Well-Posedness of a Class of Bimodal Modular Dynamical Systems

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Abstract
In this paper a class of hybrid dynamical systems with non-uniform continuous state space is discussed as a framework for modeling the modularity of system structure. A hybrid model is formulated to represent structure modifications in the form of state jumps among state spaces with different dimensions. A basic problem in the field of system theory is addressed: well-posedness analysis (existence and uniqueness of solutions). The necessary and sufficient condition for well-posedness is derived for a class of bimodal systems. The condition can be verified in the form of convex feasibility problems, hence tractable. Examples are provided for better insight with our concept.

Keyword: hybrid dynamical system, well-posedness, lexicographic inequality

1. Introduction

Hybrid dynamical systems (or HDSs for short) are dynamical systems comprising mixtures of continuous dynamics and logical decisions. Such systems arise in many applications with respect to the embedment of logical devices or heuristic rules. Examples are behavior-based robots [1], intelligent transportation systems [2], biological systems [3], to name a few. Recognizing the demand of analysis/design tools, the theory of HDSs have become an active area of research in recent years, see [4, 5] and references therein. The study of HDSs in the field of system theory tends to focus on three system classes, namely switched HDSs [6], impulsive dynamical systems [7], and linear complementarity systems [8]. These system classes are distinguished from each other due to different definitions of discontinuous dynamics. It is of particularly interest that these system classes assume the uniformity of continuous state space, i.e. continuous state space is unique and limited to n-dimensional real-valued space.

As one of the pioneer works in the field of HDSs, Branicky proposed the model of general HDSs [9] as a unified framework which captures various aspects within hybrid dynamics. It was mentioned in his work that failure situations can be modeled as hybrid dynamics by relaxing the common assumption about the dimension of continuous state space. Provided that the re-initialization of continuous states is properly defined, the relaxation is quite natural because each continuous dynamics can be defined separately from others. This idea motivates us to consider a class of HDSs whose continuous portion is composed of multiple subsystems of which some may be added or removed during the time span of interest. We refer to such systems as *modular dynamical systems* in the sense that system dynamics change according to each alteration of structure (state space) in the form of component/module attachment or detachment. The notion of modularity enables us to model several interesting phenomena, e.g. component breakdown and hot-swap of modules, which are forbidden in the conventional framework of system theory.

In this paper, we investigate the dynamics of module attachment/detachment on the aspect of well-posedness. Well-posedness of a dynamical system denotes the existence and uniqueness of its solutions from any given initial conditions. Contrary to continuous systems which require only Lipschitz continuity, one has to simultaneously verify the existence and uniqueness of both continuous and logical solutions in the case of HDSs. Existing results

on well-posedness analysis of HDSs are mainly contributed by Imura [6] for switched systems and van der Schaft [8] for linear complementarity systems. Their results proved conditions for the existence and uniqueness of hybrid solutions by predicting the infinitesimal future of hybrid states at the instant of logical transition.

The paper is organized as follows. Section 2 introduces a hybrid model for a class of modular HDSs of which each component is added or removed according to region-based rules. After the solution concept is given, the well-posedness property is defined as the existence and uniqueness of hybrid solutions in Section 3. Then the necessary and sufficient condition of well-posedness is derived in Section 4 for a class of bimodal HDSs based on geometric requirements in the form of lexicographic inequalities. To fulfill the issue of tractability, we provide a tool which transforms the well-posedness condition to a family of convex feasibility problems. An example is provided in Section 5 for better understanding of our results.

Throughout $\mathbb{R}^n$ and $\mathbb{Z}^n$ denote the n-dimensional real-valued space and the set of n-dimensional integer vectors, respectively. We use the following notation for lexicographic inequality: for $x \in \mathbb{R}^n$, if for some i, $x_j = 0$; $j = 1, 2, \ldots, i-1$, while $x_i > (<) 0$, we denote it by $x \succ (\prec) 0$. Moreover if $x = 0$ or $x \succ (\prec) 0$, we denote it by $x \succeq (\preceq) 0$. We use the font style $\mathbb{A}, \mathbb{B}, \ldots, \mathbb{Z}$ to refer to variables representing sets, spaces, or subspaces. Accordingly, we use set operations including \ as set minus, $\subset$ as subset, and $\varnothing$ as empty set.

2. Modular Dynamical Systems

In this section, we describe a hybrid model which represents a class of modular dynamical systems derived from the definition of general HDSs [10].

2.1. Dynamical Systems with Non-uniform State Space

Based on notations in the general hybrid model, an HDS with non-uniform state space can be classified as a *continuous-time-uniform, c-Euclidean, d-countable* HDS, which is denoted by a 4-tuple:

$$\mathbf{H} \triangleq [\Xi, \Gamma, \mathbf{J}, \mathbf{G}] \qquad (1)$$

where $\Xi \subset \mathbb{Z}$ is a set of logical states. $\Gamma = \{\Gamma_\xi\}_{\xi \in \Xi}$ is a collection of constituent dynamical systems. Each $\Gamma_\xi \triangleq [\mathbb{X}_\xi, \mathbb{T}, f_\xi]$ is a dynamical system with $\mathbb{X}_\xi \subseteq \mathbb{R}^{n_\xi}$ its continuous state space, $\mathbb{T} \subseteq \mathbb{R}_+$ its time set, and f_ξ its vector field. $\mathbf{J} = \{\mathbb{J}_\xi\}_{\xi \in \Xi}$ is a collection of *jump sets* $\mathbb{J}_\xi \subseteq \mathbb{X}_\xi$. $\mathbf{G} = \{g_\xi\}_{\xi \in \Xi}$ is a set of *jump transition maps* $g_\xi : \mathbb{J}_\xi \mapsto \bigcap_{\psi \in \Xi} X_\psi \times \Xi$.

The behavior of such dynamical systems can be described as follows. The system is assumed to start in some hybrid state $(x(0), \xi_0) \in \mathbb{X}_{\xi_0} \times \Xi$, then evolves along $t \in \mathbb{T}$ with respect to the vector field f_{ξ_0}. Whenever its continuous state violates a prescribed region $\mathbb{J}_{\xi_0} \subset \mathbb{X}_{\xi_0}$ at the event time τ, the *logical state* or *mode* is transitioned from ξ_0 to ξ_1 regarding $g_{\xi_0}(x(\tau))$ such that Γ_{ξ_1} becomes active. Accordingly, the system state $(x(\tau), \xi_0)$ is projected to the new state $(\tilde{x}(\tau^+), \xi_1) \in \mathbb{X}_{\xi_1} \times \Xi$. Then the system continues evolving in $\mathbb{X}_{\xi_1}$ with respect to the vector field f_{ξ_1}. Since each $\mathbb{X}_\xi$ is not assumed to be equivalent for all $\xi \in \Xi$, the addition/deduction of components/modules can be formulated in the form of state jumps among state spaces with different dimensions.

2.2. A Hybrid Model of Modular HDSS

Given a collection of M LTI subsystems $\{\Lambda_p\}_{p \in \{1, \ldots, M\}}$:

$$\dot{\mathbf{z}}_p = \mathbf{a}_p \mathbf{z}_p + \mathbf{b}_p \qquad (2)$$

where $\mathbf{a}_p \in \mathbb{R}^{m_p \times m_p}$, $\mathbf{z}_p \in \mathbb{R}^{m_p}$, and $\mathbf{b}_p \in \mathbb{R}^{m_p}$ are its parameter, state, and bias term, respectively. Let $\{\mathbf{a}_{12}, \cdots, \mathbf{a}_{M,M-1}\}$ be a family of coupled parameters among LTI subsystems.

Suppose that the set Ξ has N members or *modes*. And let each Γ_ξ ; $\xi \in \Xi$ be a composition of distinct modules from

$\left\{\Lambda_p\right\}_{p\in\{1,\dots,M\}}$. That is, Γ_{ξ} consisting of q modules is given by :

$$\dot{\mathbf{x}}^{(\xi)} = \begin{bmatrix} \mathbf{a}_{11} & \cdots & \mathbf{a}_{1q} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{q1} & \cdots & \mathbf{a}_{qq} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_q \end{bmatrix} + \begin{bmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_q \end{bmatrix} \qquad (3)$$

where $\left(\mathbf{a}_{ii},\mathbf{b}_{ii}\right) \in \left\{\left(\mathbf{a}_p,\mathbf{b}_p\right)\right\}_{p=\{1,\dots,M\}}$; $i=1,\dots q$ and $\mathbf{a}_{ij}, i \neq j$ are corresponding parameters from $\left\{\mathbf{a}_{12},\cdots,\mathbf{a}_{M,M-1}\right\}$. Hence $\mathbf{x}^{(\xi)} \in \mathbb{R}^{n_{\xi}}$ where $n_{\xi} = \sum_r m_r$; $r \in [1,\dots,M]$ are the index of subsystems which exist in Γ_{ξ} .

Let $\mathbb{J}_{\xi\psi}$; $\xi,\psi \in \Xi$ be a halfspace :

$$\mathbb{J}_{\xi\psi} = \left\{\mathbf{x} \in \mathbb{R}^{n_{\xi}} \mid J_{\xi\psi}\mathbf{x} + j_{\xi\psi} \leq 0\right\} \qquad (4)$$

where $J_{\xi\psi} \in \mathbb{R}^{n_{\xi}}$ and $j_{\xi\psi} \in \mathbb{R}$. Let $\mathbb{J}_{\xi\psi} = \varnothing$ for any mode ψ which can not be transitioned from ξ in one-step, i.e. no edge from ξ to ψ . Consequently, we can define for each mode ξ : the jump set $\mathbb{J}_{\xi} = \bigcup_{\psi\in\Xi}\mathbb{J}_{\xi\psi}$ and the space $\mathbb{X}_{\xi} = \mathbb{R}^{n_{\xi}} \setminus \mathbb{J}_{\xi}$.

We use subscript k; $0 \leq k \leq \infty$ to denote the sequence of logical state within hybrid dynamics. To map $\xi_k \mapsto \xi_{k+1}$, we suppose that logical dynamics be governed by rules associated with halfspace $\mathbb{J}_{\xi\psi}$:

$$\xi_{k+1} = \psi \quad ; \text{if} \quad \mathbf{x}(\tau) \in J_{\xi_k\psi} \qquad (5)$$

where $t_{k+1} = \tau$ is the event time at which the logical transition occurs. For each transition from ξ to ψ , the projection from space $\mathbb{R}^{n_{\xi}}$ to $\mathbb{R}^{n_{\psi}}$ is represented by the re-initialization of continuous states from $\mathbf{x}^{(\xi)} \in \mathbb{R}^{n_{\xi}}$ to $\mathbf{x}^{(\psi)} \in \mathbb{R}^{n_{\psi}}$.

$$\mathbf{x}^{(\psi)}(\tau^+) = g_{\xi\psi}\left(\mathbf{x}^{(\xi)}(\tau)\right) \qquad (6)$$

where $g_{\xi\psi} : \mathbb{R}^{n_{\xi}} \mapsto \mathbb{R}^{n_{\psi}}$. The following assumption is imposed on the dynamics of state jumps to maintain the physical meaning.

Assumption 2.1: *For each logical transition, we assume that the continuity of retained continuous state is preserved.*

Assumption 2.1 can be interpreted as follows. Let $\mathbf{x}_r$ be continuous states which present in ξ_k and ξ_{k+1} for some $k \geq 0$. Then $\mathbf{x}_r(t)$ is continuous for all $t \in (t_k, t_{k+2}]$ where t_k and t_{k+2} are event times of mode ξ_k and ξ_{k+2} , respectively. In the following, an example is given to illustrate how to model modular dynamical systems with the proposed hybrid model.

Example 2.1 Consider a large scale power system consisting of q generators of which system dynamics under decentralized control [12] are given by

$$\dot{\mathbf{x}} = A\mathbf{x} + BW(\mathbf{x}) \qquad (7)$$

where $\mathbf{x} = \begin{bmatrix} \mathbf{x}_1^T & \cdots & \mathbf{x}_q^T \end{bmatrix}^T$ are states of q generators, $W(\mathbf{x}) = \begin{bmatrix} w_1(\mathbf{x})^T & \cdots & w_q(\mathbf{x})^T \end{bmatrix}^T$ are generator controls, and $A = \mathbf{diag}(A_1,\cdots,A_q)$, $B = \mathbf{diag}(B_1,\cdots,B_q)$ are their corresponding parameters. Each generator which is given by $\dot{\mathbf{x}}_i = A_i\mathbf{x}_i + B_iw_i(\mathbf{x})$; $i=1,\dots q$ stands for the ith node within the power system in which some may be operated or shut down according to prescribed conditions.

Suppose that $q=2$ and let the second generator be operated if $w_1(\mathbf{x}) \geq \gamma_1$ and shut down if $w_2(\mathbf{x}) \geq \gamma_2$ where $\gamma_1 > \gamma_2 > 0$. This power system configuration can be modeled as an HDS consisting of two modes $\Xi = \{\alpha,\beta\}$. The dynamics of mode α and β are given by :

$$\dot{\mathbf{x}} = A_1\mathbf{x}_1 + B_1w_1(\mathbf{x}_1) \qquad (8)$$

$$\begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} B_1w_1(\mathbf{x}_1) \\ B_2w_2(\mathbf{x}_1,\mathbf{x}_2) \end{bmatrix} \qquad (9)$$
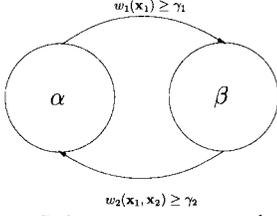
Figure. 1 A finite automaton symbolizing the activation and shutdown of a backup generator

Figure 1 depicts a hybrid automaton which symbolizes the startup/shutdown conditions as two edges between modes α and β. Suppose that the requirement for a second generator before synchronization with the power system is proportional to $\mathbf{x}_1$, e.g. $\mathbf{x}_2 = G_{\alpha\beta}\mathbf{x}_1$. Since the state $\mathbf{x}_1$ is apparently continuous, the dynamics of state $\mathbf{x}_2$ after $\alpha \mapsto \beta$ is given by :

$$\mathbf{x}_2(\tau^+) = G_{\alpha\beta}\mathbf{x}_1(\tau) + \partial_\beta(\mathbf{x}) \qquad (10)$$

where $\partial_\beta(\mathbf{x})$ represents the influence of the vector field of mode β.

2.3. Bimodal Modular HDSs

In this paper, we consider a class of bimodal modular HDSs, i.e. $\Xi = \{\alpha, \beta\}$. Let two LTI constituent systems be Γ_α and Γ_β:

$$\dot{\mathbf{x}}_1 = A_{11}\mathbf{x}_1 \qquad (11)$$

$$\begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = A_2\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \qquad (12)$$

where $\mathbf{x}_1 \in \mathbb{R}^{n_1}$ and $\mathbf{x}_2 \in \mathbb{R}^{n_2}$. Let convex sets $\mathbb{J}_\alpha$ and $\mathbb{J}_\beta$:

$$\mathbb{J}_\alpha = \{\mathbf{x}_1 \neq 0 \mid J\mathbf{x}_1 \geq 0\}, \qquad (13)$$

$$\mathbb{J}_\beta = \left\{\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \neq 0 \mid J\mathbf{x}_1 \leq 0\right\} \qquad (14)$$

be jump sets associated with Γ_α and Γ_β respectively, where $J \in \mathbb{R}^{1 \times n_1}$. That is,

$$\mathbb{X}_\alpha = \{\mathbf{x}_1 \neq 0 \mid J\mathbf{x}_1 < 0\} \cap \{\mathbf{x}_1 = 0\}, \qquad (15)$$

$$\mathbb{X}_\beta = \left\{\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \neq 0 \mid J\mathbf{x}_1 > 0\right\} \cap \left\{\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = 0\right\} \quad (16)$$

The origin of $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_1+n_2}$ are excluded from jump sets to withhold the concept of equilibrium.

Suppose that the logical transition between α and β is governed by the violation of $\mathbf{x}$ into its corresponding jump set, i.e.

$$\xi_{k+1} = \begin{cases} \alpha & ; \text{ if } \xi_k = \beta, \mathbf{x}(\tau) \in \mathbb{J}_\beta \\ \beta & ; \text{ if } \xi_k = \alpha, \mathbf{x}(\tau) \in \mathbb{J}_\alpha \end{cases} \qquad (17)$$

The dynamics of $\mathbf{x}$ after each re-initialization can be rewritten to comply with Assumption 2.1 and strict causality as follows.

$$\mathbf{x}_1^{(\alpha)}(\tau^+) = \begin{bmatrix} I & 0 \end{bmatrix}\begin{bmatrix} \mathbf{x}_1^{(\beta)}(\tau) \\ \mathbf{x}_2^{(\beta)}(\tau) \end{bmatrix} + \partial_\alpha(\mathbf{x}_1(\tau)) \quad (18)$$

$$\begin{bmatrix} \mathbf{x}_1^{(\beta)}(\tau^+) \\ \mathbf{x}_2^{(\beta)}(\tau^+) \end{bmatrix} = \begin{bmatrix} I \\ G \end{bmatrix}\mathbf{x}_1^{(\alpha)}(\tau) + \partial_\beta(\mathbf{x}(\tau)) \quad (19)$$

where $G \in \mathbb{R}^{n_2 \times n_1}$, $\Delta\tau = \tau^+ - \tau$, and

$$\partial_\xi(\mathbf{x}(\tau)) = \left.\frac{d\mathbf{x}}{dt}\right|_{\mathbf{x}(\tau),\xi}\frac{\Delta\tau}{1!} + \left.\frac{d^2\mathbf{x}}{dt^2}\right|_{\mathbf{x}(\tau),\xi}\frac{\Delta\tau^2}{2!} + \cdots$$

is the incremental term of Taylor series expansion corresponding to the vector field of mode ξ.

3. Well-posedness Analysis

Conventionally, the solution of a continuous system is obtained by integrating its vector field $f(x)$ over the time of interest, i.e. $x(t) = x(0) + \int_0^t f(x(\tau))d\tau$, which is called the *Carathéodory equation*. The Carathéodory equation can be generalized for dynamical systems with non-uniform state space by determining their corresponding vector field f_{ξ_i} of each mode ξ_i; $i = 0, \ldots, k$, consecutively.

3.1. Solution Concept

Let $\mathbf{x}(t_k, t_{k+1}]$ denotes an *interval trajectory* of continuous states between two successive event times t_k and t_{k+1} with respect to a vector field f_{ξ_k} given by the right-hand side of (3):

$$\mathbf{x}(t_k, t_{k+1}] = \left\{ \hat{\mathbf{x}} \in \mathbb{R}^{n_{\xi_k}} \middle| \begin{array}{l} \hat{\mathbf{x}} = \mathbf{x}(t_k) + \int_{t_k}^{t} f_{\xi_k}(\mathbf{x}(\tau)) d\tau, \\ t \in (t_k, t_{k+1}] \end{array} \right\} \quad (20)$$

where $\xi_k \in \Xi$.

Definition 3.1 (Hybrid Solution) *Given an HDS with non-uniform state space and its initial state* $(\mathbf{x}(0), \xi_0) \in \mathbb{X}_{\xi_0} \times \Xi$ *. Suppose that the solution* $(\mathbf{x}, \xi)$ *of the initial value problem over an arbitrary time period (0,T] can be expressed as a sequence of interval trajectories* $\langle (\mathbf{x}(t_0, t_1], \xi_0), \dots, (\mathbf{x}(t_k, t_{k+1}], \xi_k) \rangle$ *where* $t_0 = 0$ *and* $t_{k+1} = T$ *. Then the sequence* $\langle (\mathbf{x}(t_i, t_{i+1}], \xi_i) \rangle$ *;* $i = 0, \dots, k$ *, is said to be the **hybrid solution** with respect to* $(\mathbf{x}(0), \xi_0)$ *over (0,T].*

For HDSs, the integration between continuous and logical dynamics may lead to several undesirable phenomena along the evolution of time and the occurrence of events. In the following, we give the formal definition of the well-posedness property discussed throughout.

Definition 3.1 (Well-posedness) *An HDS with non-uniform state space is said to be well-posed, if its hybrid solutions can be uniquely determined with respect to an arbitrary initial state* $(\mathbf{x}(0), \xi_0) \in \mathbb{X}_{\xi_0} \times \Xi$ *over* $(0, \infty]$ *.*

To be well-posed in the sense of Definition 3.2, every hybrid solution of initial value problem must satisfy two primal requirements:

1. The uniqueness of $(\mathbf{x}(t_i, t_{i+1}], \xi_i)$ for all i; $i = 0, \dots, k$; $0 \leq k \leq \infty$.
2. The existence over infinite horizon.

Characterizing undesirable phenomena as deadlock, nondeterminism, livelock and Zeno, the following lemma gives the necessary and sufficient conditions of well-posedness for a class of modular HDSs described by our hybrid model.

Lemma 3.1 (Undesirable Phenomena) *Given a modular HDS of (2)-(6) whose hybrid solution is expressed as Definition 3.1. The HDS is well-posed if and only if none of the following statements holds.*

1. **Deadlock:** $\exists k \geq 0$, *next mode* ξ_{k+1} *can not be determined from* $(\mathbf{x}(t_{k+1}), \xi_k)$ *;* $t_{k+1} < \infty$ *such that* $\mathbf{x}(t_{k+1}, t_{k+2}]$ *does not exist.*

2. **Nondeterminism:** $\exists k \geq 0$, *there exist more than one possible modes* $\xi_{k+1} \in \{\psi_1, \dots, \psi_r\} \subseteq \Xi \setminus \{\xi_k\}$ *;* $r > 1$ *at the event time* t_{k+1} *such that* $(\mathbf{x}(t_{k+1}, t_{k+2}], \xi_{k+1})$ *is not unique.*

3. **Livelock:** $\exists k \geq 0$, *one can find an infinitely possible* $j, j > k$ *satisfying* $\xi_k = \xi_j$ *,* $\sup_{k,j} \left\| \mathbf{x}^{(\xi_j)} - \mathbf{x}^{(\xi_k)} \right\| = 0$ *, and* $\sup_{k,j} (t_j - t_k) = 0$ *.*

4. **Zeno:** *there exists a real number* ρ *;* $0 < \rho < \infty$ *satisfying* $\lim_{k \to \infty} \sum_{i=0}^{k} (t_{i+1} - t_i) = \rho$ *.*

Proof: Recall the notation of jump set in (4), the trajectory of $\mathbf{x}$ for a given ξ_i is right-continuous with constant vector field for all $t \in (t_i, t_{i+1}]$. Hence the continuous portion of each $\mathbf{x}(t_i, t_{i+1}]$; $i = 0, \dots, k$ is well-posed due to its Lipschitz continuity. Consequently, the uniqueness of hybrid solution implies the uniqueness of ξ_i for all $i = 0, \dots, k$; $0 \leq k \leq \infty$. It follows from the definition of deadlock and nondeterminism that the number of possible ξ_{k+1} are zero and $r; r > 1$, respectively. Mathematically, the existence of a right-accumulation point of event time[1] prohibits the evolution of time such that no solution can be obtained after the right-accumulation point. Both livelock and Zeno phenomena lead to the presence of right-accumulation points due to the boundedness of infinite summation. Thus the solution can not be extended to infinite horizon. These conclude the proof of sufficiency. The proof of necessity is straightforward by assuming that no undesirable phenomenon exists. And some of hybrid solutions are found which are not unique or exist over infinite horizon.

The severeness of livelock phenomenon comes as two aspects. For the mathematical aspect, the problem $\sum_{i=1}^{\infty} (t_{k+i+1} - t_{k+i})$ is ill-

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[1] A point $\tau \in \mathcal{T} \subseteq \mathbb{R}$ is a right-accumulation point of $\mathcal{T}$, if there exists $\tau_i \in \mathcal{T}, i \in \mathbb{Z}_+$ with $\tau_i < \tau$ such that $\tau = \lim_{i \to \infty} \tau_i$.

posed because $t_{k+j} - t_k$ is infinitesimal for any $j > 0$. For the practical aspect, a livelock phenomenon consumes infinite computational effort for an endless mode transitions and repetitive continuous states.

The Zeno phenomenon differs from livelock phenomenon in the respect of how to mathematically prove the existence of right-accumulation points. It was investigated in [13] that dynamical systems which exhibit Zeno phenomena are still well-behaved in physical aspects. Since the Zeno phenomenon is very hard to verify algebraically, this paper makes a supposition that the Zeno phenomenon is treated as a well-defined behavior in system dynamics.

Assumption 3.1: *A modular HDS is well-posed even it exhibits Zeno phenomena in system dynamics.*

4. Well-posedness Condition of Bimodal Systems

In this paper, we consider a class of modular HDSs consisting of two modes as described in Subsection 2.3. The following lemma shows the deduction of Lemma 3.1 with respect to limited system class.

Lemma 4.1 *Consider a modular HDS expressed by (11)-(18). The system is well-posed if and only if no livelock phenomenon exists in the hybrid solution regarding an arbitrary initial state $(\mathbf{x}(0), \xi_0)$ over $[0, \infty)$.*

Proof: Since the system class has only two modes $\Xi = \{\alpha, \beta\}$, every logical transition is unique with respect to one possible jump set for each mode. Then ξ_k and f_{ξ_k} are deterministic for any k; $0 \le k \le \infty$. This guarantees the absence of deadlock and nondeterminism phenomena. According to Assumption 3.1, undesirable phenomena in Lemma 3.1 can be reduced to only livelock phenomenon.

Consider the image of $\mathbb{J}_\alpha$ in $\mathbb{R}^{n_1+n_2}$ and $\mathbb{J}_\beta$ in $\mathbb{R}^{n_1}$ given by :

$$\hat{\mathbb{J}}_\alpha = \left\{ \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \in \mathbb{R}^{n_1+n_2} \Big| \mathbf{x}_1 \ne 0, J\mathbf{x}_1 \ge 0, \mathbf{x}_2 = G\mathbf{x}_1 \right\} \quad (21)$$

$$\hat{\mathbb{J}}_\beta = \left\{ \mathbf{x}_1 \in \mathbb{R}^{n_1} \Big| \mathbf{x}_1 \ne 0, J\mathbf{x}_1 \le 0 \right\} \quad (22)$$

The following statement shows the necessary condition for contiguous loop transitions from mode α, i.e. $\langle \xi_k = \alpha, \xi_{k-1} = \beta, \xi_{k+2} = \alpha \rangle$ and $\sup_\mathbf{x} (t_{k+2} - t_{k+1}) = 0$.

$$\left\{ \mathbf{z} \in \mathbb{R}^{n_1+n_2} \Big| \mathbf{z} = \mathbf{x} + \partial_\beta(\mathbf{x}), \mathbf{x} \in \hat{\mathbb{J}}_\alpha \right\} \cap \mathbb{J}_\beta \ne \varnothing \quad (23)$$

Similarly, a contiguous loop transition from mode β is possible if the following statement holds.

$$\left\{ \mathbf{z} \in \mathbb{R}^{n_1} \Big| \mathbf{z} = \mathbf{x} + \partial_\alpha(\mathbf{x}), \mathbf{x} \in \hat{\mathbb{J}}_\beta \right\} \cap \mathbb{J}_\alpha \ne \varnothing \quad (24)$$

Let $\overline{\mathbb{J}}_\alpha$ and $\overline{\mathbb{J}}_\beta$ be the closure of $\mathbb{J}_\alpha$ and $\mathbb{J}_\beta$, respectively. Since jump sets are characterized by $J\mathbf{x}_1 \ge (\le) 0$, we conclude that $\hat{\mathbb{J}}_\alpha \cap \mathbb{J}_\beta \ne \varnothing$ and $\hat{\mathbb{J}}_\beta \cap \mathbb{J}_\alpha \ne \varnothing$ hold regarding to $\overline{\mathbb{J}}_\beta$ and $\overline{\mathbb{J}}_\alpha$. Accordingly, the verification of (23) and (24) depends on the Taylor series expansion term $\partial_\xi(\mathbf{x})$ where $\mathbf{x} \in \overline{\mathbb{J}}_\alpha$ (or $\overline{\mathbb{J}}_\beta$). In the following, we briefly introduce the application of lexicographic inequality for the prediction of $\partial_\xi(\mathbf{x})$.

Define a matrix $W(J, A)$ as :

$$W(J, A) = \begin{bmatrix} J \\ JA \\ \vdots \\ JA^{m-1} \end{bmatrix} \quad (25)$$

where $A \in \mathbb{R}^{n \times n}$, $J \in \mathbb{R}^{1 \times n}$, $m \le n$ is the observability index of pair (J, A), i.e. $W(J, A)$ is of rank m. Consider the trajectory of system $\dot{x} = Ax$ within an area defined by $\{x \in \mathbb{R}^n | Jx \ge 0\}$. It follows from the definition of lexicographic inequality and $Ax = \dot{x}$ that for any x satisfying $W(J, A)x \succeq 0$, one of the following statements holds.

- $Jx > 0$
- $Jx = 0$ and $J\dot{x} > 0$
- $\vdots$
- $Jx^{(i)} = 0$; $i = 0, \dots, m-2$, $Jx^{(m-1)} \ge 0$.

where $x^{(i)}$ is the ith-derivative of x. Hence if $x(\tau) \in \{x | Jx \geq 0\}$ satisfies $W(J, A)x(\tau) \succeq 0$, then there exists $\varepsilon > 0$ such that $x(t) \in \{x | Jx \geq 0\}$; $\forall t \in [\tau, \tau + \varepsilon)$. That is, the infinitesimal future of x within an area defined by component-wise inequality can be predicted by the corresponding area defined by lexicographic inequality.

Define two areas $\tilde{\mathbb{J}}_\alpha \subset \mathbb{J}_\alpha$ and $\tilde{\mathbb{J}}_\beta \subset \mathbb{J}_\beta$:

$$\tilde{\mathbb{J}}_\alpha = \left\{ \mathbf{x}_1 \big| W(J, A_{11}) \mathbf{x}_1 \succeq 0 \right\} \qquad (26)$$

$$\tilde{\mathbb{J}}_\alpha = \left\{ \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \bigg| W\left(\begin{bmatrix} J & 0 \end{bmatrix}, A_2 \right) \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \preceq 0 \right\} \qquad (27)$$

Suppose that $\xi_k = \alpha$ and $\mathbf{x}(t_{k+1}) \in \overline{\tilde{\mathbb{J}}}_\alpha$. After $\xi_k \mapsto \xi_{k+1} = \beta$, it follows from the concept of lexicographic inequality that if $\left[\mathbf{x}_1(t_{k+1})^T \quad G\mathbf{x}_1(t_{k+1})^T \right]^T \in \tilde{\mathbb{J}}_\beta$, then a contiguous loop transition as expressed by (23) occurs. Similarly for $\xi_k = \beta$ with $\mathbf{x}_1(t_{k+1}) \in \tilde{\mathbb{J}}_\alpha$. Concerning mode transitions within $\overline{\tilde{\mathbb{J}}}_\alpha$ and $\overline{\tilde{\mathbb{J}}}_\beta$, we derive the necessary and sufficient condition for the existence of livelock phenomenon as follows.

Theorem 4.1 (Existence of livelock phenomena) *Given a modular HDS of (11)-(18), the existence of livelock phenomena is possible if and only if at least one of the following statements holds.*

$$\left\{ \begin{bmatrix} \mathbf{x}_1 \\ G\mathbf{x}_1 \end{bmatrix} \bigg| \mathbf{x}_1 \in \overline{\mathbb{J}}_\alpha \cap \tilde{\mathbb{J}}_\alpha \right\} \cap \tilde{\mathbb{J}}_\beta \neq \varnothing \qquad (28)$$

$$\left\{ \mathbf{x}_1 \bigg| \begin{bmatrix} \mathbf{x}_1 \\ G\mathbf{x}_1 \end{bmatrix} \in \overline{\mathbb{J}}_\beta \cap \tilde{\mathbb{J}}_\beta \right\} \cap \tilde{\mathbb{J}}_\alpha \neq \varnothing \qquad (29)$$

Proof: Suppose that $\xi_k = \alpha$ and $\exists \mathbf{x}_1(\tau)$ satisfies $J\mathbf{x}_1(\tau) = 0$ and $W(J, A_{11})\mathbf{x}_1(\tau) \succeq 0$ such that $\xi_k \mapsto \xi_{k+1} = \beta$ at time $t_{k+1} = \tau$. If (28) holds, then

$$W\left(\begin{bmatrix} J & 0 \end{bmatrix} A_2 \right) \left[\mathbf{x}_1(t_{k+1})^T \quad G\mathbf{x}_1(t_{k+1})^T \right]^T \preceq 0$$

such that a contiguous loop transition occurs. Since $\mathbf{x}_1$ is continuous, $W(J, A_{11})\mathbf{x}_1(t_{k+2}) \succeq 0$

holds after the transition to $\xi_{k+2} = \alpha$. As a consequence, an infinite sequence of contiguous loop transitions occurs without any delay between each loop transition. The proof for (29) is similar, hence proves the sufficiency. The proof of necessity is trivial by assuming some $\mathbf{x}$ in which both (28) and (29) do not hold. Then each loop transition is delayed by some finite $\delta > 0$ due to $\mathbf{x}(t) \in \mathbb{X}_\alpha$ (or $\mathbb{X}_\beta$) where $t \in [\tau, \tau + \delta)$.

Note that the underlying concept of Theorem 4.1 is general and does not depend on the definition and number of jump sets. Thus, one can easily extend Theorem 4.1 to more complicated jump sets.

The following lemma shows the relation between Theorem 4.1 and two necessary conditions of contiguous loop transitions in (23) and (24).

Lemma 4.2 *The following statements are true.*
- $\exists \mathbf{x} \in \mathbb{R}^{n_1 + n_2}$ *satisfying (28), then* $\mathbf{x}$ *satisfies both (23) and (24).*
- $\exists \mathbf{x} \in \mathbb{R}^{n_1}$ *satisfying (29), then* $\mathbf{x}$ *satisfies both (23) and (24).*
- *Statements (28) and (29) are equivalent.*

Proof: It is trivial that (23) holds for all $\mathbf{x} \in \overline{\mathbb{J}}_\beta \cap \tilde{\mathbb{J}}_\beta$. Similarly, (24) holds for all $\mathbf{x} \in \overline{\mathbb{J}}_\alpha \cap \tilde{\mathbb{J}}_\alpha$. According to the continuity of $\mathbf{x}_1 \in \overline{\mathbb{J}}_\alpha$ (and $\overline{\mathbb{J}}_\beta$), (28) and (29) state that $\mathbf{x}_1$ and $\left[\mathbf{x}_1^T \quad G\mathbf{x}_1^T \right]^T$ must lie within $\tilde{\mathbb{J}}_\alpha$ and $\tilde{\mathbb{J}}_\beta$, respectively. These prove the first two statements. Since $J\mathbf{x}_1 = 0$ characterizes both $\overline{\mathbb{J}}_\alpha$ and $\overline{\mathbb{J}}_\beta$, the left-handed side of (28) and (29) are equivalent to

$$\left\{ \left[\mathbf{x}_1^T \quad G\mathbf{x}_1^T \right]^T \big| \mathbf{x}_1 \in \tilde{\mathbb{J}}_\alpha \right\} \cap \overline{\mathbb{J}}_\beta \cap \tilde{\mathbb{J}}_\beta \text{, and}$$

$$\left\{ \mathbf{x}_1 \big| \left[\mathbf{x}_1^T \quad G\mathbf{x}_1^T \right]^T \in \tilde{\mathbb{J}}_\beta \right\} \cap \overline{\mathbb{J}}_\alpha \cap \tilde{\mathbb{J}}_\alpha \text{, respectively.}$$

Regrouping two sets $\left\{ \mathbf{x} \mid \mathbf{x} \in \overline{\mathbb{J}}_\alpha \cap \tilde{\mathbb{J}}_\alpha \right\}$ and $\left\{ \mathbf{x} \mid \mathbf{x} \in \overline{\mathbb{J}}_\beta \cap \tilde{\mathbb{J}}_\beta \right\}$, the third statement is proved.

Since each jump set is defined as a halfspace with respect to $\mathbf{x}_1$, $\overline{\mathbb{J}}_\alpha$ and $\overline{\mathbb{J}}_\beta$ are hyperplanes which are characterized by

$\mathbf{x}_1 = \mathcal{N}(J)\mathbf{z}, \mathbf{z} \in \mathbb{R}^{n-1}$ where $\mathcal{N}(J)$ is the null space of J. Applying such characterization with Theorem 4.1, we obtain the necessary and sufficient condition of well-posedness in the following theorem.

Theorem 4.2 (Well-posedness condition) *For a modular HDS described by (11)-(18), the system is well-posed if and only if there does not exist any* $\mathbf{y} \in \mathbb{R}^{n-1}$ *satisfying*

$$W(J, A_{11})\mathcal{N}\mathbf{y} \succeq 0 \qquad (30)$$

$$W\left([J \quad 0], A_2\right)\begin{bmatrix} I \\ G \end{bmatrix}\mathcal{N}\mathbf{y} \preceq 0 \qquad (31)$$

where $\mathcal{N} = \text{null}(J)$.

Proof: It follows from Lemma 4.2 that both (28) and (29) are equivalent and refer to the set of $\mathbf{x}_1$ satisfying $J\mathbf{x}_1 = 0$, $\mathbf{x}_1 \in \tilde{\mathbb{J}}_\alpha$, and $\begin{bmatrix} \mathbf{x}_1^T & G\mathbf{x}_1^T \end{bmatrix}^T \in \tilde{\mathbb{J}}_\beta$. Since (30) and (31) denote respectively $\overline{\mathbb{J}}_\alpha \cap \tilde{\mathbb{J}}_\alpha$ and $\overline{\mathbb{J}}_\beta \cap \tilde{\mathbb{J}}_\beta$, the absence of $\mathbf{y}$ satisfying (30) and (31) implies that livelock phenomenon does not exist. Then we conclude from Lemma 4.1 that the system is well-posed.

To verify well-posedness property by Theorem 4.2, one has to check if two sets defined by lexicographic inequalities are empty. We provide the following lemma as a tool for proving that (30) and (31) can not be fulfilled by any $\mathbf{y} \in \mathbb{R}^{n-1}$.

Lemma 4.3 *Given* $U \in \mathbb{R}^{m \times n}$ *and* $V \in \mathbb{R}^{p \times n}$ *with an assumption that none of the row vectors of* U *and* V *is zero. The set defined by* $\{x \in R^n \mid Ux \succeq 0, Vx \succeq 0\}$ *is not empty if the following convex feasibility problems are feasible.*

$$\begin{cases} \exists z_1 \in \mathbb{R}^n; & U_1 z_1 \geq 0, V_1 z_1 \geq 0 \\ \exists z_2 \in \mathbb{R}^n; & U_2 \tilde{U}_2 z_2 \geq 0, V_2 \tilde{V}_2 z_2 \geq 0 \\ \qquad \vdots \\ \exists z_k \in \mathbb{R}^{n-k+1}; & U_k \tilde{U}_k z_k \geq 0, V_k \tilde{V}_k z_k \geq 0 \end{cases} \qquad (32)$$

where $z_i \neq 0$; $i = 0,\dots,k$, $k = \min(m,p)$, $\tilde{U}_i = \text{null}(U_i)$, $\tilde{V}_i = \text{null}(V_i)$,

$$U = \begin{bmatrix} U_1 \\ \vdots \\ U_m \end{bmatrix} \text{, and } V = \begin{bmatrix} V_1 \\ \vdots \\ V_p \end{bmatrix}.$$

Proof: The proof is straightforward from the definition of lexicographic inequality.

If any of the convex feasibility problems is infeasible, then one can conclude that no x satisfies the corresponding convex constraints [14]. Therefore Lemma 4.3 provides a tractable method for checking the emptiness of areas defined by lexicographic inequalities. In addition, the procedure can be efficiently solved by optimization tools. Hence it is applicable with (30) and (31) in Theorem 4.2.

5. Example

This section provides an example to show the application of our results in well-posedness analysis. Note that the given example belongs to a system class which is more general than those described by (11)-(18) in two aspects. First, the system exhibits autonomous switching phenomena corresponding to a different vector field for each mode. Second, the definition of jump sets includes a bias term in the form of $Jx \geq \gamma$. These illustrate the generality of our results.

Consider a system with continuous-time dynamics of modes α and β defined as follows.

- Mode α:

$$\dot{x}_1 = 1$$

- Mode β:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -0.1 & 10 \\ 1 & -0.1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Each mode transition between α and β is governed by the following rule:

$$\xi_{k+1} = \begin{cases} \beta & ;\xi_k = \alpha \text{ and } x_1 \geq \gamma \\ \alpha & ;\xi_k = \beta \text{ and } x_1 \leq \gamma \end{cases}$$

The re-initialization of states x_1 and x_2 for each mode transition is defined by :

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_1 \text{ and } x_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

In the following, we consider the influence of γ in three cases.

1. $\gamma > 0$: we obtain from the definition of jump set and dynamics of mode α and β that :

$$\tilde{\mathbb{J}}_\alpha = \left\{ x_1 \middle| x_1 \geq \gamma \right\}$$

$$\tilde{\mathbb{J}}_\beta = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| x_1 \leq \gamma, x_2 \geq 0 \right\}$$

It follows from $G = 1$ that $\hat{\mathbb{J}}_\alpha \cap \overline{\mathbb{J}}_\beta$ and $\hat{\mathbb{J}}_\beta \cap \overline{\mathbb{J}}_\alpha$ the points $\left\{ \begin{bmatrix} \gamma & \gamma \end{bmatrix}^T \right\}$ and $\{\gamma\}$, respectively. Therefore $\hat{\mathbb{J}}_\alpha \cap \tilde{\mathbb{J}}_\beta$ is empty such that (28) does not hold. Then we conclude that the system is well-posed. The simulation of the system with $\gamma = 5$ is shown in Figure 2.

x2 is not defined for x1 < 5
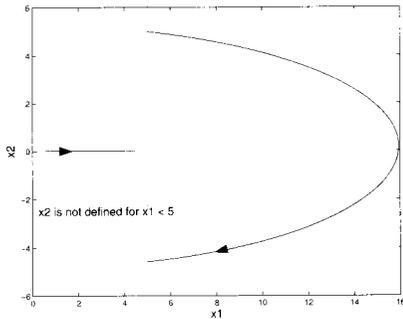
Figure 2: Simulation with $\gamma = 5$ and

$$\left(x_1(0), \xi_0 \right) = \left(0, \alpha \right)$$

2. $\gamma < 0$: since $\hat{\mathbb{J}}_\alpha \cap \tilde{\mathbb{J}}_\beta$ is $\left\{ \begin{bmatrix} \gamma & \gamma \end{bmatrix}^T \right\}$, then (28) holds. This indicates the presence of livelock phenomenon corresponding to an infinite transition between (γ, α) and $\left(\begin{bmatrix} \gamma & \gamma \end{bmatrix}^T, \beta \right)$. Therefore we conclude that those systems with $\gamma < 0$ are not well-posed.

3. $\gamma = 0$: the points $x_1 = 0$ and $\begin{bmatrix} x_1 & x_2 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ are the equilibrium points of mode α and β. Then the points $\{\gamma\}$ and $\left\{ \begin{bmatrix} \gamma & \gamma \end{bmatrix}^T \right\}$ can be excluded from our consideration. Therefore we conclude that the system with $\gamma = 0$ is well-posed.

6. Conclusion

In this paper, we have presented the necessary and sufficient conditions of well-posedness for a class of bimodal modular HDSs. Our conditions guarantee that every hybrid solution of the system class of interest can be uniquely determined over infinite time horizon. An advantage of our results is that the well-posedness condition can be reformulated into a family of convex feasibility problems, hence tractable and efficiently solvable. The illustrated example indicates that our results can be easily extended to cover broader classes of systems.

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