principal, then* F *itself is principal.*

Proof. Let $F \vee M = [a)$ and $F \wedge M = [b)$. Then $a \geq m \wedge f$ for some $m \in M$, $f \in F$. Since $f \in F \subseteq [a)$, $f \geq a$. Hence by Theorem 1.3, $a = m_1 \wedge f$ for some $m_1 \in M$.Thus, $[a) = [m_1) \vee [f) \subseteq M \vee [f) \subseteq M \vee [f \wedge b) \subseteq M \vee F = [a)$. Therefore, $M \vee I = M \vee [f \wedge b)$. Again, $[b) = M \wedge F \supseteq M \wedge [f \wedge b) \supseteq M \wedge [b) = [b)$.Therefore, $M \wedge F = M \wedge [f \wedge b)$. Since $[f \wedge b) \subseteq F$, by the definition of modularity, $F = [f \wedge b)$.

2. Dual Standard elements and Standard filters

An element s of S is called a *dual standard* if $x \geq s \wedge t$ $(x, t \in S)$ which implies the existence of $s_1 \geq s$, $t_1 \geq t$ $(s_1, t_1 \in S)$ such that $x = s_1 \wedge t_1$.

Now we give some properties of dual standard elements.

Lemma 2.1. *Every dual standard element of* S *is a dual distributive element of* S.

Proof. Let s be a dual standard element of S. Suppose $x \geq s \wedge a$ and $x \geq s \wedge b$ $(x, a, b \in S)$, Since s is dual standard, there exist $s_1 \geq s$, $a_1 \geq a$ such that $x = s_1 \wedge a_1$ $(s_1, a_1 \in S)$. Now $a_1 \geq x \geq s \wedge b$. This implies $a_1 = s_2 \wedge b_1$ $(b_1, s_2 \in S)$, $s_2 \geq s$, $b_1 \geq b$. Therefore $x = s_1 \wedge s_2 \wedge b_1 \geq s \wedge b_1$, where $b_1 \geq a_1 \geq a$, $b_1 \geq b$. Hence s is a dual distributive element of S.

Thus by [1, Theorem 1] a meet semilattice possessing a dual standard element is of course directed above. So in this section we will always assume the meet semilattices S is directed above.

Theorem 2.2. *Let* $s \in S$ *be a dual standard element. If* $a \vee s$ *exists for some* $a \in S$, *then* $a \vee s$ *is a dual standard in* [a).

Proof. Here, s is dual standard. Suppose $a \vee s$ exists, then $a \vee s \in [a)$. Let $x \in [a)$ and $x \geq (a \vee s) \wedge t$, $t \in [a)$. Now $x \geq s \wedge t$. Since s is dual standard in S, $x = s_1 \wedge t_1$ for some $s_1 \geq s$, $t_1 \geq t$. Then $s_1 \geq x \geq a$ implies $s_1 \geq s \vee a$. Therefore, $x = s_1 \wedge t_1$, $s_1 \geq s \vee a$, $t_1 \geq t$. Hence, $a \vee s$ is dual standard in [a).

Theorem 2.3. *Let* $s_1, s_2 \in S$ *be dual standard, then* $s_1 \wedge s_2$ *is dual standard.*

Proof. Let $x \geq (s_1 \wedge s_2) \wedge t = s_1 \wedge (s_2 \wedge t)$. Since s_1 is dual standard, there exist $a \geq s_1$, $b \geq s_2 \wedge t$ such that $x = a \wedge b$. Now $b \geq s_2 \wedge t$. Since s_2 is dual standard, there exist $c \geq s_2$, $d \geq t$ such that $b = c \wedge d$. So $x = (a \wedge c) \wedge d$ where $a_1 \geq s_1$, $c \geq s_2$, $d \geq t$. That is, $a \wedge c \geq s_1 \wedge s_2$, $d \geq t$. Hence $s_1 \wedge s_2$ is dual standard.

A filter F of a meet semilattice S is called *standard* if F is a standard element of D(S).

Theorem 2.4. *An element* s *of* S *is dual standard if and only if* [s) *is a standard filter of* S.

Proof. Suppose s is dual standard. Then $[a) \wedge ([s) \vee [t)) = [a) \wedge [s \wedge t) = [a \vee (s \wedge t)) = [(a \vee s) \wedge (a \vee t) = [(a \vee s) \vee [(a \vee t) = ([a) \wedge [s)) \vee ([a) \wedge [t))$. Therefore [s) is standard.

Conversely, suppose [s) is standard. Let $x \geq s \wedge t$, then $[x) \subseteq [s \wedge t) = [s) \vee [t)$. So, $[x) = [x) \wedge ([s) \vee [t)) = ([x) \wedge [s)) \vee ([x) \wedge [t))$ as [s) is standard.

Thus, $x \in ([x) \wedge [s)) \vee ([x) \wedge [t))$, which implies $x \geq a \wedge b \geq x$ where $a \in [x) \wedge [s)$, $b \in [x) \wedge [t)$. Therefore, $x = a \wedge b$, $a \geq s$, $b \geq t$ and s is dual standard.

Now we give a characterization of standard filters.

Theorem 2.5. *A filter* T *of* S *is standard if and only if,* $x \geq t \wedge a$ *for some* $t \in T$ $(x, a \in S)$ *implies the existence of* $t_1 \in T$ *and* $a_1 \geq a$ *such that* $x = t_1 \wedge a_1$.

Proof. Suppose T is standard and let $x \geq t \wedge a$ for some $t \in T$ $(x, a \in S)$. Then $x \in [x) \wedge (T \vee [a)) = ([x) \wedge T) \vee ([x) \wedge [a))$. This implies $x \geq t_1 \wedge a_1$ for some $t_1 \geq x$, $t_1 \in T$ and $a_1 \geq x$, $a_1 \geq a$. Now $t_1 \geq x$ and $a_1 \geq x$ imply $t_1 \wedge a_1 \geq x$, which implies $x = t_1 \wedge a_1$ with $t_1 \in T$ and $a_1 \geq a$.

Conversely, let the given condition hold and let $x \in I \wedge (T \vee J)$ for any $I, J \in D(S)$. Then $x \in I$ and $x \geq t \wedge j$ for some $t \in T$ and $j \in J$. Then there exist $t_1 \in T$ and $j_1 \geq j$ such that $x = t_1 \wedge j_1$. Since $t_1 \geq x$, $j_1 \geq x$ and $x \in I$, $x \in (I \wedge T) \vee (I \wedge J)$. This implies T is standard.

Following result gives characterizations for standard ideals in meet semilattices directed above.

Theorem 2.6. *For a filter* T *of* S, *the following conditions are equivalent*:

(i) T *is a standard filter*

(ii) *For any filter* I, $T \vee I = \{t \wedge i \mid t \in T, i \in I\}$

(iii) *For any principal filter* I, $T \vee I = \{t \wedge i \mid t \in T, i \in I\}$

(iv) *For any* $x, y \in S$, $[x) \wedge (T \vee [y)) = ([x) \wedge T) \vee ([x) \wedge [y))$.

Proof. (i) $\Rightarrow$ (ii). Suppose T is standard and $x \in T \vee I$. Then $x \geq t \wedge i$ for some $t \in T$ and $i \in I$. Since T is standard, so by Theorem 2.5, there exist $t_1 \in T$, $i_1 \geq i$ such that $x = t_1 \wedge i_1$. Thus (ii) holds.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (iv). Suppose (iii) holds and $p \in [x) \wedge (T \vee [y))$. Then $p \geq x$ and $p = t \wedge y_1$ for some $t \in T$, $y_1 \in [y)$. Now $t \geq p \geq x$ and $y_1 \geq p \geq x$ implies $t \in [x) \wedge T$ and $y_1 \in [x) \wedge [y)$. Thus $p \in ([x) \wedge T) \vee ([x) \wedge [y))$ and hence (iv) holds.

(iv) $\Rightarrow$ (i) . Suppose (iv) holds and $x \geq t \wedge a$ for some $t \in T$ $(x, a \in S)$. Then $x \in T \vee [a)$, and so $[x) = [x) \wedge (T \vee [a)) = ([x) \wedge T) \vee ([x) \wedge [a))$. Then $x \geq p \wedge q$ for some $p \in T$, $q \geq a$ with $p, q \geq x$. This implies $x = p \wedge q$ and $p \in T$, $q \geq a$. Hence by Theorem 2.5, T is standard.

Theorem 2.7. *Let* I *be an arbitrary filter and* T *a standard filter of* S. *If* $I \vee T$ *and* $I \wedge T$ *are both principal, then* I *itself is principal.*

Proof. Let $I \vee T = [a)$ and $I \wedge T = [b)$. Then $a = t \wedge x$ $(t \in T, x \in I)$. We claim that $I = [x \wedge b)$. Indeed, $[a) = [t) \vee [x) \subseteq T \vee [x) \subseteq T \vee [x \wedge b) \subseteq T \vee I = [a)$, so $T \vee I = T \vee [x \wedge b)$. Further, $[b) = T \wedge I \supseteq T \wedge [x \wedge b) \supseteq T \wedge [b) = [b)$ and so $T \wedge I = T \wedge [x \wedge b)$. Since T is standard, by [7, Theorem 1], $I = [x \wedge b)$.

The following theorem gives a characterization of standard filters in terms of a congruence.

Theorem 2.8. *A filter* T *of* S *is standard if and only if*
$\theta = \{(x, y) \in S \times S \mid x \wedge t = y \wedge t = x \wedge y$ *for some* $t \in T\}$ *is a congruence relation and* $(x, y) \in \theta$, $z \geq y$, which *implies that there exists* $w \geq x, z$ *such that* $(w, z) \in \theta$.

Proof. Let T be a standard filter. It will be shown that $\theta = \{(x, y) \in S \times S \mid x \wedge t = y \wedge t$ for some $t \in T\}$ $(= \theta_T)$ so that θ is a congruence relation. Clearly $\theta \subseteq \theta_T$. So let $(x, y) \in \theta_T$ so that $x \wedge t = y \wedge t$ and hence $x \in [x) \cap ([y) \vee T) = [x) \cap [y)) \vee ([x) \cap T)$. Thus there exist $y_1 \geq y, x$, $t_1 \in T$, $t_1 \geq x$ such that $x \geq y_1 \wedge t_1$ and hence $x = y_1 \wedge t_1$. By a similar argument there exist $x_1 \geq x$, $t_2 \in T$ such that $y = x_1 \wedge t_2$. Therefore $x \wedge y = y_1 \wedge t_1 \wedge x_1 \wedge t_2$ so that $x \wedge y = x \wedge t_1 \wedge t_2 = y \wedge t_1 \wedge t_2$ and $t_1 \wedge t_2 \in T$. Thus $(x, y) \in \theta$. Let $(x, y) \in \theta$ and $z \geq y$ so that $x \wedge t = y \wedge t$ for some $t \in T$ and hence $z \geq z \wedge t \geq y \wedge t = x \wedge t$. Thus $z \in [z) \cap ([x) \vee T) = ([z) \cap [x)) \vee ([z) \cap T)$ so that $z \geq x_1 \wedge t_1$ where $x_1 \geq z, x$, $t_1 \geq z$, $t_1 \in T$ and hence $z = x_1 \wedge t_1$. Thus $z \wedge t_1 = x_1 \wedge t_1$ so that $(z, x_1) \in \theta_T = \theta$.

Conversely let $x \in A \cap (T \vee B)$ so that $x \in A$ and $x \geq t \wedge b$ for some $t \in T$ and $b \in B$. Since $(t \wedge b, b) \in \theta$, there exists $w \geq x, b$ such that $(w, x) \in \theta$. Hence $w \wedge t_1 = x \wedge t_1 = w \wedge x = x$ for some $t \in T$. Thus T is standard.

Thus we have the following corollary.

Corollary 2.9. *An element* $t \in S$ *is dual standard if and only if* $\theta = \{(x, y) \in S \times S \mid x \wedge t_1 = y \wedge t_1 = x \wedge y$ *for some* $t_1 \geq t\}$ *is a congruence relation and* $(x, y) \in \theta$. $z \geq y$ *implies that there exists* $w \geq x, z$ *such that* $(w, z) \in \theta$.

Finally we include the following characterization of dual standard elements.

Theorem 2.10. *An element* s *of* S *is dual standard, if and only if,*

(i) s *is dual distributive*

and (ii) s *is modular.*

Proof. Suppose s is dual standard. Then (i) holds by Lemma 2.1.

For (ii), let $y \geq s \wedge x$ with $y \leq x$. Since s is dual standard, so $y = s_1 \wedge x_1$ for some $s_1 \geq s$, $x_1 \geq x$ (s_1, $x_1 \in S$). Thus $y = x \wedge y = x \wedge s_1 \wedge x_1 = s_1 \wedge x$. Therefore S is modular.

Conversely, suppose (i) and (ii) hold and let $y \geq s \wedge x$ ($x, y \in S$). Since $y \geq s \wedge y$ and s is dual distributive, there exists $t \in S$ with $t \geq x, y$ such that $y \geq s \wedge t$. Then by modularity of s, there exists $s_1 \geq s$ such that $y = s_1 \wedge t$. This implies that s is dual standard in S.

Thus we have the following results:

Corollary 2.11. *In a modular semilattice, every dual distributive element is dual standard.*

Corollary 2.12. *A filter of a semilattice is standard if and only if it is both modular and distributive.*

Conclusion

In this paper we have proved that a filter in a directed above meet semilattice is standard if and only if it is both modular and distributive.

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