Numerical Prediction of Non-Isothermal Flow Through a Rotating Curved Duct with Square Cross-Section

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Abstract

Non-isothermal flow through a rotating curved duct with square cross section, driven by a pressure gradient along the duct, is studied numerically by using the spectral method over a wide range of the Taylor number, Tr, $0 \le Tr \le 2000$. A temperature difference is applied across the vertical sidewalls for Grashof number Gr = 100, where the outer wall of the duct is heated and the inner one is cooled. The rotation of the duct about the center of curvature is imposed. The effects of rotation (Coriolis force) on the flow characteristics are investigated. Though the present study covers a wide range of the Dean number, Dn, $0 \le Dn \le 2000$, in the present paper, three cases of the Dean numbers, Dn = 500, Dn = 1000 and Dn = 2000 are discussed in detail. Steady solutions are obtained by the Newton-Raphson iteration method and their linear stability is investigated. When there is no stable steady solution, time evolution calculations as well as their spectral analyses show that the periodic oscillations, obtained for small Tr and at moderate Dn, turn into steady state if Tr is increased. For larger Dn, however, the flow undergoes ``steady \rightarrow periodic \rightarrow chaotic \rightarrow steady", if Tr is increased.

Keywords: Rotating curved duct, Secondary flow, Dean number, Taylor number, Time evolution

1. Introduction

The study of flow through a curved duct is of fundamental interest because of its importance in chemical. mechanical and biological engineering. Due to engineering applications and their intricacy, the flow in a rotating curved duct has become one of the most challenging research fields of fluid mechanics. Since rotating machines were introduced into engineering applications, such as gas turbines, electric generators, rotating heat exchangers, cooling systems and some separation processes, scientists have paid considerable attention to the characteristics of the flows in these rotating systems. The readers are referred to Berger et al. [1], Nandakumar and Masliyah [2] and Ito [3] for some outstanding reviews on curved duct flows.

One of the interesting phenomena of the flow through a curved duct is the bifurcation of the flow because generally there exist many steady solutions due to channel curvature. Many researchers have performed experimental and numerical investigations on developing and fully developed curved duct flows. An early complete bifurcation study of two-dimensional (2-D) flow through a curved duct of square cross section was conducted by Winters [4]. Very recently, Mondal et al. [5] a performed a comprehensive numerical study on fully developed bifurcation structure and stability of 2-D flow through a curved duct with square cross section and found a close relationship between the unsteady solutions and the bifurcation diagram of steady solutions. The flow through a curved duct with differentially heated vertical sidewalls has other aspects because secondary flows promote fluid

mixing and heat transfer in the fluid (Chandratilleke and Nursubyakto [6]). Recently, Mondal et al. [7] and Yanase et al. [8] performed numerical investigations of nonisothermal flows through curved ducts with rectangular cross sections square and respectively, where they studied the flow characteristics with the effects of secondary flows on convective heat transfer. They also studied the transitional behavior of the unsteady solutions by time evolution calculations.

The flow through a rotating curved duct is subject. which has attracted another considerable attention because of its importance in engineering devices. Early works on rotating curved duct flows were constrained to two simplified limiting cases with strong or weak rotations. Ludwieg [9] first analyzed the flow in a co-rotating (the rotating angular velocity and the axial velocity are in the same direction) curved duct by integrating the momentum equations. Miyazaki [10] studied the characteristics of the flow and heat transfer in a rotating curved rectangular duct with positive rotation. Wang and Cheng [11], employing a finite volume method, examined the flow characteristics and heat transfer in curved ducts for positive cases and found reverse secondary flow for co-rotation cases. Selmi and Nandakumar [12] and Yamamoto et al. [13] performed extensive works on the rotating curved duct flows and their bifurcations. Yamamoto et al. [13], employing the spectral method, examined the flow structure and the flow rate ratio for the flow in a rotating curved square duct and found a six-cell phenomenon in the secondary flow. In their paper, they predicted there should be some multiple solutions but they did not obtain then. Yang and Wang [14] performed comprehensive numerical study on bifurcation structure and stability of solutions for laminar mixed convection in a rotating curved duct of square cross section. Transient behavior of the unsteady solutions, such as periodic, multi-periodic or chaotic solutions are yet unresolved for the nonisothermal flow in a rotating curved duct. This paper is, therefore, an attempt to fill up this gap with the study of stability analysis of multiple solutions.

It is well known that, the fluid flowing in a rotating curved duct is subjected to two forces: the Coriolis force due to rotation and the centrifugal force due to curvature. These two forces affect each other, so complex behaviors of the secondary flow and the axial flow can be obtained. For isothermal flows of a constant property fluid, the Coriolis force tends to produce vorticity while centrifugal force is purely hydrostatic. When a temperature induced variation of fluid density occurs for nonisothermal flows, both Coriolis and centrifugaltype buoyancy forces can contribute to the generation of vorticity (Wang and Cheng [11]). These two effects of rotation either enhance or counteract each other in a non-linear manner depending on the direction of duct rotation, direction of wall heat flux and the flow domain. Buoyancy force is proportional to the square of the rotation speed while Coriolis force increases proportionally with the rotation speed itself (Wang and Cheng [11]). Therefore, the effect of system rotation is more subtle and complicated and yields new; richer features of flow and heat transfer in general, bifurcation and stability in particular, for non-isothermal flows. While some of such new features are revealed by recent analytical and numerical works (Wang and Cheng [11]; Yang and Wang [14]), there is no known study on bifurcation and stability for forced convection in a rotating curved duct with the study of time-dependent behavior.

In the present paper, a comprehensive numerical study is presented for fully developed bifurcation structure and stability of 2-D viscous incompressible fluid through a rotating curved square duct whose outer wall is heated and inner one is cooled. Flow characteristics are studied over a wide range of the Dean number and the Taylor number by finding the steady solutions, investigating their linear stability and calculating nonlinear behavior of the unsteady solutions by time evolution calculations, spectral analysis and phase spaces.

2. Governing Equations

Consider a hydrodynamically and thermally fully developed two-dimensional flow of viscous incompressible fluid through a rotating curved duct with square cross section. Let 2hand 2l be the height and the width of the cross section, although we are considering only the case of h = l in the present study. Figure 1 shows the coordinate system, where C is the center of the duct cross-section and L is the radius of curvature of the duct. The x' and y' axes are taken to be in the horizontal and vertical directions respectively, and z' is the coordinate along the center-line of the duct, i.e., the axial direction. The system rotates at a constant angular velocity Ω_T around the y' axis. It is assumed that the outer wall of the duct is heated while the inner one is cooled. The temperature of the outer wall is $T_0 + \Delta T$ and that of the inner wall is $T_0 - \Delta T$, where $\Delta T > 0$. It is also assumed that the flow is uniform in the axial direction, and that it is driven by a constant pressure gradient $G\left(G = -\frac{\partial P'}{\partial z'}\right)$ along the

center-line of the duct. Then the continuity, Navier-Stokes and energy equations, in terms of dimensional variables, are expressed as:

Continuity equation:

$$\frac{\partial u'}{\partial r'} + \frac{\partial v}{\partial y'} + \frac{u'}{r'} = 0 \tag{1}$$

Momentum equations:

$$\frac{\partial u'}{\partial t'} + u' \frac{\partial u'}{\partial r'} + v' \frac{\partial u'}{\partial y'} - \frac{w'^2}{r'} - 2\Omega_T w'$$

$$= -\frac{1}{\rho} \frac{\partial P'}{\partial r'} + v \left[\frac{\partial^2 u'}{\partial r'^2} + \frac{\partial^2 u'}{\partial y'^2} + \frac{1}{r'} \frac{\partial u'}{\partial r'} - \frac{u'}{r'^2} \right],$$
(2)

$$\frac{\partial v'}{\partial t'} + u' \frac{\partial v'}{\partial r'} + v' \frac{\partial v'}{\partial y'} = -\frac{1}{\rho} \frac{\partial P'}{\partial y'} + v \left[\frac{\partial^2 v'}{\partial r'^2} + \frac{1}{r'} \frac{\partial v'}{\partial r'} + \frac{\partial^2 v'}{\partial y'^2} \right] + g\beta T',$$
(3)

$$\frac{\partial w'}{\partial t'} + u' \frac{\partial w'}{\partial r'} + v' \frac{\partial w'}{\partial y'} + \frac{u'w'}{r'} + 2\Omega_{\tau}u'$$

$$= -\frac{1}{\rho} \frac{1}{r'} \frac{\partial P'}{\partial z'} + v \left[\frac{\partial^2 w'}{\partial r'^2} + \frac{\partial^2 w'}{\partial y'^2} + \frac{1}{r'} \frac{\partial w'}{\partial r'} - \frac{w'}{r'^2} \right]$$
(4)

Energy equation:

$$\frac{\partial T'}{\partial t'} + u' \frac{\partial T'}{\partial r'} + v' \frac{\partial T'}{\partial y'} = \kappa \left[\frac{\partial^2 T'}{\partial r'^2} + \frac{\cdot 1}{r'} \frac{\partial T'}{\partial r'} + \frac{\partial^2 T'}{\partial y'^2} \right],$$
(5)

where r' = L + x', and u', v' and w' are the dimensional velocity components in the x', y', and z' directions respectively, and these

velocities are zero at the wall. Here, P' is the dimensional pressure, T' the dimensional temperature and t' is the dimensional time. In the above formulations, ρ , ν , β , κ and g are the density, the kinematic viscosity, the coefficient of thermal expansion, the coefficient of thermal diffusivity and the gravitational acceleration, respectively. Thus, in Eqs. (1) to (5) the variables with prime denote the dimensional quantities.

The dimensional variables are then nondimensionalized by using the representative length l, the representative velocity $U_0 = \nu/l$, where ν is the kinematic viscosity of the fluid. We introduce the non-dimensional variables defined as:

$$u = \frac{u'}{U_0}, \quad v = \frac{v'}{U_0}, \quad w = \frac{\sqrt{2\delta}}{U_0}w',$$
$$x = \left(\frac{x'}{l} - \frac{1}{\delta}\right), \quad \overline{y} = \frac{y'}{l}, \quad z = \frac{z'}{l}$$
$$T = \frac{T'}{\Delta T}, \quad t = \frac{U_0}{d}t', \quad \delta = \frac{d}{L},$$
$$P = \frac{P'}{\rho U_0^2},$$

where u, v and w are the non-dimensional velocity components in the x, \overline{y} and z directions, respectively; t is the nondimensional time, P the non-dimensional pressure, δ is the nondimensional curvature defined as $\delta = \frac{l}{L}$, and temperature is nondimensionalized by ΔT . Henceforth, all the variables are nondimensionalized if not specified.

Since the flow field is uniform in the zdirection, the sectional stream function ψ is introduced as:

$$u = \frac{1}{1 + \delta x} \frac{\partial \psi}{\partial \overline{y}}, \quad v = -\frac{1}{1 + \delta x} \frac{\partial \psi}{\partial x}.$$
 (6)

A new coordinate variable y is introduced in the \overline{y} direction as $\overline{y} = ay$, where $a = \frac{h}{l}$ is the aspect ratio of the duct cross section. In the present study, we consider the case for a = l(square duct). Then, the basic equations for w, ψ and T are expressed in terms of nondimensional variables as:

$$(1 + \delta x)\frac{\partial w}{\partial t} + \frac{\partial (w, \psi)}{\partial (x, y)} - Dn + \frac{\delta^2 w}{1 + \delta x}$$
$$= (1 + \delta x)\Delta_2 w - \frac{\partial}{(1 + \delta x)}\frac{\partial \psi}{\partial y}w + \delta \frac{\partial w}{\partial x} - \delta Tr \frac{\partial \psi}{\partial y},$$
(7)

$$\begin{split} &\left(\Delta_{2} - \frac{\delta}{1+\delta x}\frac{\partial}{\partial x}\right)\frac{\partial\psi}{\partial t} \\ = -\frac{1}{(1+\delta x)}\frac{\partial(\Delta_{2}\psi,\psi)}{\partial(x,y)} + \frac{\delta}{(1+\delta x)^{2}} \\ &\times \left[\frac{\partial\psi}{\partial y}\left(2\Delta_{2}\psi - \frac{3\delta}{1+\delta x}\frac{\partial\psi}{\partial x} + \frac{\partial^{2}\psi}{\partial x^{2}}\right) - \frac{\partial\psi}{\partial x}\frac{\partial^{2}\psi}{\partial x\partial y}\right] \\ &+ \frac{\delta}{(1+\delta x)^{2}}\times \left[3\delta\frac{\partial^{2}\psi}{\partial x^{2}} - \frac{3\delta^{2}}{1+\delta x}\frac{\partial\psi}{\partial x}\right] - \frac{2\delta}{1+\delta x}\frac{\partial}{\partial x}\Delta_{2}\psi \\ &+ w\frac{\partial w}{\partial y} + \Delta_{2}^{-2}\psi + \frac{1}{2}Tr\frac{\partial w}{\partial y}, \end{split}$$
(8)

$$\frac{\partial T}{\partial t} + \frac{1}{\left(1 + \delta x\right)} \frac{\partial (T, \psi)}{\partial (x, y)} = \frac{1}{\Pr} \left(\Delta_2 T + \frac{\delta}{1 + \delta x} \frac{\partial T}{\partial x} \right)$$
(9)

where

$$\Delta_2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \frac{\partial(f,g)}{\partial(x,y)} \equiv \frac{\partial f \,\partial g}{\partial x \,\partial y} - \frac{\partial f \,\partial g}{\partial y \,\partial x}$$
(10)

The Dean number Dn, the Grashof number Gr, the Taylor number Tr and the Prandtl number Pr, which appear in Eqs. (7) to (8) are defined as:

$$Dn = \frac{Gl^3}{\mu v} \sqrt{\frac{2l}{L}}, \quad Gr = \frac{\beta g \Delta Tl^3}{v^2}, \quad Tr = \frac{2\sqrt{2\delta}\Omega_T l^3}{v\delta},$$
$$\Pr = \frac{v}{k} \tag{11}$$

where μ is the viscosity of the fluid. In the present study, only *Dn* and *Tr* varied while δ , *Gr* and *Pr* are fixed as $\delta = 0.1$, *Gr* = 100, 500 and 1000 and *Pr* = 7.0 (water).

Rigid boundary conditions for w and ψ are used:

$$w(\pm 1, y) = w(x, \pm 1) = \psi(\pm 1, y) = \psi(x, \pm 1)$$
$$= \frac{\partial \psi}{\partial x}(\pm 1, y) = \frac{\partial \psi}{\partial y}(x, \pm 1) = 0$$
, (12)

and the temperature T is assumed to be constant on the walls as:

$$T(1, y) = 1, T(-1, y) = -1, T(x, \pm 1) = x.$$
 (13)

It should be noted that Eqs. (7) to (9) are invariant under the transformation of the variables:

$$\begin{array}{c|c}
y \Rightarrow -y, \\
w(x, y, t) \Rightarrow w(x, -y, t), \\
\psi(x, y, t) \Rightarrow -\psi(x, -y, t), \\
T(x, y, t) \Rightarrow -T(x, -y, t).
\end{array}$$
(14)

Therefore, the case of heating the inner sidewall and cooling the outer sidewall can be deduced directly from the results obtained in this paper.

3. Numerical Calculations 3.1 Method of Numerical Calculation

The method adopted in the present numerical calculation is the spectral method. This is the method which is thought to be the best numerical method for solving the Navier-Stokes equations (Gottlieb and Orszag [15]). By this method, the variables are expanded in a series of functions consisting of Chebyshev polynomials. That is, the expansion functions $\Phi_n(x)$ and $\Psi_n(x)$ are expressed as:

$$\Phi_n(x) = (1 - x^2) C_n(x), \quad \Psi_n(x) = (1 - x^2)^2 C_n(x),$$
(15)

where $C_n(x) = \cos(n \cos^{-1}(x))$ is the n^{th} order Chebyshev polynomial. w(x, y, t), $\psi(x, y, t)$ and T(x, y, t) are expanded in terms of the expansion functions $\Phi_n(x)$ and $\Psi_n(x)$ as:

$$w(x, y, t) = \sum_{m=0}^{M} \sum_{n=0}^{N} w_{mn}(t) \Phi_m(x) \Phi_n(y),$$

$$\psi(x, y, t) = \sum_{m=0}^{M} \sum_{n=0}^{N} \Psi_{mn}(t) \Psi_m(x) \Psi_n(y),$$

$$T(x, y, t) = \sum_{m=0}^{M} \sum_{n=0}^{N} T_{mn}(t) \Phi_m(x) \Phi_n(y) + x$$
(16)

where *M* and *N* are the truncation numbers in the *x* and *y* directions respectively. The expansion coefficients w_{mn} , ψ_{mn} and T_{mn} are then substituted into the basic Eqs. (7) and (9) and the collocation method is applied. As a result, nonlinear algebraic equations for w_{mn} , ψ_{mn} and T_{mn} are obtained. The collocation points are taken to be:

$$x_{i} = \cos\left[\pi\left(1 - \frac{i}{M+2}\right)\right], \quad i = 1, ..., M+1$$
$$y_{j} = \cos\left[\pi\left(1 - \frac{j}{N+2}\right)\right], \quad j = 1, ..., N+1$$
$$(17)$$

where i = 1,..., M + 1 and j = 1,..., N + 1. The steady solutions are obtained by the Newton-Raphson iteration method assuming that all the coefficients are time independent. The convergence is assured by taking $\varepsilon_p < 10^{-10}$, where subscript *p* denotes the iteration number and ε_p is defined as:

$$\varepsilon_{p} = \sum_{m=0}^{M} \sum_{n=0}^{N} \left[\left(w_{mn}^{(p+1)} - w_{mn}^{p} \right)^{2} + \left(\psi_{mn}^{(p+1)} - \psi_{mn}^{p} \right)^{2} \right].$$

Linear stability of the steady solutions is two-dimensional (zinvestigated against independent) perturbations. To do this, the eigenvalue problem is solved, which is constructed by the application of the function expansion method, together with the collocation method, to the perturbation equations obtained from Eqs. (7) to (9). It is assumed that the time dependence of the perturbation is $e^{\sigma t}$, where $\sigma = \sigma_r + i\sigma_i$ is the eigenvalue with σ_r the real part, σ_i the imaginary part and $i = \sqrt{-1}$. If all the real parts of the eigenvalue σ are negative, the steady solution is linearly stable, but if there exists at least one positive real part of the eigenvalue, it is linearly unstable. In the unstable region, the perturbation grows monotonically for $\sigma_i = 0$ and oscillatorily for $\sigma_i \neq 0$. Finally, in order to calculate the unsteady solutions, the Crank-Nicolson and Adams-Bashforth methods together with the function expansion (16) and the collocation methods are applied to Eqs. (7) to (9).

3.2 Numerical Accuracy

numerical the present In calculations, M = 20 and N = 20 have been used for sufficient accuracy of the solutions. The values of Q and w(0,0), obtained for Gr = 100, Dn = 1000 and Tr = 100 at $\delta = 0.1$ are shown in Table 1 for various values of M, where Q is the flux through the duct and w(0,0) is the axial velocity of the steady solutions at (x, y) = (0, 0). From Table 1, it is found that Q changes 0.0055% from M = 14 to M = 16, 0.0046% from M = 16 to M = 18, 0.0005% from M = 18 to M =20 and 0.000003% from M = 20 to M = 22; w(0.0) changes 0.1406% from M = 14 to M =16, 0.0641% from M = 16 to M = 18, 0.0325% from M = 18 to M = 20, and 0.0351% from M =20 to M = 22. Therefore, it is concluded that the values M = 20 and N = 20 give sufficient accuracy for the present numerical calculations.

4. Flux through the duct

The dimensional total flux Q' through the duct in the rotating coordinate system is calculated by:

$$Q' = \int_{-d}^{d} \int_{-d}^{d} w' dx' dy' = v dQ$$
 (18)

where

$$Q = \int_{-1}^{1} \int_{-1}^{1} w dx dy$$
 (19)

is the dimensionless total flux. The mean axial velocity \overline{w}' is expressed as:

$$\overline{w}' = \frac{Qv}{4d} \tag{20}$$

In the present paper, Q is used to denote the steady solution branches and to pursue the time evolution of the unsteady solutions.

5. Results

We take a curved duct with square cross section and rotate it around the center of curvature with an angular velocity Ω_T . According to the definition of Tr, positive Trmeans that the rotational direction is the same as that of the main flow. In the present study, we investigate the flow characteristics only for the case of positive rotation of the duct (positive Tr) and discuss the flow phenomena for three cases of the Dean numbers, *Case I: Dn* = 500, *Case II:* Dn = 1000 and *Case III: Dn* = 2000, over a wide range of the Taylor number $0 \le Tr \le 2000$. Thus, an interesting and complicated flow behavior will be expected if duct rotation is involved for these cases.

Case I: Dn = 500

Steady solutions and their linear stability analysis

With the present numerical calculations, a single branch of the steady solution is obtained for Dn = 500 by using the path continuation technique as discussed in Keller [16]. In this regard, it should be noted that Mondal et al. [7] obtained two branches of steady solutions for the non-isothermal flow through a curved square duct without rotation. Figure 2(a) shows the flux Q through the duct versus the Taylor number Tr for Dn = 500. As seen in Fig. 2(a), the branch is very simple and is smoothly extending to larger Tr. The branch starts from point a (Tr = 0) and goes to the direction of increasing Tr as Q decreases, and arrives at point b (Tr = 2000) without any turning on its way.

We now discuss the variation of the secondary flow and temperature profile at several values of Tr on this branch for constant ψ and T. We look at the figures from upstream. Therefore, from these figures we can understand the structures of the secondary flow and temperature distribution in a cross-section of the duct. To draw the contours of ψ and T, we use the increments $\Delta \psi = 0.6$ and $\Delta T = 0.2$ for all the figures in this paper, if not specified. The right-hand side of each box is in a outside direction of the duct curvature. In the figures of secondary flow, solid lines $(\psi \ge 0)$ show that the secondary flow is in a counterclockwise direction while dotted lines ($\psi < 0$) are in a clockwise direction. In the figures of temperature field, on the other hand, solid lines

are those for $T \ge 0$ and dotted ones for T < 0. As seen in Fig. 2(b), the steady solution branch consists of nearly symmetric two-vortex solutions only. Linear stability of the steady solution branch is then investigated. It is found that the branch is linearly stable for any Tr in the range investigated in this paper. Then, in order to study the non-linear behavior of the unsteady solutions, time-evolution calculations of Q are performed for $0 \le Tr \le 2000$. Since the steady solution branch is linearly stable for any value of Tr in the range, time evolution of Q also shows that the value of Q quickly approaches steady state.

Case II: Dn = 1000

Steady solutions and their linear stability analysis

We obtain two branches of steady solutions = 1000 over the Taylor for Dn number $0 \le Tr \le 2000$. It is found that there exists no bifurcating relationship between the two branches. The bifurcation diagram is shown in Fig. 3 for Dn = 1000 using Q, the representative quantity of the solutions. The two steady solution branches are named the first steady solution branch (first branch, thin solid line) and the second steady solution branch (second branch, dashed line), respectively. In the following, the two steady solution branches as well as the flow patterns on the respective branches are discussed.

The first steady solution branch

The first steady solution branch for Dn =1000 is solely depicted in Fig. 4(a) for $0 \le Tr \le 2000$. It should be remarked that between the two branches of steady solutions, only this branch exists throughout the whole range of Tr. As seen in Fig. 4(a), the branch is very similar to the branch obtained for Dn = 500. The branch starts from point a (Tr = 0) and extends to the direction of increasing Tr and decreasing Q up to point c (Tr = 2000) without any turning throughout its way. The change of the flow characteristics contours of typical secondary flow and temperature profile are shown in Fig. 4(b) for several values of Tr, where the contours of ψ and T are drawn with the same increments as discussed in Case I. As seen in Fig. 4(b), the first branch is composed of two-vortex solutions. The secondary flow is an asymmetric two-vortex solution for small Tr.

However, as Tr increases, the asymmetry gradually decreases and the flow pattern is nearly symmetric due to weak Coriolis force. With strong centrifugal force, the flow pattern becomes asymmetric; as Tr increases the Coriolis force becomes strong, which balances the centrifugal force, and the flow pattern is approximately symmetric.

Linear stability of the first steady solution branch is then investigated. It is found that the branch is linearly unstable if Tris small (Tr < 138.6). However, as Tr increases, the steady solution becomes stable and remains stable onwards for larger Tr. Thus we find that the steady solution is linearly stable for $138.6 \le Tr \le 2000$. The eigenvalues of the first steady solution branch are listed in Table 2 where the eigenvalues with the maximum real part of σ (first eigenvalue) are presented. Those for the linearly stable solutions are printed in bold letters. As seen in Table 2, the stability region exists for $138.6 \le Tr \le 2000$ and the perturbation grows monotonically ($\sigma_i = 0$) for larger Tr. Therefore, the Hopf bifurcation occurs at $Dn \approx 138.6$. The region of linearly stable steady solution is shown with a thick solid line in Fig. 4(a).

The second steady solution branch

The second steady solution branch for Dn =1000, shown by a dashed line in Fig. 3, is solely depicted in Fig. 5(a). As seen in Fig. 5(a), the branch starts from point a (Tr = 0) and goes to the direction of increasing Tr and decreasing Qup to point b (Tr = 2000), where it experiences a smooth turning and goes to the direction of increasing Q and decreasing Tr up to point c (Tr= 0). The change of the flow patterns, contours of typical secondary flow and temperature profile are shown in Fig. 5(b) for several values of Tr. As seen in Fig. 5(b), the branch is composed of two- and four-vortex solutions. It is found that the secondary flow is an asymmetric two-vortex solution from point a, and as Tr increases $(Tr \ge 500)$ an additional pair of secondary vortices appear in the central part of the right-hand side of the duct cross-section. These additional vortices are called Dean vortices, which play an important role in the enhancement of heat transfer. From point b (Tr = 742) to point c (Tr = 0), the secondary flow is a symmetric four-vortex solution. Linear stability of the steady solution shows that the

branch is linearly unstable everywhere, for any value of *Tr*.

Time evolution

In order to study the non-linear behavior of the unsteady solutions, following Mondal [17]. time-evolution calculations as well as their spectral analysis are performed for Tr in the range $0 \le Tr \le 2000$ at Dn = 1000. Time evolution of Q for Tr = 100 is shown in Fig. 6(a). It is found that the flow is time periodic. In the same figure, the relationship between the periodic solution and the steady states, the values of O for the steady solution branches at Tr = 100 are also shown by straight lines using the same kind of lines as were used in the bifurcation diagram in Fig. 3. As seen in Fig. 6(a), the periodic solution at Tr = 100 oscillates in the region between the upper and lower parts of the second steady solution branch, and the upper part of the second branch or the first branch plays a role of an envelope of this periodic oscillation. Then, in order to see the change of the flow characteristics as time proceeds, contours of typical secondary flow and temperature distribution are shown in Fig. 6(b) for one period of oscillation at Tr = 100, where it is seen that the periodic solution at Tr =100 oscillates between asymmetric two- and four-vortex solutions. The periodic oscillation obtained for Tr = 100 at Dn = 1000 is well justified by the power spectrum of the time change of Q as shown in Fig. 7. In this figure, the line spectrum of the fundamental frequency (f = 20 Hz) as well as its harmonics are seen which indicates that the flow is time periodic. If Tr is increased further, the periodic oscillation turns into steady state. Time evolution of Q is then performed for $Tr \ge 139$, where the steady solution is linearly stable on the first branch, and it is found that the value of Q quickly approaches that of the stable solution on the first branch.

Case III: Dn = 2000

Steady solutions and their linear stability analysis

We obtain four branches of steady solutions for Dn = 2000 over a wide range of Tr for $0 \le Tr \le 2000$. The bifurcation diagram of steady solutions is shown in Fig. 8. The four steady solution branches are named the *first steady solution branch* (first branch, thick solid line), the *second steady solution branch* (second branch, dashed line), the third steady solution branch (third branch, thin solid line) and the fourth steady solution branch (fourth branch, dash dotted line), respectively. It is found that the steady solution branches are independent and there exists no bifurcating relationship among the branches in the parameter range investigated in this paper. The steady solution branches are obtained by the path continuation technique with various initial guesses as discussed in Mondal [17] and are distinguished by the nature and number of secondary flow vortices appearing in the cross section of the duct. In the following, the four steady solution branches along with the flow patterns on the respective branches are discussed in detail.

The first steady solution branch

The first steady solution branch for Dn =2000 is solely depicted in Fig. 9(a) for $0 \le Tr \le 2000$. It is found that the branch is very similar to the first branch obtained for Tr =1000. The branch starts from point a (Tr = 0)and goes to the direction of increasing Tr as Odecreases which extends up to point d(Tr =2000) without any turning. Then, in order to observe the change of the flow patterns on the first branch, contours of typical secondary flow and temperature profile are drawn at several values of Tr as shown in Fig. 9(b), where it is seen that the branch is composed of only twovortex solutions which are symmetric with respect to the horizontal plane y = 0. Three types of forces. Coriolis force, strong centrifugal force and buoyancy force act on the fluid at the same time, which make the flow patterns symmetric.

Linear stability of the first branch shows an interesting result. It is found that the branch is linearly stable in a couple of intervals of Tr, one for small $Tr \ (0 \le Tr \le 329.7)$ and another one for larger Tr (907.2 $\leq Tr \leq 2000$). Thus the branch is linearly unstable for the region $(329.8 \le Tr \le 907.1)$. The eigenvalues of the first steady solution branch are listed in Table 3, where the eigenvalues with the maximum real part of σ (first eigenvalue) are presented. Those for the linearly stable solutions are printed in bold letters. As seen in Table 3, the perturbation grows oscillatorily $(\sigma_i \neq 0)$ for $329.7 \le Tr \le 907.2$ and monotonically ($\sigma_i = 0$) for $Tr \ge 907.2$. Therefore, the *Pitchfork bifurcation* occurs at Tr = 329.7 and the *Hopf* *bifurcation* at $Dn \approx 907.2$. Linearly stable steady solution regions are shown with thick solid lines in Fig. 9(a).

The second steady solution branch

We draw the second steady solution branch for Dn = 2000 separately in Fig. 10(a). The branch has a similarity with the second branch obtained for Dn = 1000, and the only difference is that it extends up to larger Tr. As seen in Fig. 10(a), the branch starts from point a (Tr = 0) and goes to the direction of increasing Tr as Qdecreases and arrives at point b (Tr = 1409.12), where it turns to the opposite direction with a gentle turning at point b. The branch then goes to the direction of increasing Q and decreasing Tr up to point c (Tr = 0). To observe the change of the flow patterns, contours of typical secondary flow and temperature profile on this branch are shown in Fig. 10(b) for several values of Tr. As seen in Fig. 10(b), the branch consists of asymmetric two- and nearly symmetric four-vortex solutions. It is found that the secondary flow is a two-vortex solution from point a to point b, but when the branch turns at point b down to point c the secondary flow becomes a four-vortex solution. Linear stability of the steady solution shows that the branch is linearly unstable for any value of Tr.

The third steady solution branch

The third steady solution branch for Dn =2000, shown by a thin solid line in Fig. 8, is exclusively depicted in Fig. 11(a). As seen in Fig. 11(a), the branch is very entangled with many turning points on its way, like the third branch obtained by Yanase et al. [8] for isothermal flow without rotation. We draw the contours of secondary flow and temperature profile at several values of Tr on this branch in Fig. 11(b), where to draw the contours of ψ and T we use the increments $\Delta \psi = 1.2$ and $\Delta T = 0.4$, respectively. As seen in Fig. 11(b), the branch is comprised of two- and four-vortex solutions but are different from those of the second steady solution branch. Linear stability of the third branch shows that the branch is also unstable everywhere over the region of Tr investigated in this paper.

The fourth steady solution branch

We draw the fourth steady solution branch for Dn = 2000 solely in Fig. 12(a). As seen in Fig. 12(a), the branch exists for a small region of Tr, and the upper and lower parts of branch pass very close to each other. The branch starts from point a (Tr = 0) and goes to the direction of increasing Tr and decreasing O up to point b (Tr = 256), where it experiences a reverse turning and goes to the direction of increasing O and decreasing Tr up to point c (Tr = 0). To observe the change of the flow patterns and temperature distributions, contours of typical secondary flow and temperature profile on this branch are shown in Fig. 12(b) at several values of Tr. where it is seen that the branch is composed of only four-vortex solutions. Linear stability of the fourth branch shows that the branch is linearly unstable everywhere.

Time evolution

We perform time-evolution calculations of the unsteady solutions for Dn = 2000and $0 \le Tr \le 2000$. Time evolution of O for Dn = 2000 and $Tr \leq 329$, at which the steady solution is linearly stable on the first branch, shows that the value of Q quickly approaches that of the stable solution on this branch no matter what the initial conditions we use. Then, in order to see what happens when all the steady are solutions linearly unstable in the region $330 \le Tr \le 907$, time evolutions of Q are then performed for Tr = 500 and 800. Figure 13(a) shows the time-evolution result for Tr =500 where it is seen that the flow oscillates multi-periodically. In the same figure, to observe the relationship between the periodic solution and the steady states, the values of Qfor the steady solution branches at Tr = 500 are also shown by straight lines using the same kind of lines as were used in the bifurcation diagram in Fig. 8. As seen in Fig. 13(a), the periodic solution at Tr = 500 oscillates in the region below the upper, parts but above the lower parts of the steady solution, there are branches, that is, in the middle region of the steady solution there are branches. To observe the periodic change of flow characteristics and the temperature distributions, contours of typical secondary flow and temperature profile for one period of oscillation, there are at $19.25 \le t \le 19.32$ are shown in Fig. 13(b), where it is seen that the multi-periodic oscillation at Tr = 500 is a twovortex solution. In this regard, it is interesting to note that though the unsteady flow presented in Fig. 13(b) for Tr = 500 seems to be multiperiodic, it is actually periodic, which is justified

by the power spectrum as shown in Fig. 14. It is found that only the line spectrum of the fundamental frequency (f = 14.5 Hz) and its harmonics are seen which indicates that the flow is time periodic. Next, the time evolution of Qtogether with the values of O for the steady solution branches, indicated by straight lines, are shown in Fig. 15(a) for Tr = 800. It is found that the flow oscillates periodically in the region along the values of Q on the upper parts of the third steady solution branch. The associated secondary flow patterns and temperature profiles are shown in Fig. 15(b). It is found that the unsteady flow at Tr = 800 also oscillates between the asymmetric two-vortex solutions. Then, in order to investigate whether the flow is periodic, multi-periodic or chaotic, a power spectrum of the time change of Q is performed as shown in Fig. 16. It is found that not only the line spectrum of the fundamental frequency (f =14.5 Hz, the same frequency as obtained for Tr = 500) and its harmonics, but also other line spectra at small frequencies are seen. This result shows that the oscillation presented in Fig. 15(a) may be multi-periodic or chaotic, which is justified by the phase diagram as shown in Fig 17. If Tr is increased further, the flow turns into a steady state. Time evolutions of Q are then performed at several values of Trfor $908 \le Tr \le 2000$, and it is found that the value of Q approaches steady state. The reason is that the steady flow is stable on the first steady solution branch in this region.

We now discuss the transitional behavior of the unsteady solutions by drawing phase spaces at larger Dn numbers, Dn = 2000. The change of the flow state from multi-periodic oscillation to chaotic state is explicitly exhibited by drawing the orbit of the solution in the phase spaces as shown in Fig. 17(a) for Tr = 500 and in Fig. 17(b) for Tr = 800, where the abscissa is Q and the ordinate is γ . The orbits are drawn by tracing the time evolution of a solution. As seen in Fig. 17(a), a multi-periodic orbit is seen for Tr= 500; for Tr = 800, however, a chaotic orbit is observed, which was not clearly observed by the power spectrum of the solutions as presented in Fig. 16. Phase spaces are, therefore, found to be more helpful for the investigation of chaotic flow behavior. This type of flow behavior was also investigated by Mondal et al. [7] for the non-isothermal flow through a curved square duct and termed as transitional chaos.

Thus, by the time evolution calculations as well as by the power spectrum of the solutions, it is found that stable a steady solutions occur in the regions for $0 \le Tr \le 329$ and $908 \le Tr \le 2000$, and periodic or chaotic solutions for $330 \le Tr \le 907$. Linear stability analysis indicates that a stable steady solution exists for $0 \le Tr \le 329.7$ and $907.2 \le Tr \le 2000$. Therefore, the results of the linear stability analysis and those of the time evolution calculations are consistent.

6. Discussion

In this section, a brief discussion on the plausibility of applying 2-D calculations as well as the accuracy of the present numerical calculations will be given by comparing with some other works. There have been a lot of numerical and experimental studies that showed that curved duct flows easily attain asymptotic fully developed 2-D states at most 270[°] from the Wang and Yang [18] performed inlet. experimental as well as numerical prediction of fully developed bifurcation structure and periodic oscillation in curved square duct flows. They showed that even periodic flows can be analyzed by 2-D calculations and these periodic between oscillations take place the symmetric/asymmetric 2-cell and 4-cell flows where there are no stable steady solutions. They also showed that for an oscillating flow, there exists a close similarity between the flow observation at 270° and the 2-D calculation. Nearly similar types of flow characteristics are observed in the present case of rotating curved duct flows. In fact, the periodic oscillation observed in the cross section of their duct was a traveling wave advancing in the downstream direction. Therefore, it is found that 2-D calculations can predict the existence of threedimensional travelling wave solutions as an appearance of 2-D periodic oscillation presented in this paper.

It is noteworthy that even curved duct flow, which does not attain an asymptotic state, can be analyzed by 2-D calculations. Yamamoto et al. [19] performed numerical prediction (2-D calculations) of isothermal flows through a rotating curved square duct (rotation of the duct except the outer wall) and obtained many branches of steady solutions. They investigated the secondary flow characteristics with changing Dn for constant rotational speed (Tr). Very recently, Yamamoto et al. [20] performed experimental investigation of isothermal flows through a rotating curved square duct by using the same configuration as were used in their numerical predictions. They compared their numerical results with the experimental data and found good agreement between the two investigations. In the present case of the nonisothermal flows, however, we also obtain nearly similar behavior of the flows as were obtained by Yamamoto et al. [20]. For the flow through a spiral duct, on the other hand, Mees et al. [21] observed that the flow exhibits an oscillation between two-vortex and four-vortex flows first, but turns to a steady two-vortex flow downstream, which can be explained by the present bifurcation study. In the present paper, we present the bifurcation diagram where it is shown that there exists a region of stable steady solutions (two-vortex solutions) in the lowest Dn region, and oscillating solutions appear if Dnis increased (an oscillation of two- and fourvortex solutions). If Dn is increased further, a region of stable steady solutions again appears two-vortex solutions are observed. when Therefore, it is found that our numerical results may give good agreement with the experimental observations discussed so far.

There is some other evidence showing that the occurrence of chaotic or turbulent flow may be predicted by 2-D analysis. Yamamoto et al. [22] investigated linear stability of helical pipe flows with respect to 2-D perturbations and compared the results with their experimental data. There was good agreement between the numerical results and the experimental data, which shows that even the transition to turbulence can be predicted by 2-D analysis to some extent, though it is true that the full transition process cannot be elucidated by 2-D analysis. The transition process from the periodic oscillation to chaotic state, obtained by the 2-D calculation in the present paper, may correspond to destabilization of travelling waves in the curved duct flows like that of Tollmien-Schlichting waves in a boundary layer. Our 2-D analysis, therefore, may contribute to the study of curved duct flows by giving a complete outline for not only fully developed but also developing curved duct flows.

7. Conclusions

In this paper, a detailed numerical study on fully developed two-dimensional flow of

viscous incompressible fluid through a rotating curved duct with square cross section has been analyzed by using the spectral method over a range of the Taylor number, wide $0 \le Tr \le 2000$ and the Dean number. . $0 \le Dn \le 2000$ for the curvature $\delta = 0.1$ Though the present study covers a wide range of Dn, in this paper, however, three cases of the Dean numbers, Dn = 500, Dn = 1000 and Dn =2000 have been discussed in detail with a temperature difference between the vertical sidewalls for the Grashof number Gr = 100, where the outer wall is heated and the inner one cooled.

After a comprehensive survey over the parametric ranges, a single branch of asymmetric steady solution is obtained for Dn =500; for Dn = 1000 and Dn = 2000, on the other hand, we obtain two and four branches of solutions, symmetric/asymmetric steady respectively. It is found that there exist two- and four-vortex solutions on various branches. These vortices are generated due to the centrifugal force and Coriolis force or by their combinations. It is found that as Dn increases the number of steady solutions also increases. Linear stability of the steady solutions reveals an interesting result. It is found that the single branch, obtained for Dn = 500, is linearly stable for any value of Tr in the range. For Dn = 1000, however, the same branch is linearly unstable at small Tr, and if Tr is increased further the steady solution becomes stable. For Dn = 2000, on the other hand, it is found that among four branches of steady solutions, only the first branch, which exists throughout the whole range of Tr, is linearly stable in a couple of intervals of Tr, one for small Tr and another for larger Tr, and thus the flow undergoes *'steady-stable* unsteady \rightarrow steady-stable', if Tr is increased. It is found that the Hopf bifurcation occurs at Tr on the boundary between the stable and unstable solutions.

Time evolution calculations as well as their spectral analyses show that in the unstable region for Dn = 1000, the unsteady flow becomes periodic before turning to steady state. In the unstable region for Dn = 2000, on the other hand, the unsteady flow becomes periodic first, then multi-periodic, then chaotic and finally turns into steady state again, if Tr is increased. In order to investigate the transition from multi-periodic oscillations to chaotic states more explicitly, the orbit of the solution is

drawn in phase space. Both spectral analysis and phase space are found to be very useful for the investigation of chaotic flow behavior. In this regard, it should be worth mentioning that irregular oscillation of the flow through a curved duct has been observed experimentally by Ligrani and Niver [23] for a large aspect ratio and by Wang and Yang [18] for the square duct.

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Figure 1: Coordinate system of the rotating curved duct.



Figure 2: (a)Steady solution branch for Dn = 500. (b) Contours of secondary flow (top) and temperature profile (bottom) on the steady solution branch at several values of Tr.



Figure 3: Steady solution branches for Dn = 1000.



Figure 4: (a) First steady solution branch with the region of linear stability (bold line) for Dn = 1000. (b) Contours of secondary flow (top) and temperature profile (bottom) on the first steady solution branch at several values of Tr.



Figure 5: (a) Second steady solution branch for Dn = 1000. (b) Contours of secondary flow (top) and temperature profile (bottom) on the second steady solution branch at several values of Tr.



Figure 6: Unsteady solutions for Dn = 1000 and Tr = 100. (a) Time evolution of Q and the values of Q for the steady solutions for $10 \le t \le 20$. (b) Contours of secondary flow (top) and temperature profile (bottom) for one period of oscillation at $17.20 \le t \le 17.58$.



Figure 7: Power-spectrum of the time evolution of Q for Dn = 1000 and Tr = 100.



Figure 8: Steady solution branches for Dn = 2000.



Figure 9: (a) First steady solution branch with the region of linear stability (bold line) for Dn = 2000. (b) Contours of secondary flow (top) and temperature profile (bottom) for the first steady solution branch at several values of Tr.



Figure 10: (a) Second steady solution branch for Dn = 2000. (b) Contours of secondary flow (top) and temperature profile (bottom) for the second steady solution branch at different values of Tr (from upper branch to the lower).



Figure 11: (a) Third steady solution branch for Dn = 2000. (b) Contours of secondary flow (top) and temperature profile (bottom) for the third steady solution branch at different values of Tr (from upper branch to the lower).



Figure 12: (a) Fourth steady solution branch for Dn = 2000. (b) Contours of secondary flow (top) and temperature profile (bottom) for the fourth steady solution branch at different values of Tr (from upper branch to the lower).



Figure 13: Unsteady solutions for Dn = 2000 and Tr = 500. (a) Time evolution of Q and the values of Q for the steady solutions for $18 \le t \le 20$. (b) Contours of secondary flow (top) and temperature profile (bottom) for one period of oscillation at $19.25 \le t \le 19.32$.

(b)

(b)



Figure 14: Power-spectrum of the time evolution of Q for Dn = 2000 and Tr = 500.



Figure 15: Unsteady solutions for Dn = 2000 and Tr = 800. (a) Time evolution of Q and the values of Q for steady solutions for $17 \le t \le 20$. (b) Contours of secondary flow (top) and temperature profile (bottom) for $18.70 \le t \le 19.30$.



Figure 16: Power-spectrum of the time evolution of Q for Dn = 2000 and Tr = 800.



Figure 17: Phase plots in the $Q - \gamma$ plane for Dn = 2000, where $\gamma = \iint \psi dx dy$. (a) Tr = 500, (b) Tr = 800.

М	N	Q	w(0,0)
14	14	320.983357	358.116399
16	16	321.001250	358.620968
18	18	320.986431	358.851160
20	20	320.988227	358.967855
22	22	320.988235	359.094040

Table 1: The values of Q and w(0,0) for various M and N (=M) at Gr = 100, Dn = 1000 and Tr = 100.

Tr	Q	σ_r	σ_i
0	344.008843	5.8088	$\pm 1.8039 \times 10$
50	331.635861	4.5957	$\pm 1.7751 \times 10$
100	320.988227	2.5179	$\pm 1.5867 \times 10$
138.5	313.685868	5.8056×10^{-3}	$\pm 1.3145 \times 10$
138.6	313.667760	-2.5583×10 ⁻²	±1.3136×10
300	288.738963	-2.0775	0
500	266.093461	-2.2484	0
1000	227.949935	-2.6308	0
1500	202.620767	-2.9665	0
2000	184.083345	-3.2511	0

Table 2: Linear stability of the first steady solution branch for Dn = 1000.

Tr	\mathcal{Q}	σ_r	σ_{i}
0	581.688020	-2.0567	0
100	578.707647	-2.1672	0
200	532.350027	-2.2628	0
329.7	508.168972	-2.5298×10 ⁻³	± 8.6611 ×10
329.8	508.151959	4.4347×10^{-4}	$\pm 8.6614 \times 10$
500	482.006081	4.3244	$\pm 8.9715 \times 10$
800	445.715705	2.7692	$\pm 9.2969 \times 10$
907.1	434.818632	5.5975×10^{-4}	±9.1320×10
907.2	434.808854	-2.6167×10 ⁻³	± 9.1318 ×10
1200	. 408.932261	-2.7602	0
1500	386.450693	-3.1309	0
2000	356.047961	-3.4141	0

Table 3: Linear stability of the first steady solution branch for Dn = 2000.