Numerical Simulation of the Three-Dimensional Heat Equation and Applications

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Abstract

This paper extends the fourth order time-split finite-difference methods, namely the locally one-dimensional (**LOD**) method and the alternating direction implicit (**ADI**) method for solving the three-dimensional time-dependent heat equation. The numerical solutions are compared with the analytic ones.

Keywords:LOD and ADI methods, three-dimensional heat equation

1. Introduction

Finite difference methods (**FDMs**) have been widely used for a few decades in teaching and modeling [1,2]. In this research, we will study the new splitting **FDM** in [3] by B.J. Noye and K.J. Hayman and use it to apply to three-dimensional time-dependent heat equations subject to a constant coefficient α_x and α_y which is expressed in the general form as:

$$u_t = \alpha_x u_{xx} + \alpha_y u_{yy},$$
$$0 \le x \le L,\ 0 \le y \le B,\ 0 \le t \le T. \qquad (1)$$

The ADI method is conditionally stable while the LOD method needs the condition $0 < s_x \le 2/3$ and $0 < s_y \le 2/3$ for stability. The ADI method has advantages over the LOD method.

We will extend these methods with the three-dimensional time-dependent heat equation, which is expressed in the form :

$$u_t = \alpha(u_{xx} + u_{yy} + u_{zz})\ ,0 \le x \le A,\ 0 \le y \le B$$
$$0 \le z \le C, 0 < t \le T \qquad (2)$$

With the initial condition :

$$u(x,y,z,0) = F(x,y,z), 0 \le x \le A,\ 0 \le y \le B,$$
$$0 \le z \le C, 0 < t \le T \qquad (3)$$

and Dirichlet boundary conditions which are expressed in the following form :

$$u(0,y,z,t) = G_1(y,z),\ 0 \le y \le B, 0 \le z \le C, 0 < t \le T$$

$$u(A,y,z,t) = G_2(y,z),\ 0 \le y \le B, 0 \le z \le C, 0 < t \le T$$

$$u(x,0,z,t) = G_3(x,z),\ 0 \le x \le A, 0 \le z \le C, 0 < t \le T$$

$$u(x,B,z,t) = G_4(x,z),\ 0 \le x \le A, 0 \le z \le C, 0 < t \le T$$

$$u(x,y,0,t) = G_5(x,y),\ 0 \le x \le A, 0 \le y \le B, 0 < t \le T$$

$$u(x,y,C,t) = G_6(x,y),\ 0 \le y \le B, 0 \le y \le B, 0 < t \le T \qquad (4)$$

where α is the thermal diffusivity .

2. The LOD Method

In order to solve the equation (2) by using the new **LOD** method we split this equation into three one-dimensional heat equations,

$$\frac{1}{3}\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2} \qquad (5)$$

$$\frac{1}{3}\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial y^2} \qquad (6)$$

$$\frac{1}{3}\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial z^2}. \qquad (7)$$

Each of these equations is then solved one third of the time step used for the complete three-dimensional equation. First of all we introduce some basic notation. Let $U^n_{i,j,k}$ denote an approximation of function $u(x,y,z,t)$. We use the FDM to approximate its values at gridpoint $(i\Delta x, j\Delta y, k\Delta z, n\Delta t)$ for $i=1,2,\ldots I$, $j=1,2,\ldots J, k=1,2,\ldots K$ and $n=1,2,\ldots N$. The grid spacing $\Delta x, \Delta y, \Delta z$ and time step size Δt are computed by $\Delta x = A/I$, $\Delta y = B/J$, $\Delta z = C/K$, $\Delta t = T/N$, where I,J,K and N are integers. We can compute the grid Fourier numbers in x, y and z directions by using the following formulas :

$s_x = \alpha\Delta t/(\Delta x)^2$, $s_y = \alpha\Delta t/(\Delta y)^2$ and $s_z = \alpha\,\Delta t/(\Delta z)^2$, respectively.

Next, we shall now show that the formula used in solving equation (5) is :

$$U^{n+1/3}_{i,j,k} = \frac{s_x}{12}(6s_x - 1)\left(U^n_{i-2,j,k} + U^n_{i+2,j,k}\right) + \frac{2s_x}{3}(2 - 3s_x)\left(U^n_{i-1,j,k} + U^n_{i+1,j,k}\right) + \frac{1}{2}\left(2 - 5s_x + 6s_x^2\right)U^n_{i,j,k} \tag{8}$$

where $i = 2,3,\ldots ,I-2$ for all $j = 0,1,2,\ldots,J$ and $k = 0,1,2,\ldots,K$. The problem of finding values of $U^{n+1/3}_{1,j,k}$ which can not be derived by using (8) can be overcome by using a re-arrangement of the unconditionally stable inverted version of the equation (8), which is obtained by setting Δt for $-\Delta t$ in (8). Putting $i = 3$ and re-arranging gives :

$$U^{n+1/3}_{1,j,k} = \frac{8(2+3s_x)}{(6s_x+1)}\left(U^{n+1/3}_{2,j,k} + U^{n+1/3}_{4,j,k}\right) - \frac{6\left(2+5s_x+6s_x^2\right)}{s_x(6s_x+1)}U^{n+1/3}_{3,j,k} - U^{n+1/3}_{5,j,k} + \frac{12}{s_x(6s_x+1)}U^n_{3,j,k}. \tag{9}$$

This gives values of $U^{n+1/3}_{1,j,k}$ for all $j = 0,1,2,\ldots,J$ and $k = 0,1,2,\ldots,K$, because all the values on its right hand side are known. The values of $U^{n+1/3}_{J-1,j,k}$ can be calculated by using a similar formula obtained when replacing Δt by $-\Delta t$ and setting $i = I-3$ in the equation (8) to obtain:

$$U^{n+1/3}_{I-1,j,k} = \frac{8(2+3s_x)}{(6s_x+1)}\left(U^{n+1/3}_{I-2,j,k} + U^{n+1/3}_{I-4,j,k}\right) - \frac{6\left(2+5s_x+6s_x^2\right)}{s_x(6s_x+1)}U^{n+1/3}_{I-3,j,k} - U^{n+1/3}_{I-5,j,k} + \frac{12}{s_x(6s_x+1)}U^n_{I-3,j,k} \tag{10}$$

for all $j = 0,1,2,\ldots,J$ and $k = 0,1,2,\ldots,K$. Similarly, for computing values of $U^{n+2/3}_{i,j,k}$ from the values of $U^{n+1/3}_{i,j,k}$ in the y-sweep used in the second stage, the formula used with $j = 2,3,4,\ldots,J-2$ for each $i = 1,2,\ldots,I-1$ and $k = 1,2,\ldots,K-1$ is given by :

$$U^{n+2/3}_{i,j,k} = \frac{s_y}{12}(6s_y - 1)\left(U^{n+1/3}_{i,j-2,k} + U^{n+1/3}_{i,j+2,k}\right) + \frac{2s_y}{3}(2 - 3s_y)\left(U^{n+1/3}_{i,j-1,k} + U^{n+1/3}_{i,j+1,k}\right) + \frac{1}{2}\left(2 - 5s_y + 6s_y^2\right)U^{n+1/3}_{i,j,k} \tag{11}$$

The formulas for computing the values $U^{n+2/3}_{i,1,k}$ and $U^{n+2/3}_{i,J-1,k}$ are given as :

$$U^{n+2/3}_{i,1,k} = \frac{8(2+3s_y)}{(6s_y+1)}\left(U^{n+2/3}_{i,2,k} + U^{n+2/3}_{i,4,k}\right) - \frac{6\left(2+5s_y+6s_y^2\right)}{s_y(6s_y+1)}U^{n+2/3}_{i,3,k} - U^{n+2/3}_{i,5,k} + \frac{12}{s_y(6s_y+1)}U^{n+1/3}_{i,3,k} \tag{12}$$

$$U^{n+2/3}_{i,J-1,k} = \frac{8(2+3s_y)}{(6s_y+1)}\left(U^{n+2/3}_{i,J-2,k} + U^{n+2/3}_{i,J-4,k}\right) - \frac{6\left(2+5s_y+6s_y^2\right)}{s_y(6s_y+1)}U^{n+2/3}_{i,J-3,k} - U^{n+2/3}_{i,J-5,k} + \frac{12}{s_y(6s_y+1)}U^{n+1/3}_{i,J-3,k} \tag{13}$$

To compute the values of $U^{n+1}_{i,j,k}$ from the values of $U^{n+2/3}_{i,j,k}$ in the z-sweep used in the third stage, the formula used with $k = 2,3,4,\dots,K-2$ for each $i = 1,2,3,\dots,I-1$ and $j = 1,2,3,\dots,J-1$ is :

$$
\begin{aligned}
U^{n+1}_{i,j,k} &= \frac{s_z}{12}(6s_z-1)\left(U^{n+2/3}_{i,j,k-2}+U^{n+2/3}_{i,j,k+2}\right) \\
&\quad + \frac{2s_z}{3}(2-3s_z)\left(U^{n+2/3}_{i,j,k-1}+U^{n+2/3}_{i,j,k+1}\right) \\
&\quad + \frac{1}{2}\left(2-5s_y+6s_y^2\right)U^{n+2/3}_{i,j,k}
\end{aligned} \tag{14}
$$

Similarly, the formula for computing values of $U^{n+1}_{i,j,1}$ is given by :

$$
\begin{aligned}
U^{n+1}_{i,j,1} &= \frac{(6s_z+1)}{8(2+3s_z)}\left(U^{n+1}_{i,j,0}+U^{n+1}_{i,j,4}\right) \\
&\quad + \frac{3(2+5s_z+6s_z^2)}{4s_z(2+3s_z)}U^{n+1}_{i,j,2} \\
&\quad - U^{n+1}_{i,j,3} - \frac{3}{2s_z(2+3s_z)}U^{n+2/3}_{i,j,2}
\end{aligned} \tag{15}
$$

At the end of the complete procedure involving heat distribution in x, y and z directions, the known boundary values $U^{n+1}_{i,j,0}$ are used. We include the values on the right hand side of equation (15), which all values are known. The values of $U^{n+1}_{i,j,K-1}$ are found by using a similar equation as in equation (16).

$$
\begin{aligned}
U^{n+1}_{i,j,K-1} &= \frac{(6s_z+1)}{8(2+3s_z)}\left(U^{n+1}_{i,j,K}+U^{n+1}_{i,j,K-4}\right) \\
&\quad + \frac{3(2+5s_z+6s_z^2)}{4s_z(2+3s_z)}U^{n+1}_{i,j,K-2} \\
&\quad - U^{n+1}_{i,j,K-3} - \frac{3}{2s_z(2+3s_z)}U^{n+2/3}_{i,j,K-2}
\end{aligned} \tag{16}
$$

We will determine the von Neumann stability [1], which is applied to the problem (8). First let $U^n_{p,q,r} = e^n e^{i\beta_1 b_1 h} e^{i\beta_2 b_2 h} e^{i\beta_3 b_3 h}$. Then we substitute it in the equation (8), to yield :

$$
\begin{aligned}
&\xi^{n+\frac{1}{3}} e^{i\beta_1 pk} e^{i\beta_2 qk} e^{i\beta_3 rk} = \\
&\frac{s_x}{12}(6s_x-1)\left(\xi^n e^{i\beta_1 (p-2)k} e^{i\beta_2 qk} e^{i\beta_3 rk} + \xi^n e^{i\beta_1 (p+2)k} e^{i\beta_2 qk} e^{i\beta_3 rk}\right) + \frac{2s_x}{3}(2-3s_x) \\
&\left(\xi^n e^{i\beta_1 (p-1)k} e^{i\beta_2 qk} e^{i\beta_3 rk} + \xi^n e^{i\beta_1 (p+1)k} e^{i\beta_2 qk} e^{i\beta_3 rk}\right) \\
&+ \frac{1}{2}\left(2-5s_x+6s_x^2\right)\xi^n e^{i\beta_1 pk} e^{i\beta_2 rk} e^{i\beta_3 rk}
\end{aligned} \tag{17}
$$

Canceling all common terms, by dividing (17) with $\xi^n e^{i\beta_1 pk} e^{i\beta_2 qk} e^{i\beta_3 rk}$, we get :

$$
\xi^{\frac{1}{3}} = \frac{s_x}{12}(6s_x-1)\left(e^{-2i\beta_1 k}+e^{2i\beta_1 k}\right) + \frac{2s_x}{3}(2-3s_x)\left(e^{-i\beta_1 k}+e^{i\beta_1 k}\right) + \frac{1}{2}\left(2-5s_x+6s_x^2\right) \tag{18}
$$

Then using $\frac{e^{i\beta k}+e^{-i\beta k}}{2} = \cos\beta k$, the amplification factor is :

$$
\xi^{\frac{1}{3}} = \frac{s_x(6s_x-1)}{3}\cos^2\beta_1 k + \frac{4s_x(2-3s_x)}{3}\cos\beta_1 k + \frac{3-7s_x+6s_x^2}{3} \tag{19}
$$

where $\beta_1 = m\pi; m \in I$ and let $\xi^{\frac{1}{3}} = G$, the condition is :

$$
|G| \le 1 \tag{20}
$$

This is required for stability. For all positive values of s_x we obtain that **FDE** (8) is stable over the interval [4]. Similarly, we obtain that the FDEs are stable over all s_y and s_z in the interval $(0, 2/3]$. Thus this method is stable, in the von Neumann sense, for :

$$
0 < s_x, s_y, s_z \le 2/3. \tag{21}
$$

3. The ADI Method

In order to solve the problem (2) by the new ADI method, we use a three stage procedure. The first stage to use in the z–direction sweep is :

$$
\begin{aligned}
&(6s_z-1)U^{n+1}_{i,j,k-1} - 4(1+3s_z)U^{n+1}_{i,j,k} + (6s_z-1)U^{n+1}_{i,j,k+1} \\
&= -s_x\left(U^n_{i-1,j,k-1}+U^n_{i-1,j,k+1}+U^n_{i+1,j,k-1}+U^n_{i+1,j,k+1}\right) \\
&\quad - s_y\left(U^n_{i,j-1,k-1}+U^n_{i,j-1,k+1}+U^n_{i,j+1,k-1}+U^n_{i,j+1,k+1}\right)
\end{aligned}
$$

$$-4s_x\left(U^n_{i-1,j,k}+U^n_{i+1,j,k}\right)-4s_y\left(U^n_{i,j-1,k}+U^n_{i,j+1,k}\right)$$
$$+\left(2s_x+2s_y-1\right)\left(U^n_{i,j,k-1}+4U^n_{i,j,k}+U^n_{i,j,k+1}\right) \quad (22)$$

The second stage in the next time step, the formula in the y–direction sweep is :

$$(6s_y-1)U^{n+2}_{i,j-1,k}-4(1+3s_y)U^{n+2}_{i,j,k}+(6s_y-1)U^{n+2}_{i,j+1,k}$$
$$=-s_x\left(U^{n+1}_{i-1,j-1,k}+U^{n+1}_{i-1,j+1,k}+U^{n+1}_{i+1,j-1,k}+U^{n+1}_{i+1,j+1,k}\right)$$
$$-s_z\left(U^{n+1}_{i,j-1,k-1}+U^{n+1}_{i,j-1,k+1}+U^{n+1}_{i,j+1,k-1}+U^{n+1}_{i,j+1,k+1}\right)$$
$$-4s_x\left(U^{n+1}_{i-1,j,k}+U^{n+1}_{i+1,j,k}\right)-4s_z\left(U^{n+1}_{i,j,k-1}+U^{n+1}_{i,j,k+1}\right)$$
$$+(2s_x+2s_z-1)\left(U^{n+1}_{i,j-1,k}+4U^{n+1}_{i,j,k}+U^{n+1}_{i,j+1,k}\right) \quad (23)$$

Finally, the formula to use in the x–direction sweep is :

$$(6s_x-1)U^{n+3}_{i-1,j,k}-4(1+3s_x)U^{n+3}_{i,j,k}+(6s_x-1)U^{n+3}_{i+1,j,k}$$
$$=-s_y\left(U^{n+2}_{i-1,j-1,k}+U^{n+2}_{i-1,j+1,k}+U^{n+2}_{i+1,j-1,k}+U^{n+2}_{i+1,j+1,k}\right)$$
$$-s_z\left(U^{n+2}_{i-1,j,k-1}+U^{n+2}_{i-1,j,k+1}+U^{n+2}_{i+1,j,k-1}+U^{n+2}_{i+1,j,k+1}\right)$$
$$-4s_y\left(U^{n+2}_{i,j-1,k}+U^{n+2}_{i,j+1,k}\right)-4s_z\left(U^{n+2}_{i,j,k-1}+U^{n+2}_{i,j,k+1}\right)$$
$$+\left(2s_y+2s_z-1\right)\left(U^{n+2}_{i-1,j,k}+4U^{n+2}_{i,j,k}+U^{n+2}_{i+1,j,k}\right) \quad (24)$$

In order to show the stability of the **ADI** method, we replace $U^n_{p,q,r}=\xi^n e^{i\beta_1 pk}e^{i\beta_2 qk}e^{i\beta_3 rk}$ into(22) and then divide by $\xi^n e^{i(\beta_1 p+\beta_2 q+\beta_3 r)k}$ We get :

$$(6s_z-1)\xi\left(e^{-i\beta_3 k}+e^{i\beta_3 k}\right)-4(1+3s_z)\xi$$
$$=-s_x\left\{e^{-i\beta_1 k}\left(e^{-i\beta_3 k}+e^{i\beta_3 k}\right)+e^{i\beta_1 k}\left(e^{-i\beta_3 k}+e^{i\beta_3 k}\right)\right\}$$
$$-s_y\left\{e^{-i\beta_2 k}\left(e^{-i\beta_3 k}+e^{i\beta_3 k}\right)+e^{i\beta_2 k}\left(e^{-i\beta_3 k}+e^{i\beta_3 k}\right)\right\}$$
$$-4s_x\left\{e^{-i\beta_1 k}+e^{i\beta_1 k}\right\}-4s_y\left\{e^{-i\beta_2 k}+e^{i\beta_2 k}\right\}$$
$$+\left(2s_x+2s_y-1\right)\left\{4+e^{-i\beta_3 k}+e^{i\beta_3 k}\right\} \quad (25)$$

Using $e^{-i\beta_i k}+e^{i\beta_i k}=2\cos\beta_i k$, for $i=1,2,3$, to obtain :

$$\xi=-\frac{4s_x(\cos\beta_1 k\cos\beta_3 k)}{2(6s_z-1)\cos\beta_3 k-(4+12s_z)}-\frac{4s_y(\cos\beta_2 k\cos\beta_3 k)}{2(6s_z-1)\cos\beta_3 k-(4+12s_z)}+\frac{(-8s_x\cos\beta_1 k-8s_y\cos\beta_2 k)}{2(6s_z-1)\cos\beta_3 k-(4+12s_z)}$$
$$+\frac{((2s_x+2s_y-1)(4+2\cos\beta_3 k))}{2(6s_z-1)\cos\beta_3 k-(4+12s_z)} \quad (26)$$

We then apply $\cos\beta k=1-2\sin^2\frac{\beta k}{2}$, $a_i=\sin^2\frac{\beta_i k}{2}; i=1,2,3$ and put $\xi=\xi_z$ in (26) to obtain :

$$\xi_z=\left\{\frac{(24-16a_3)(a_1s_x+a_2s_y)+2(2a_3-3)}{-24a_3s_z+2(2a_3-3)}\right\}$$
$$=\left\{\frac{(2a_3-3)(1-4(a_1s_x+a_2s_y))}{-12a_3s_z+(2a_3-3)}\right\} \quad (27)$$

Using the same argument together with (23) and (24) yields :

$$\xi_y=\left\{\frac{(24-16a_2)(a_1s_x+a_3s_z)+2(2a_2-3)}{-24a_2s_y+2(2a_2-3)}\right\}$$
$$=\left\{\frac{(2a_2-3)(1-4(a_1s_x+a_3s_z))}{-12a_2s_y+(2a_2-3)}\right\}. \quad (28)$$

$$\xi_x=\left\{\frac{(24-16a_1)(a_2s_y+a_3s_z)+2(2a_1-3)}{-24a_1s_x+2(2a_1-3)}\right\}$$
$$=\left\{\frac{(2a_1-3)(1-4(a_2s_y+a_3s_z))}{-12a_1s_x+(2a_1-3)}\right\} \quad (29)$$

Let $s=s_x=s_y=s_z$ which yields :

$$G=\xi_x\xi_y\xi_z=$$
$$\left\{\frac{(2a_3-3)(1-4(a_1+a_2)s)}{-12a_3s+(2a_3-3)}\right\}\times\left\{\frac{(2a_2-3)(1-4(a_1+a_3)s)}{-12a_2s+(2a_2-3)}\right\}\times\left\{\frac{(2a_1-3)(1-4(a_2+a_3)s)}{-12a_1s+(2a_1-3)}\right\}. \quad (30)$$

The condition (20) is also required for stability. In order to find the values of s which satisfy the condition in (20), we sudstitute $G=\xi_x\xi_y\xi_z$ into (20) This yields :

$$|\xi_x\xi_y\xi_z|=|\xi_x||\xi_y||\xi_z|\le 1 \quad (31)$$

Let us consider the case $|\xi_x| \le 1$, $|\xi_y| \le 1$ and $|\xi_z| \le 1$ which satisfy (31).

We will show the values of s which satisfy the condition $|\xi_z| \le 1$. Consider $i\,a_i = \sin^2\frac{\beta_i k}{2}; i = 1,2,3$ where $\beta_i = m\pi; m \in I$. If $k \in I$ we then obtain $\sin^2\frac{\beta_i k}{2} = 0$ and $\sin^2\frac{\beta_i k}{2} = 1$. We now consider the values of ξ_z when $k \notin I$. Then $\left\{\frac{(2a_3 - 3)(1 - 4(a_1 + a_2)s)}{-12a_3 s + (2a_3 - 3)}\right\}$ can be expressed as a function of $\sin\frac{\beta_3 k}{2}$ as :

$$\xi_z\left(\sin\frac{\beta_3 k}{2}\right) = \left\{\frac{\left(2\sin^2\frac{\beta_3 k}{2} - 3\right)(1 - 4(a_1 + a_2)s)}{-12s\sin^2\frac{\beta_3 k}{2} + \left(2\sin^2\frac{\beta_3 k}{2} - 3\right)}\right\} \qquad (32)$$

Its extreme value of $\xi_z\left(\sin\frac{\beta_3 k}{2}\right)$ occurs at $\sin\frac{\beta_3 k}{2} = 0$, we now substitute this value into equation (32) to obtain equation (33).

$$\xi_z\left(\sin\frac{\beta_3 k}{2}\right) = (1 - 4(a_1 + a_2)s). \qquad (33)$$

Hence the condition $\left|\xi_z\left(\sin\frac{\beta_3 k}{2}\right)\right| = |(1 - 4(a_1 + a_2)s)| \le 1$ is required for stability. This condition requires that $0 \le s \le \frac{1}{2(a_1 + a_2)}$. Choosing the minimum value of $\frac{1}{2(a_1 + a_2)}$ we obtain $0 \le s \le 1/4$. Combining all the results leads to the conclusion that $0 \le s \le 1/4$.

Similarly, applying the above procedure with ξ_x in (23) and ξ_y in (24), we obtain that $0 < s \le 1/4$. Hence the **FDEs** (22)-(24) are stable over the interval $(0,1/4]$. This shows that the method is conditionally stable in the three-dimensional problem.

4. Numerical Test

We will use the LOD method and the ADI Method for solving the following equation :

$$u_t = \alpha(u_{xx} + u_{yy} + u_{zz}),$$
$$0<x<0.2, 0<y<0.2, 0<z<0.1 \qquad (34)$$

Subject to the initial conditions:

$$u(x,y,z,0) = 300,$$
$$0 \le x \le 0.2,\ 0 \le y \le 0.2,\ 0 \le z \le 0.1 \qquad (35)$$

and the boundary conditions are :

$$u(0,y,z,t) = u(0.2,y,z,t) = u(x,0,z,t) = 0$$
$$= u(x,0.2,z,t) = u(x,y,0,t) = u(x,y,0.1,t) \qquad (36)$$

when $\alpha = 1.01183\times10^{-6}\ m^2/s$. The analytic solution of this problem is :

$$u(x,y,z,t) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty} A_{(2p+1)(2q+1)(2r+1)} \sin\left(\frac{(2p+1)\pi x}{0.2}\right)\times$$
$$\sin\left(\frac{(2q+1)\pi y}{0.2}\right)\sin\left(\frac{(2r+1)\pi z}{0.1}\right)\times$$
$$\exp\left[-\alpha\pi^2\left(\frac{(2p+1)^2}{0.2^2} + \frac{(2q+1)^2}{0.2^2} + \frac{(2r+1)^2}{0.1^2}\right)t\right] \qquad (37)$$

when $A_{(2p+1)(2q+1)(2r+1)} = \frac{4.8\times10^6}{(2p+1)(2q+1)(2r+1)\pi^3}$

Figure 1 presents values of the analytic solution, $u(0.1,0.1,z,t)$ and numerical solution $U(0.1,0.1,z,n\Delta t)$. We let $\Delta x = \Delta y = \Delta z = 0.01\,m$ and $\Delta t = \underline{49.4154156330}\ s$. We can observe that the **LOD** method gives a better approximation than that of the **ADI** method.

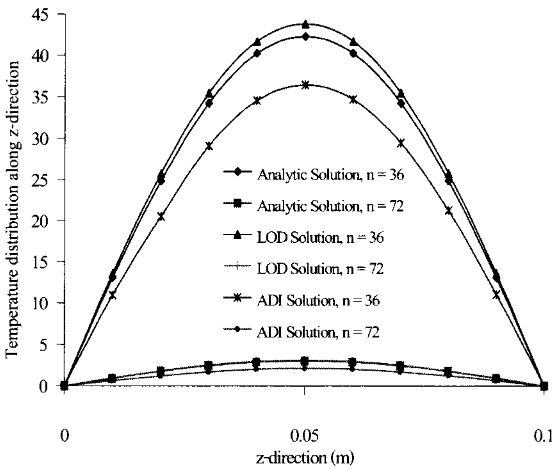

Figure 1. Temperature decay along z-direction when $s = 1/2$.

This guarantees that the ADI method is conditionally stable when the Grid Fourier number s is in the interval $(0, 1/4]$.

In order to present the order of accuracy of the **LOD** method, we use the function $u(x, y, z, t) = \exp(-3\alpha\pi^2 t)\sin\pi x\sin\pi y\sin\pi z + 100xyz$ to find the values at the grid point $(0.5, 0.5, 0.5, 1)$, where $\alpha = 0.01$ and $\Delta x = \Delta y = \Delta z$.

Table 1 presents errors of **LOD** solutions when compared with the exact solution, $u(0.5, 0.5, 0.5, 1) = \underline{13.2437218794}$ where m denotes spatial grid separation.

Table 1. Errors of the numerical solution in different number of grid spatial separation.

m	Error when $s = 1/2$ $\times 10^{-4}$	Error when $s = 1/3$ $\times 10^{-5}$	Error when $s = 1/4$ $\times 10^{-5}$	Error when $s = 1/5$ $\times 10^{-5}$	Error when $s = 1/6$ $\times 10^{-5}$
10	0.2437	0.4410	0.2340	0.3669	0.5658
20	0.0150	0.0257	0.0110	0.0178	0.0295
30	0.0030	0.0050	0.0020	0.0033	0.0058
40	0.0009	0.0016	0.0006	0.0010	0.0019
50	0.0004	0.0006	0.0002	0.0004	0.0008
60	0.0002	0.0003	0.0001	0.0002	0.0004

Consider the relation between error and spatial grid separation for the LOD method in Figure 2. This guarantees and shows that the order of accuracy of the LOD method is four when s satisfies the stability condition (21).

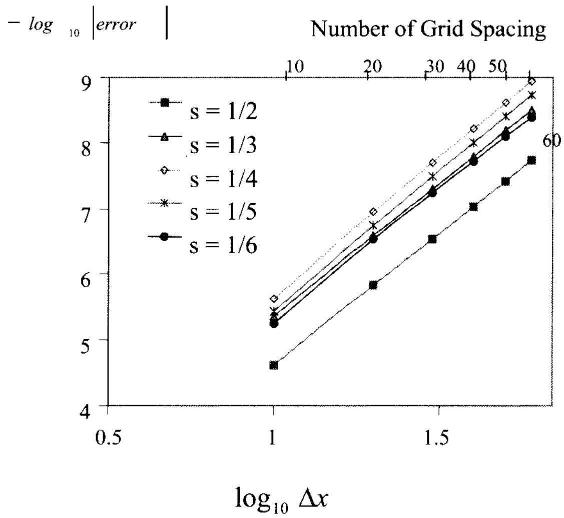

Figure 2 The relation between error and spatial grid separation for the LOD method.

5. Discussion and Conclusion

In the two-dimensional heat equation, the ADI is unconditionally stable [3]. This proves the usefulness since Δt can be made large without loss of stability. Thus, the ADI method has an advantage over the LOD method.

The result of our findings for solving three-dimensional equations is that the ADI method loses its advantage. Hence for solving three-dimensional heat equations, using the LOD method is better than that of the ADI method.

In applications, the temperature distribution of the samples may be used for practical work such as heat sensor for a black box to quickly search for the exact location of an aircraft accident, etc.

6.References

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